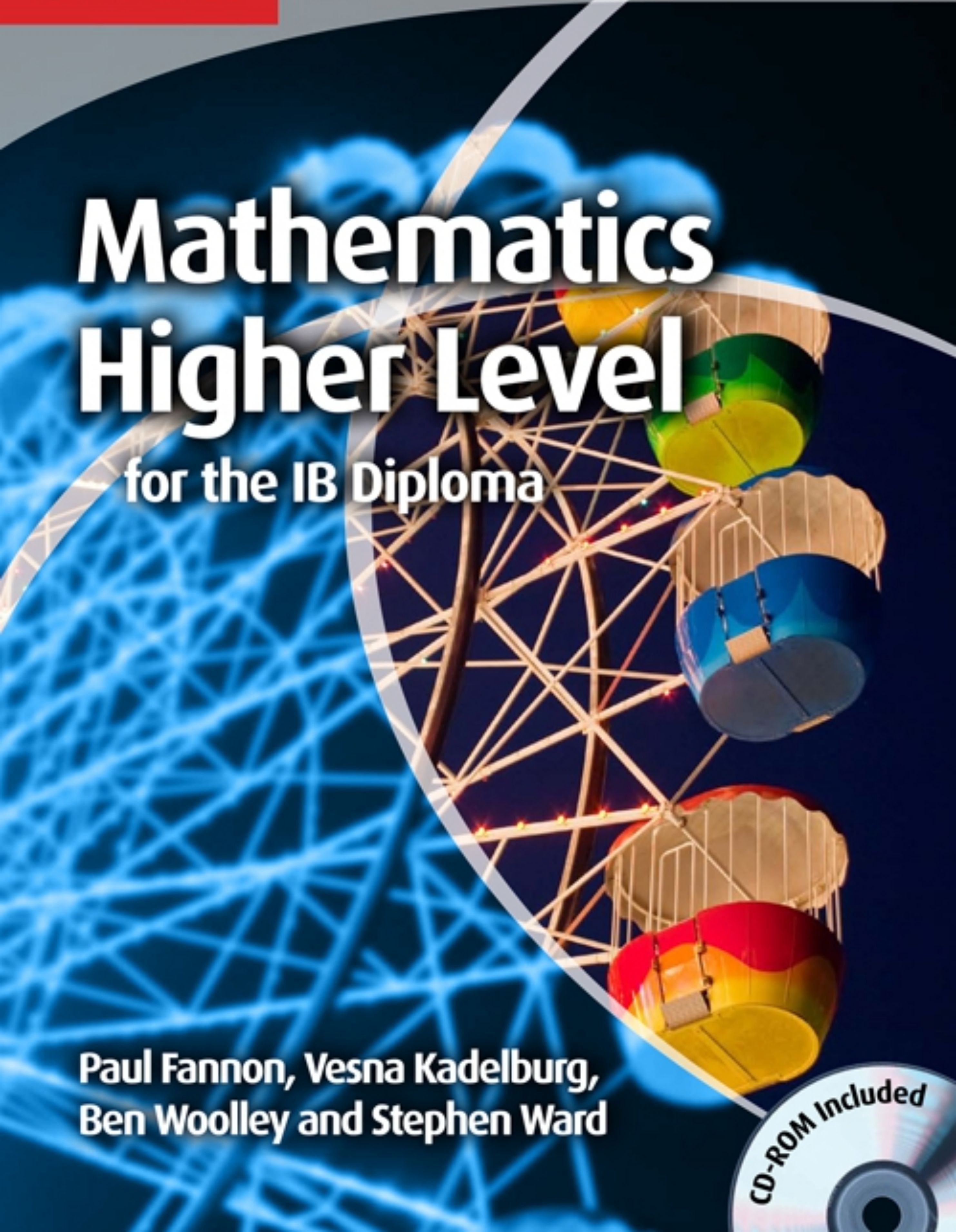
Cambridge

Mathematics Higher Level

for the IB Diploma



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Introduction

Structure of the book

The book is split roughly into four blocks: chapters 2 to 8 cover algebra and functions; chapters 9 to 14 cover geometry; chapters 16 to 20 cover calculus; and chapters 21 to 24 cover probability and statistics. Chapters 1, 15 and 25; on counting principles, complex numbers and induction (respectively), bring together several areas of the course and chapter 26 includes questions that mix different parts of the course – a favourite trick in International Baccalaureate® (IB) examinations.

You do not have to work through the book in the order presented, but (given how much the International Baccalaureate® likes to mix up topics) you will find that several questions refer to material in previous chapters. In these cases a 'rewind panel' will tell you that the material has been covered previously so that you can either remind yourself or decide to move on.

We have tried to include in the book only the material that will be examinable. There are many proofs and ideas which are useful and interesting, and these are on the CD-ROM if you would like to explore them.

Each chapter starts with a list of learning objectives to give you an idea about what the chapter contains. There is also an introductory problem that illustrates what you will be able to do after you have completed the chapter. Some introductory problems relate to 'real life' situations, while others are purely mathematical. You should not expect to be able to solve the problem, but you may want to think about possible strategies and what sort of new facts and methods would help you. The solution to the introductory problem is provided at the end of the chapter, after the summary of the chapter contents.

Key point boxes

The most important ideas and formulae are emphasised in the 'KEY POINT' boxes. When the formulae are given in the Formula booklet, there will be an icon: ; if this icon is not present, then the formulae are ***not*** in the Formula booklet and you may need to learn them or at least know how to derive them.

Worked examples

Each worked example is split into two columns. On the right is what you should write down. Sometimes the example might include more detail then you strictly need, but it is designed to give you an idea of what is required to score full method marks in examinations. However, mathematics is about much more than examinations and remembering methods. So, on the left of the worked examples are notes that describe the thought processes and suggest which route you should use to tackle the question. We hope that these will help you with any exercise questions that differ from the worked examples. It is very deliberate that some of the questions require you to do more than repeat the methods in the worked examples. Mathematics is about thinking!

Signposts

There are several boxes that appear throughout the book.

Theory of knowledge issues

Every lesson is a Theory of knowledge lesson, but sometimes the links may not be obvious. Mathematics is frequently used as an example of certainty and truth, but this is often not the case. In these boxes we will try to highlight some of the weaknesses and ambiguities in mathematics as well as showing how mathematics links to other areas of knowledge.

From another perspective

The International Baccalaureate® encourages looking at things in different ways. As well as highlighting some international differences between mathematicians these boxes also look at other perspectives on the mathematics we are covering: historical, pragmatic and cultural.

Research explorer

As part of your course, you will be asked to write a report on a mathematical topic of your choice. It is sometimes difficult to know which topics are suitable as a basis for such reports, and so we have tried to show where a topic can act as a jumping-off point for further work. This can also give you ideas for an Extended essay. There is a lot of great mathematics out there!

Exam hint

EXAM HINT

Although we would encourage you to think of mathematics as more than just learning in order to pass an examination, there are some common errors it is useful for you to be aware of. If there is a common pitfall we will try to highlight it in these boxes. We also point out where graphical calculators can be used effectively to simplify a question or speed up your work, often referring to the relevant calculator skills sheet on the CD-ROM.

Fast forward / rewind

Mathematics is all about making links. You might be interested to see how something you have just learned will be used elsewhere in the course, or you may need to go back and remind yourself of a previous topic. These boxes indicate connections with other sections of the book to help you find your way around.

How to use the questions

The colour-coding

The questions are colour-coded to distinguish between the levels.

Black questions are drill questions. They help you practise the methods described in the book, but they are usually not structured like the questions in the examination. This does not mean they are easy, some of them are quite tough.

Each differently numbered drill question tests a different skill. Lettered subparts of a question are of increasing difficulty. Within each lettered part there may be multiple roman-numeral parts ((i), (ii),...) , all of which are of a similar difficulty. Unless you want to do lots of practice we would recommend that you only do one roman-numeral part and then check your answer. If you have made a mistake then you may want to think about what went wrong before you try any more. Otherwise move on to the next lettered part.

Green questions are examination-style questions which should be accessible to students on the path to getting a grade 3 or 4.

Blue questions are harder examination-style questions. If you are aiming for a grade 5 or 6 you should be able to make significant progress through most of these.

Red questions are at the very top end of difficulty in the examinations. If you can do these then you are likely to be on course for a grade 7.

Gold questions are a type that are *not* set in the examination, but are designed to provoke thinking and discussion in order to help you to a better understanding of a particular concept.

At the end of each chapter you will see longer questions typical of the second section of International Baccalaureate® examinations. These follow the same colour-coding scheme.

Of course, these are just **guidelines**. If you are aiming for a grade 6, do not be surprised if you find a green question you cannot do. People are never equally good at all areas of the syllabus. Equally, if you can do all the red questions that does not guarantee you will get a grade 7; after all, in the examination you have to deal with time pressure and examination stress!

These questions are graded relative to our experience of the final examination, so when you first start the course you will find all the questions relatively hard, but by the end of the course they should seem more straightforward. Do not get intimidated!

Calculator versus non-calculator questions

In the final examination there will be one paper in which calculators are not allowed. Some questions specifically need a calculator but most could appear in either the calculator or the non-calculator paper.

Some questions are particularly common in the non-calculator paper and you must be able to know how to deal with these. They are highlighted by the non-calculator symbol.

Some questions can be done in a clever way using a calculator or cannot be realistically done without using a calculator. These questions are marked with a calculator symbol.

In the final examination you will not get this indication, so you must make sure you learn which types of questions have an easy calculator method. The calculator skills sheets on the CD-ROM can help with this.

For the questions that do not have either calculator icon, you should mix up practising with and without a calculator. Be careful not to become too reliant on your calculator. Half of the core examination is done without one!

On the CD-ROM

On the CD-ROM there are various materials that you might find useful.

Prior learning

The International Baccalaureate® syllabus lists what candidates should know before taking the examination. The topics on the list are not all explicitly covered in the syllabus, but their knowledge may be required to answer examination questions. Don't worry, you do not need to know all this before starting the course, and we have indicated in the rewind boxes where a particular concept or skill is required. On the CD-ROM you will find a self-assessment test to check your knowledge, and some worksheets to help you learn any skills that you might be missing or need to revise.

Option chapters

Each of the four options is covered by several chapters on the CD-ROM.

Coursebook support

Supporting worksheets include:

- calculator skills sheets that give instructions for making optimal use of some of the recomended graphical calculators
- extension worksheets that go in difficulty beyond what is required at International Baccalaureate®
- fill-in proof sheets to allow you to re-create proofs that are not required in the examination
- self-discovery sheets to encourage you to investigate new results for yourself
- supplementary sheets exploring some applications, international and historical perspectives of the mathematics covered in the syllabus.

e-version

A flat pdf of the whole coursebook (for days when you don't want to carry the paperback!).

We hope you find Higher Level Mathematics for the IB diploma an interesting and enriching course. You might also find it quite challenging, but do not get intimidated, frequently topics only make sense after lots of revision and practice. Persevere and you will succeed.

The author team.

1 Counting principles

In this chapter you will learn:

- how to break down complicated questions into parts that are easier to count, and then combine them together
- how to count the number of ways to arrange a set of objects
- the algebraic properties of a useful new tool called the factorial function
- in how many ways you can choose objects from a group
- strategies for applying these tools to harder problems.

Introductory problem

If a computer can print a line containing all 26 letters of the alphabet in 0.01 seconds, estimate how long it would take to print all possible permutations of the alphabet.

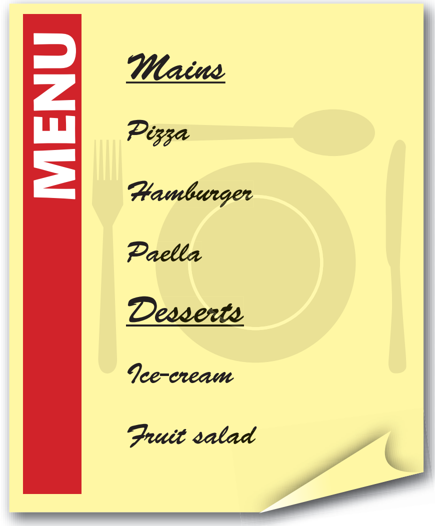
Counting is one of the first things we learn in mathematics and at first it seems very simple. If you were asked to count how many people there are in your school, this would not be too tricky. If you were asked how many chess matches would need to be played if everyone were to play everyone else, this would be a little more complicated. If you were asked how many different football teams could be chosen, you might find that the numbers become far too large to count without coming up with some clever tricks. This chapter aims to help develop strategies for counting in such difficult situations.

1A The product principle and the addition principle

Counting very small groups is easy. So, we need to break down more complicated problems into counting small groups. But how do we then combine these together to come up with an answer to the overall problem? The answer lies in using the product principle and the addition principle, which can be illustrated using the following menu.

Counting sometimes gets extremely difficult. Are there more whole numbers or odd numbers; fractions or decimals? Have a look at the work of Georg Cantor, and the result may surprise you!

The word analysis literally means 'breaking up'. When a problem is analysed it is broken down into simpler parts. One of the purposes of studying mathematics is to develop an analytical mind, which is considered very useful in many different disciplines.



Anna would like to order a main course *and* a dessert. She can choose one of three main courses and one of two desserts. How many different choices could she make? Bob would like to order *either* a main course *or* a dessert. He can choose one of the three main courses or one of the two desserts; how many different orders can he make?

$n(A)$ means the size of the set of options of A. See Prior Learning Section G on the CD-ROM.

We can use the notation $n(A)$ to represent the number of ways of making a choice about A.

The **product principle** tells us that when we want to select one option from A *and* one option from B we *multiply* the individual possibilities together.

KEY POINT 1.1

The Product principle (AND rule)

The number of ways of in which both choice A *and* choice B can be made is the product of the number of options for A and the number of options for B.

$n(A \text{ AND } B) = n(A) \times n(B)$

The **addition principle** tells us that when we wish to select one option from A *or* one option from B we *add* the individual possibilities together.

The addition principle has one essential restriction. You can only use it if there is no overlap between the choices for A and the choices for B. For example, you cannot apply the addition principle to counting the number of ways of getting an odd number or a prime number on a die. If there is no overlap between the choices for A and for B, the two events are **mutually exclusive**.

KEY POINT 1.2

The Addition Principle (OR rule)

The number of ways of in which either choice A *or* choice B can be made is the sum of the number of options for A and the number of options for B.

If A and B are mutually exclusive then
$n(A \text{ OR } B) = n(A) + n(B)$

The hardest part of applying either the addition or product principle is breaking the problem down and deciding which principle to use. You must make sure that you have included all of the cases and checked that they are mutually exclusive. It is often useful to rewrite questions to emphasise what is required, 'AND' or 'OR'.

Worked example 1.1

An examination has ten questions in section A and four questions in section B. How many different ways are there to choose questions if you must:

(a) choose one question from each section?
(b) choose a question from either section A or section B?

Describe the problem accurately

(a) Choose one question from A (10 ways)
AND
one from B (4 ways)

'AND' means we should apply the product principle: $n(A) \times n(B)$

Number of ways $= 10 \times 4$
$= 40$

Describe the problem accurately

(b) Choose one question from A (10 ways)
OR
one from B (4 ways)

'OR' means we should apply the addition principle: $n(A) + n(B)$

Number of ways $= 10 + 4$
$= 14$

In the example above we cannot answer a question twice so there are no repeated objects; however, this is not always the case.

Worked example 1.2

In a class there are awards for best mathematician, best sportsman and nicest person. Students can receive more than one award. In how many ways can the awards be distributed if there are twelve people in the class?

Describe the problem accurately

Choose one of 12 people for the best mathematician (12 ways)
AND
one of the 12 for best sportsmen (12 ways)
AND
one of the 12 for nicest person. (12 ways)

Apply the product principle

$12 \times 12 \times 12 = 1728$

This leads us to a general idea.

KEY POINT 1.3

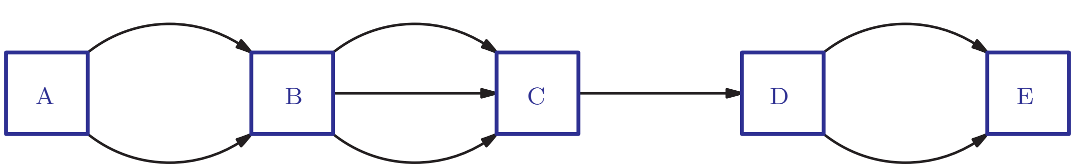
The number of ways of selecting something r times from n objects is n^r.

Exercise 1A

1. If there are 10 ways of doing A, 3 ways of doing B and 19 ways of doing C, how many ways are there of doing
   (a) (i) both A and B? (ii) both B and C?
   (b) (i) either A or B? (ii) either A or C?

2. If there are 4 ways of doing A, 7 ways of doing B and 5 ways of doing C, how many ways are there of doing
   (a) all of A, B and C?
   (b) exactly one of A, B or C?

3. How many different paths are there
   (a) from A to C?
   (b) from C to E?
   (c) from A to E?



4. Jamil is planting out his garden and needs one new rose bush and some dahlias. There are 12 types of rose and 4 varieties of dahlia in his local nursery. How many possible selections does he have to choose from? *[3 marks]*

5. A lunchtime menu at a restaurant offers 5 starters, 6 main courses and 3 desserts. How many different choices of meal can you make if you would like

(a) a starter, a main course and a dessert?

(b) a main course and either a starter or a dessert?

(c) any two different courses? *[6 marks]*

6. Five men and three women would like to represent their club in a tennis tournament. In how many ways can one mixed doubles pair be chosen? *[3 marks]*

7. A mathematics team consists of one student from each of years 7, 8, 9 and 10. There are 58 students in year 7, 68 in year 8, 61 in year 9 and 65 in year 10.

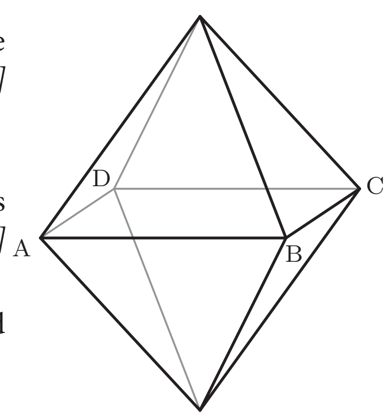
(a) How many ways are there of picking the team?

Year 10 is split into three classes: 10A (21 students), 10B (23 students) and 10C (21 students).

(b) If students from 10B cannot participate in the competition, how many ways are there of picking the team? *[4 marks]*

8. Student passwords consist of three letters chosen from A to Z, followed by four digits chosen from 1–9. Repeated characters are allowed. How many possible passwords are there? *[4 marks]*

9. A beetle walks along the edges from the base to the tip of an octahedral sculpture, visiting exactly two of the middle vertices (A, B, C or D). How many possible routes are there? *[6 marks]*



10. Professor Square has 15 different ties (seven blue, three red and five green), four waistcoats (red, black, blue and brown) and 12 different shirts (three each of red, pink, white and blue). He always wears a shirt, a tie and a waistcoat.

(a) How many different outfits can he make?

Professor Square never wears any outfit that combines red with pink.

(b) How many different outfits can he make with this limitation? *[6 marks]*

11. How many different three digit numbers can be formed using the digits 1,2,3,5,7

(a) once only?

(b) if digits can be repeated? *[6 marks]*

12. In how many ways can

(a) four toys be put into three boxes?

(b) three toys be put into five boxes? *[6 marks]*

If you look in other textbooks you may see permutations referred to in other ways. This is an example of a word that can take slightly different meanings in different countries. The definition here is the one used in the International Baccalaureate® (IB).

$n!$ occurs in many other mathematical situations. You will see it in the Poisson distribution (see Section 23D), and if you study option 9 on calculus, you will see that it is also important in a method for approximating functions called Taylor series.

1B Counting arrangements

The word 'ARTS' and the word 'STAR' both contain the same letters, but arranged in a different order. They are both arrangements (also known as **permutations**) of the letters R, A, T and S. We can count the number of different permutations.

There are four possibilities for the first letter, then for each choice of the first letter there are three options for the second letter (because one of the letters has already been used). This leaves two options for the third letter and then the final letter is fixed. Using the 'AND rule' the number of possible permutations is $4 \times 3 \times 2 \times 1 = 24$.

The number of permutations of n different objects is equal to the product of all positive integers less than or equal to n. This **expression** is abbreviated to $n!$ (pronounced 'n factorial').

KEY POINT 1.4

The number of ways of arranging n objects is $n!$

$n! = n(n-1)(n-2)\ldots \times 2 \times 1$

Worked example 1.3

A test has 12 questions. How many different arrangements of the questions are possible?

Describe the problem accurately

Permute (arrange) 12 items

Number of permutations $= 12!$

$= 479\,001\,600$

In examination questions you might have to combine the idea of permutations with the product and addition principles.

Worked example 1.4

A seven-digit number is formed by using each of the digits 1–7 exactly once. How many such numbers are even?

Describe the problem accurately: even numbers end in 2, 4 or 6	Pick the final digit to be even (3 ways)
Only 6 digits left to arrange	AND then permute the remaining 6 digits (6! ways)
Apply the product principle	$3 \times 720 = 2160$ possible even numbers

This example shows a very common situation where there is a constraint, in this case we have to end with an even digit. It can be more efficient to fix each part of the constraint separately, instead of searching all the possibilities for the ones which are allowed.

EXAM HINT

Factorials get very large very quickly. Although you should know factorials up to 6!, most of the time you will use your calculator. See Calculator skills sheet 3 on the CD-ROM.

Worked example 1.5

How many permutations of the word SQUARE start with three vowels?

Describe the problem accurately	Permute the three vowels at the beginning (3! ways) AND Permute the three consonants at the end (3! ways)
Apply the product principle	Number of ways $= 3! \times 3! = 36$

Exercise 1B

1. Evaluate:

   (a) (i) $5!$ (ii) $6!$

   (b) (i) $2 \times 4!$ (ii) $3 \times 5!$

   (c) (i) $6! - 5!$ (ii) $6! - 4 \times 5!$

2. Evaluate:

   (a) (i) $8!$ (ii) $11!$

   (b) (i) $9 \times 5!$ (ii) $9! \times 5$

   (c) (i) $12! - 10!$ (ii) $9! - 7!$

3. Find the number of ways of arranging:

   (a) 6 objects (b) 8 objects (c) 26 objects.

4. (a) How many ways are there of arranging seven textbooks on a shelf?

   (b) In how many of those arrangements is the single biggest textbook not at either end? *[5 marks]*

5. (a) How many five-digit numbers can be formed by using each of the digits 1–5 exactly once?

   (b) How many of those numbers are divisible by 5? *[5 marks]*

6. A class of 16 pupils and their teacher are queuing outside a cinema.

   (a) How many different arrangements are there?

   (b) How many different arrangements are there if the teacher has to stand at the front? *[5 marks]*

7. A group of nine pupils (five boys and four girls) are lining up for a photograph, with all the girls in the front row and all the boys at the back. How many different arrangements are there? *[5 marks]*

8. (a) How many six-digit numbers can be made by using each of the digits 1–6 exactly once?

   (b) How many of those numbers are smaller than 300 000? *[5 marks]*

9. A class of 30 pupils are lining up in three rows of ten for a class photograph.
How many different arrangements are possible? *[6 marks]*

10. A baby has nine different toy animals. Five of them are red and four of them are blue. She arranges them in a line so that the colours are arranged symmetrically. How many different arrangements are possible? *[7 marks]*

1C Algebra of factorials

To solve more complicated counting problems we often need to simplify expressions involving factorials. This is done using the formula for factorials, which you saw in Key point 1.4:

$$n! = n(n-1)(n-2)\ldots \times 2 \times 1$$

Worked example 1.6

(a) Evaluate $9! \div 6!$

(b) Simplify $\dfrac{n!}{(n-3)!}$

(c) Write $10 \times 11 \times 12$ as a ratio of two factorials.

Write in full and look for common factors in denominator and numerator

(a) $\dfrac{9\times 8\times 7\times 6\times 5\ldots\times 2\times 1}{6\times 5\ldots\times 2\times 1}$

$= 9\times 8\times 7$

$= 504$

Write in full and look for common factors in denominator and numerator

(b) $\dfrac{n\times(n-1)\times(n-2)\times(n-3)\ldots\times 2\times 1}{(n-3)\times(n-4)\ldots\times 2\times 1}$

$= n(n-1)(n-2)$

Reverse the ideas from (b)

(c) $10\times 11\times 12 = \dfrac{1\times 2\ldots\times 9\times 10\times 11\times 12}{1\times 2\ldots\times 9}$

$= \dfrac{12!}{9!}$

You can usually solve questions involving sums or differences of factorials by looking for common factors of the terms. It is important to understand the link between one factorial and the next:

KEY POINT 1.5

$$(n+1)! = (n+1) \times n!$$

Worked example 1.7

Simplify:

(a) $9! - 7!$ (b) $(n+1)! - n! + (n-1)! = 72 \times 7!$

Link 9! and 7!

(a) $9! = 9 \times 8! = 9 \times 8 \times 7!$

Take out the common factor of 7!

$9! - 7! = 72 \times 7! - 7$
$= (72-1) \times 7! = 71 \times 7!$

Link $(n+1)!$, $n!$ and $(n-1)!$

(b) $n! = n \times (n-1)!$
$(n+1)! = (n+1) \times n! = (n+1) \times n \times (n-1)!$

Take out the common factor of $(n-1)!$

$(n+1)! - n! + (n-1)! = n(n+1)(n-1)! - n(n-1)! + (n-1)!$
$= (n-1)!(n^2 + n - n + 1) = (n-1)!(n^2+1)$

0! is defined to be 1. Why might this be?

$\frac{1}{2}!$ is $\frac{\sqrt{\pi}}{2}$

To explore why you need to look at the Gamma function.

Being able to simplify expressions involving $n!$ is useful because $n!$ becomes very large very quickly. Often factorials cannot be evaluated even using a calculator. For example, a standard scientific calculator can only calculate up to $69! = 1.71 \times 10^{98}$ (to 3 significant figures).

EXAM HINT

Remember that you will lose one mark per paper in the IB if you give any answers to less than three significant figures (3SF) (unless the answer is exact of course!). See Prior learning Section B on the CD-ROM if you need a reminder about significant figures.

Exercise 1C

1. Fully simplify the following fractions:

(a) (i) $\frac{7!}{6}$ (ii) $\frac{12!}{11!}$

(b) (i) $\frac{8!}{6!}$ (ii) $\frac{11!}{8!}$

(c) (i) $\frac{100!}{98!}$ (ii) $\frac{51!}{49!}$

(d) (i) $\frac{n!}{(n-1)!}$ (ii) $\frac{(a+1)!}{a!}$

(e) (i) $\frac{a!}{(a-2)!}$ (ii) $\frac{(b+1)!}{(b-1)!}$

(f) (i) $\frac{(x+6)!}{(x+8)!}$ (ii) $\frac{(x-7)!}{(x-4)!}$

2. Write these as a ratio of 2 factorials:

(a) (i) $7 \times 8 \times 9$ (ii) $6 \times 5 \times 4 \times 3$

(b) (i) $1013 \times 1014 \times 1015 \times 1016$

(ii) $307 \times 308 \times 309$

(c) (i) $(n+2)(n+3)(n+4)(n+5)$

(ii) $n(n-1)(n+1)(n+2)$

3. Explain why $6! - 5! = 5 \times 5!$

Simplify in a similar fashion:

(a) (i) $10! - 9!$ (ii) $12! - 10!$

(b) (i) $14! - 3 \times 13!$ (ii) $16! + 5 \times 15!$

(c) (i) $11! + 11 \times 9!$ (ii) $12! - 22 \times 10!$

(d) (i) $n! - (n-1)!$ (ii) $(n+2)! - n!$

4. Solve $\frac{n!}{(n-2)!} = 20$ where n is a positive integer. *[5 marks]*

5. Solve $\frac{(n+1)!}{(n-2)!} = 990.$ *[6 marks]*

6. Solve $n! - (n-1)! = 16(n-2)!$ for $n \in \mathbb{N}$. *[6 marks]*

1D Counting selections

Suppose that three pupils are to be selected from a class of 11 to attend a meeting with the Head Teacher. How many different groups of three can be chosen?

In this example we need to *choose* three pupils out of 11, but they are not to be arranged in any specified order. The order is not important; the selection of Ali, Bill and then Cathy is the same as the selection of Bill, Cathy and then Ali. This sort of selection is called a **combination**. In general, the formula for

the number of ways of choosing r objects out of n is given the symbol $\binom{n}{r}$, or ${}^nC_r$ (pronounced 'n C r' or 'n choose r').

EXAM HINT

The Formula booklet tells you the formula, but not what it is used for.

KEY POINT 1.6

The number of ways of choosing r objects out of n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Does learning the terminology of $n!$ and $\binom{n}{r}$ give you new mathematical knowledge?

Does this mathematical language clarify or muddle the principles behind it? Is it necessary to have 'expert language' in all subjects? Does giving something a label help you think about it?

Worked example 1.8

A group of 12 friends want to form a team for a five-a-side football tournament.

(a) In how many different ways can a team of five be chosen?

(b) Rob and Amir are the only goalkeepers and they cannot play in any other position. If the team of five has to contain exactly one goalkeeper, how many different ways can the team be chosen?

Describe the problem accurately

(a) Choosing 5 players from 12

Number of ways $= \binom{12}{5} = 792$

Describe the problem accurately

(b) Pick a goalkeeper, then fill in the rest of the team

Split into cases to satisfy the condition

Amir is in goal (1 way)
AND
Choose four other players (? ways)
OR
Rob is in goal. (1 way)
AND
Choose four other players (? ways)

With Amir picked there are four remaining slots, but Rob cannot be picked so ten players remain

Number of ways with Amir in goal $= \binom{10}{4}$

Number of ways with Rob in goal $= \binom{10}{4}$

Apply the addition principle and product principle

$$\text{Total number of ways} = 1\times\binom{10}{4} + 1\times\binom{10}{4}$$
$$= 210 + 210$$
$$= 420$$

EXAM HINT

You can calculate $\binom{n}{r}$ on your calculator using a button like this; see Calculator skills sheet 3 on the CD-ROM.

We will use the idea of combinations to help us expand brackets in chapter 8 and to find probabilities in Section 23C.

Exercise 1D

1. Evaluate:

(a) (i) $\binom{7}{2}$ (ii) $\binom{12}{5}$

(b) (i) $3\times\binom{6}{3}$ (ii) $10\times\binom{6}{5}$

(c) (i) $\binom{5}{0}\times\binom{9}{5}$ (ii) $\binom{10}{8}\times\binom{3}{1}$

(d) (i) $\binom{5}{2}+\binom{9}{5}$ (ii) $\binom{6}{0}+\binom{7}{3}$

2. (a) (i) In how many ways can six objects be selected from eight?

(ii) In how many ways can five objects be selected from nine?

(b) (i) In how many ways can either three objects be selected from ten, or seven objects be selected from 12?

(ii) In how many ways can either two objects be selected from five, or three objects be selected from four?

(c) (i) In how many ways can five objects be selected from seven and three different objects be selected from eight?

(ii) In how many ways can six objects be selected from eight and three different objects be selected from seven?

3. Solve:

(a) (i) $\binom{n}{2}=91$ (ii) $\binom{n}{2}=351$

(b) (i) $\binom{n}{3}=1330$ (ii) $\binom{n}{3}=680$

4. An exam paper consists of 15 questions. Students can select any nine questions to answer. How many different selections can be made? *[4 marks]*

5. Suppose you are revising for seven subjects, but you can only complete three in one evening.

(a) In how many ways can you select three subjects to do on Monday evening?

(b) If you have to revise mathematics on Tuesday, in how many ways can you select the subjects to do on Tuesday evening? *[5 marks]*

6. In the 'Pick'n'Mix' lottery, players select seven numbers out of 39. How many different selections are possible? *[4 marks]*

7. There are 16 boys and 12 girls in a class. Three boys and two girls are needed to take part in the school play. In how many different ways can they be selected? *[5 marks]*

8. A soccer team consists of one goalkeeper, four defenders, four midfielders and two forwards. A manager has three goalkeepers, eight defenders, six midfielders and five forwards in the squad. In how many ways can she pick the team? *[5 marks]*

9. A school is planning some trips over the summer. There are 12 places on the Greece trip, ten places on the China trip and ten places on the Disneyland trip. There are 140 pupils in the school who are all happy to go on any of the three trips.

In how many ways can the spaces be allocated? *[5 marks]*

10. A committee of three boys and three girls is to be selected from a class of 14 boys and 17 girls. In how many ways can the committee be selected if:

(a) Ana has to be on the committee?

(b) the girls must include either Roberta or Priya, but not both? *[6 marks]*

11. Raspal's sweet shop stocks seven different types of 2¢ sweets and five different types of 5¢ sweets. If you want no more than one of each sweet, how many different selections of sweets can be made when spending:

(a) exactly 6¢?

(b) exactly 7¢?

(c) exactly 10¢?

(d) at most 5¢? *[6 marks]*

12. An English examination has two sections. Section A has five questions and section B has four questions. Four questions must be answered in total.

(a) How many different ways are there of selecting four questions to answer if there are no restrictions?

(b) How many different ways are there of selecting four questions if there must be at least one question answered from each section? *[6 marks]*

13. 15 points lie around a circle. Each point is connected to every other point by a straight line. How many lines are formed in this way? *[4 marks]*

14. Ten points are drawn on a sheet of paper so that no three lie in a straight line. By connecting up the points:

(a) how many different triangles can be drawn?

(b) how many different quadrilaterals can be drawn? *[7 marks]*

15. At a party, everyone shakes hands with everyone else. If 276 handshakes are exchanged, how many people are there at the party? *[6 marks]*

16. A group of 45 students are to be seated in three rows of 15 for a school photograph. Within each row, students must sit in alphabetical order according to name, but there is no restriction determining the row in which a student must sit.

How many different seating arrangements are possible, assuming no students have identical names? *[6 marks]*

1E Exclusion principle

The **exclusion principle** is a way to count what you are interested in by first counting what you are *not* interested in. This is often needed for counting where a certain property is prohibited (not allowed).

KEY POINT 1.7

The Exclusion principle

Count what you are not interested in and subtract it from the total.

Imagine that a five-digit code is formed using each of the digits 1–5 exactly once. If we wanted to count how many such codes do *not* end in '25' we could work out *all* of the possible options that do not end in '25':

Worked example 1.9a

How many five-digit codes formed by using each of the digits 1–5 exactly once do not end in '25'?

Describe the problem accurately	Pick the final digit from {1, 2, 3, 4} (4 ways) AND arrange the remaining four digits (4! ways) OR Pick the final digit as 5 (1 way) AND pick the penultimate digit from {1, 3, 4} (3 ways) AND arrange the remaining three digits (3! ways)
Use the product and addition principles	$(4 \times 4!) + (1 \times 3 \times 3!) = 114$

This method works, but it is long and complicated. An easier way is to use the exclusion principle.

Worked example 1.9b

How many five-digit codes formed by using each of the digits 1–5 exactly once do not end in '25'?

Describe the problem accurately	Count permutations of five digit codes (5! ways) then EXCLUDE cases where the last two digits are '25' (1 way) AND arrange the remaining three digits (3! ways)
Use the product and exclusion principles	$5! - 1 \times 3! = 114$

Many questions ask us to use the exclusion principle in a situation containing an 'at least' or 'at most' restriction.

Worked example 1.10

Theo has eight different jigsaws and five different toy bears. He chooses four things to play with. In how many ways can he make a selection with at least two bears?

Describe the problem accurately	Count combinations of four toys from thirteen $\binom{13}{4}$ ways

continued . . .

Then EXCLUDE

Combinations with no bears $\binom{5}{0}$ ways

AND

Four jigsaws $\binom{8}{4}$ ways

OR

Combinations with one bear $\binom{5}{1}$ ways

AND

Three jigsaws $\binom{8}{3}$ ways

Apply the product and addition principles

$$\text{Number of ways} = \binom{13}{4} - \left(\binom{5}{0}\binom{8}{4} + \binom{5}{1}\binom{8}{3}\right)$$

$$= 714 - (1 \times 70 + 5 \times 56) = 365$$

EXAM HINT

This example highlights the ambiguity of the English language. Can you see why we exclude (no bears and four jigsaws) OR (one bear and three jigsaws) rather than (no bears and four jigsaws) AND (one bear and three jigsaws)?

Be careful when you read questions. For example, the opposite of a word beginning in a consonant OR ending in a consonant is not a word beginning in a vowel OR ending in a vowel. For example, the word 'LOVE' fits both descriptions. If you are interested there is formal work on this, called De Morgan's laws, in the Sets, Relations and Groups option (option 8).

Exercise 1E

1. How many of the numbers between 101 and 800 inclusive are not divisible by 5? *[4 marks]*

2. How many permutations of the letters of the word JUMPER do not start with a J? *[5 marks]*

3. A bag contains 12 different chocolates, four different mints and six different toffees. Three sweets are chosen. How many ways are there of choosing

(a) all not chocolates? (b) not all chocolates? *[6 marks]*

4. How many permutations of the letters of KITCHEN

(a) do not begin with KI?

(b) do not have K and I in the first two letters? *[6 marks]*

5. A committee of five people is to be selected from a class of 12 boys and nine girls. How many such committees include at least one girl? *[6 marks]*

6. In a word game there are 26 letter tiles, each with a different letter. How many ways are there of choosing seven tiles so that at least two are vowels? *[6 marks]*

7. A committee of six is to be selected from a group of ten men and 12 women. In how many ways can the committee be chosen if it has to contain at least two men and one woman? *[6 marks]*

8. Seven numbers are chosen from the integers 1–19 inclusive. How many have

(a) at most two even numbers?

(b) at least two even numbers? *[7 marks]*

9. How many permutations of the letters DANIEL do not begin with D or do not end with L? *[6 marks]*

1F Counting ordered selections

Sometimes we want to choose a number of objects from a bigger group but the order in which they are chosen *is important*. For example, finding the possibilities for the first three finishers in a race or forming numbers from a fixed group of digits. The strategy for dealing with these situations is first to choose from the larger group and then permute the chosen objects.

Worked example 1.11

A class of 28 pupils has to select a committee consisting of a class representative, a treasurer, a secretary and a football captain. Each post needs to be a different person. In how many different ways can the four posts be filled?

Select four people and then allocate them to different jobs. Note that this is the same as selecting four people and then permuting them

Choose 4 from 28 $\binom{28}{4}$ ways

AND

Permute those 4 (4! ways)

Apply the product principle

Number of ways $=\binom{28}{4}\times 4! = 491400$

Suppose that the whole set has size n and we are selecting and ordering a subset of size r. The number of ways of doing this is given the symbol ${}^nP_r$ and is described as 'the number of permutations of r objects out of n'. We can use the formula for $\binom{n}{r}$ to deduce a formula for ${}^nP_r$: there are $\binom{n}{r}$ ways of choosing the objects in the subset, and these can be arranged in $r!$ ways. Therefore, ${}^nP_r = \binom{n}{r} \times r!$ or:

KEY POINT 1.8

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

This can be written, using factorial algebra, to:

$$ {}^nP_r = n(n-1)(n-2)\ldots(n-r+1) $$

Written in this form you can see that the formula for ${}^nP_r$ is an application of the 'AND rule'. For example, suppose that there are five people in a race and we want to count the number of possibilities for the first three positions. We can reason that if there are five options for the winner then, for each winner, there are four options for the second place and three options for the third place. This gives $5 \times 4 \times 3 = {}^5P_3$.

Exercise 1F

1. Evaluate:

(a) (i) ${}^6P_1$ (ii) ${}^5P_1$

(b) (i) ${}^8P_2$ (ii) ${}^{11}P_2$

(c) (i) ${}^{10}P_3$ (ii) ${}^{12}P_3$

2. Find the number of permutations of:

(i) 4 objects out of 10 (ii) 6 objects out of 7.

3. Find the number of ways of selecting when order matters:

(i) 3 objects out of 5 (ii) 2 objects out of 15.

4. Solve the following equations:

(a) (i) ${}^{n}P_{2} = 42$ (ii) ${}^{n}P_{2} = 90$

(b) (i) ${}^{n}P_{3} = 9 \times {}^{n}P_{2}$ (ii) ${}^{n}P_{3} = 12 \times {}^{n}P_{2}$

5. In a 'Magic Sequence' lottery draw there are 39 balls numbered 1–39. Seven balls are drawn at random. The result is a sequence of seven numbers which must be matched in the correct order to win the grand prize. How many possible sequences can be made? *[4 marks]*

6. A teacher needs to select four pupils from a class of 24 to receive four different prizes. How many possible ways are there to award the prizes? *[4 marks]*

7. How many three-digit numbers can be formed from digits 1–9 if no digit can be repeated? *[4 marks]*

8. Eight athletes are running a race. In how many different ways can the first three places be filled? *[4 marks]*

9. An identification number consists of two letters followed by four digits chosen from 0–9. No digit or letter may appear more than once. How many different identification numbers can be made? *[6 marks]*

10. Prove that ${}^{n}P_{n-1} = {}^{n}P_{n}$. *[3 marks]*

11. Three letters are chosen from the word PICTURE and arranged in order. How many of the possible permutations contain at least one vowel? *[6 marks]*

12. Eight runners compete in a race. In how many different ways can the three medals be awarded if James wins either a gold or a silver? *[6 marks]*

13. A class of 18 needs to select a committee consisting of a President, a Secretary and a Treasurer. How many different ways are there to form the committee if Rajid does not want to be President? *[6 marks]*

14. Solve the equation ${}^{2n}P_{3} = {}^{6}P_{n}$, justifying that you have found all solutions. *[6 marks]*

1G Keeping objects together or separated

How many letter arrangements of the word 'SQUARE' have the Q and U next to each other? How many have the vowels all separated?

When a problem has constraints like this we need some clever tricks to deal with them.

The first type of problem we will look at is where objects are forced to stay together. The trick is to imagine the letters in the word SQUARE as being on tiles. If the Q and the U need to be together, we are really dealing with five tiles, one containing QU:

We must also remember that the condition is satisfied if Q and U in are in the other order.

Is mathematics about finding answers or having strategies and ideas? Has the balance between the two changed as you have progressed through learning mathematics?

Worked example 1.12

How many permutations of the word 'SQUARE' have the Q and the U next to each other?

Describe the problem accurately

Find the number of permutations of the five 'tiles' (S, QU, A, R, E) (5! ways)
AND
Find the number of permutations of the letters QU on the 'double tile' (2! ways)

Apply the product principle

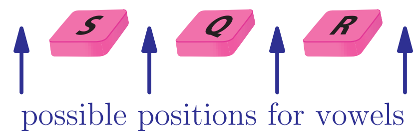
Number of ways $= 5! \times 2! = 240$

KEY POINT 1.9

If a group of items have to be kept together, treat the items as one object. Remember that there may be permutations of the items within this group too.

Another sort of constraint is where objects have to be kept apart. This is *not* the opposite of objects being kept together unless we are separating only two objects. For example, the opposite of three objects staying together includes having two of them together and the third one apart. So when dealing with keeping objects apart we need to focus on the gaps that the critical objects can fit into.

Consider the question of how many permutations of the word SQUARE have none of the vowels together. We first permute all of the consonants. One such permutation is



There are four gaps in which we can put the vowels. We only have three vowels and so only need to choose three of the gaps, and then decide in what order to insert the vowels.

Worked example 1.13

How many permutations of the word 'SQUARE' have none of the vowels together?

Describe the problem accurately

Permute 3 consonants — $3!$ ways

AND

Choose 3 out of the four gaps — $\binom{4}{3}$ ways

AND

Permute the three vowels to put into the gaps — $3!$ ways

Apply the product principle

Total number of ways $3! \times 4 \times 3! = 144$

KEY POINT 1.10

If k objects have to be kept apart, permute all the other objects and then count the permutations of k 'gaps'.

Exercise 1G

1. In how many ways can 14 people be arranged in a line if Joshua and Jolene have to stand together? *[6 marks]*

2. Students from three different classes are standing in the lunch queue. There are six students from 10A, four from 10B and four from 10C. In how many ways can the queue be arranged if students from the same class have to stand together? *[6 marks]*

3. In how many ways can six Biology books and three Physics books be arranged on the shelf if the three Physics books are always next to each other? *[6 marks]*

4. In a photo there are three families (six Greens, four Browns and seven Grays) arranged in a row. The Browns have had an argument so no Brown will stand next to another Brown. How many different permutations are permitted? *[6 marks]*

5. In a cinema there are 15 seats in a row. In how many ways can seven friends be seated in the same row if
   (a) there are no restrictions?
   (b) they all want to sit together? *[6 marks]*

6. Five women and four men stand in a line.
   (a) In how many arrangements will all four men stand next to each other?
   (b) In how many arrangements will all the men stand next to each other and all the women stand next to each other?
   (c) In how many arrangements will all the men be apart?
   (d) In how many arrangements will all the men be apart and all the women be apart? *[6 marks]*

Summary

- The **product principle** ('AND rule') states the total number of possible outcomes when there are two choices (A and B) to be made is the product of the number of outcomes for each choice:

$$n(A \text{ AND } B) = n(A) \times n(B)$$

- The **addition principle** ('OR rule') states that the number of outcomes when we are interested in either the first choice A *OR* the second choice B being made is the sum of the number of outcomes for each choice. A and B must be mutually exclusive.

$$n(A \text{ OR } B) = n(A) + n(B)$$

- The number of **permutations** (specific arrangements) of n different objects is $n!$; $n! = n\,(n-1)\,(n-2)\ldots \times 2 \times 1$
- The link between one factorial and the next can be shown algebraically as: $(n+1)! = (n+1) \times n!$

- If we are selecting r out of n objects, and the order in which we do so does *not* matter, we describe this as a **combination** and there are $\binom{n}{r}$ or ${}^nC_r$ ways to do this. This has the formula

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

- There are ${}^nP_r$ ways of selecting r out of n objects when the order in which we select *does* matter.

$${}^nP_r = \frac{n!}{(n-r)!}$$

- The **exclusion principle** says that to find the number of permutations which do not have a certain property, find the number of permutations that do have the property and subtract it from the number of all possible permutations.
- If a group of items have to be kept together, treat them as one object. Remember that there may be permutations within this group too.
- If objects have to be kept apart, permute all the other objects and then count the permutations of the 'gaps'.

Introductory problem revisited

If a computer can print a line containing all 26 letters of the alphabet in 0.01 seconds, estimate how long it would take to print all possible permutations of the alphabet.

There are 26! ways to permute all the letters of the alphabet. A calculator tells us that this is approximately 4.03×10^{26} ways. At 0.01 seconds per line, this is 4.03×10^{24} seconds; 4.67×10^{19} days or 1.28×10^{17} years. The age of the universe is estimated at around 14 billion years, so this is approximately 10 million times longer than the universe has existed – it is hard to understand such large numbers!

According to the Guinness World Records™ (Book of Records) the largest number which has ever been found to have a use comes from the field of counting. It is called Graham's Number and it is so large that it is not possible to write it out in normal notation even using all the paper in the world, so a new system of writing numbers had to be invented to describe it.

Mixed examination practice 1

Short questions

1. Seven athletes take part in the 100 m final of the Olympic games. In how many ways can three medals be awarded? *[4 marks]*

2. In how many ways can five different letters be put into five different envelopes? *[5 marks]*

3. In how many ways can ten cartoon characters stand in a queue if Mickey, Bugs Bunny and Jerry must occupy the first three places in some order? *[5 marks]*

4. How many three digit numbers contain no zeros? *[6 marks]*

5. A committee of four children is chosen from eight children. The two oldest children cannot both be chosen. Find the number of ways the committee may be chosen. *[6 marks]*

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6. Solve the equation $(n+1)! = 30(n-1)!$ for $n \in \mathbb{N}$.
(Remember: $\mathbb{N}$ is the set of natural (whole non-negative) numbers.) *[5 marks]*

7. How many permutations of the word 'CAROUSEL' start and end in a consonant? *[5 marks]*

8. Solve the equation $\binom{n}{2} = 105$. *[6 marks]*

9. A group of 15 students contains seven boys and eight girls. In how many ways can a committee of five be selected if it must contain at least one boy? *[6 marks]*

10. Abigail, Bahar, Chris, Dasha, Eustace and Franz are sitting next to each other in six seats in a cinema. Bahar and Eustace cannot sit next to each other. In how many different ways can they permute themselves? *[6 marks]*

11. A committee of five is to be selected from a group of 12 children. The two youngest cannot both be on the committee. In how many ways can the committee be selected? *[6 marks]*

12. A car registration number consists of three different letters followed by five digits chosen from 1–9 (the digits can be repeated). How many different registration numbers can be made? *[6 marks]*

13. A van has eight seats, two at the front, a row of three in the middle and a row of three at the back. If only 5 out of a group of 8 people can drive, in how many different ways can they be arranged in the car? *[6 marks]*

14. Ten people are to travel in one car (taking four people) and one van (taking six people). Only two of the people can drive. In how many ways can they be allocated to the two vehicles? (The permutation of the passengers in each vehicle is not important.) *[7 marks]*

Long questions

1. Five girls, Anya, Beth, Carol, Dasha and Elena, stand in a line. How many possible permutations are there in which

(a) Anya is at one end of the line?

(b) Anya is not at either end?

(c) Anya is at the left of the line or Elena is on the right, or both? *[9 marks]*

2. In how many ways can five different sweets be split amongst two people if

(a) each person must have at least one sweet?

(b) one person can take all of the sweets?

(c) one of the sweets is split into two to be shared, and each person gets two of the remaining sweets? *[9 marks]*

3. In a doctor's waiting room, there are 14 seats in a row. Eight people are waiting to be seen.

(a) In how many ways can they be seated?

(b) Three of the people are all in the same family and they want to sit together. How many ways can this happen?

(c) The family no longer have to sit together, but there is someone with a very bad cough who must sit at least one seat away from anyone else. How many ways can this happen? *[8 marks]*

4. (a) Explain why the number of ways of arranging the letters RRDD, given that all the R's and all the D's are indistinguishable is $\binom{4}{2}$.

(b) How many ways are there of arranging n R's and n D's?

(c) A miner is digging a tunnel on a four by four grid. He starts in the top left box and wants to get to the gold in the bottom right box. He can only tunnel directly right or directly down one box at a time. How many different routes can he take?

(d) What will be the general formula for the number of routes when digging on an n by m grid? *[10 marks]*

5. 12 people need to be split up into teams for a quiz.

(a) Show that the number of ways of splitting them into two groups of the same size is $\frac{1}{2}\binom{12}{6}$.

(b) How many ways are there of splitting them into two groups of any size (but there must be at least one person in each group)?

(c) How many ways are there of splitting them into three groups of four people? *[9 marks]*

6. (a) How many different ways are there to select a group of three from a class of 31 people?

(b) In another class there are 1540 ways of selecting a group of three people. How many people are there in the class?

(c) In another class the teacher noted that the number of ways to select a group of size three is 100 times larger than the number of people in the class. How many people are in the class? *[9 marks]*

2 Exponents and logarithms

In this chapter you will learn:

- some rules for dealing with exponents
- about a function where the unknown is in the exponent
- about the value e and some of its properties
- how to undo exponential functions using an operation called a logarithm
- the rules of logarithms
- about graphs of logarithms
- to use logarithms to find exact solutions to simple exponential equations.

Introductory problem

A radioactive substance has a half-life of 72 years. A 1 kg block of the substance is found to have a radioactivity of 25 million Becquerel (Bq). How long, to the nearest 10 years, would it take for the radioactivity to have fallen below 10 000 Bq?

Many mathematical models (biological, physical and financial in particular) involve the concept of continuous growth or decay where the rate of growth/decay of the population is linked to the size of that population. You may have met similar situations already, for example when a bank account earns compound interest: the increase in amount each year depends upon how much is in the account. Any similar situation is governed by an exponential function, which you will learn about in this chapter.

We shall also look at how we can find out how long an exponential process has been occurring, using a function called a logarithm.

2A Laws of exponents

See Prior learning Section C on the CD-ROM which looks at exponents.

The exponent of a number shows you how many times the number is to be multiplied by itself. You will have already met some of the rules for dealing with exponents before, and in this section we shall revisit and extend these rules. (In other courses, the exponent might have been called the 'index' or 'power'.)

A number written in **exponent form** is one which explicitly looks like:

$$a^n$$

n is referred to as the **exponent** or **power**
a is referred to as the **base**

a^n *is* pronounced 'a to the exponent n' or, more simply, 'a to the n'.

To investigate the rules of exponents let us consider an example:

> Mathematics is often considered a subject without ambiguity. However, the value of 0^0 is undetermined; it depends upon how you get there!

Worked example 2.1

Simplify.

(a) $a^3 \times a^4$ (b) $a^3 \div a^4$ (c) $(a^4)^3$ (d) $a^4 + a^3$

(a) $a^3 \times a^4 = (a \times a \times a \times a) \times (a \times a \times a) = a^7$

(b) $a^3 \div a^4 = \dfrac{a \times a \times a}{a \times a \times a \times a} = \dfrac{1}{a} = a^{-1}$

Use the idea from part (a)

(c) $(a^4)^3 = a^4 \times a^4 \times a^4 = a^{12}$

(d) $a^4 + a^3 = a^3(a+1)$

The example above suggests some rules of exponents.

> It is questionable whether in part (d) we have actually simplified the expression. Sometimes the way mathematicians choose to simplify expressions is governed by how it looks as well as how it is used.

KEY POINT 2.1

$$a^m \times a^n = a^{m+n}$$

KEY POINT 2.2

$$a^m \div a^n = a^{m-n}$$

KEY POINT 2.3

$$(a^m)^n = a^{m \times n}$$

We can use Key point 2.3 to justify the interpretation of $a^{\frac{1}{n}}$ as the nth root of a, since $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \times n} = a$. This is exactly the property we require of the nth root of a. So, we get the rule: $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$.

EXAM HINT

These rules are NOT given in the formula booklet. Make sure that you can use them in both directions, e.g. if you see 2^6 you can rewrite it as $(2^3)^2$ and if you see $(2^3)^2$ you can rewrite it as 2^6. Both ways will be important!

Worked example 2.2

Evaluate $64^{\frac{2}{3}}$.

Use $a^{m/n} = \left(a^{\frac{1}{n}}\right)^m$ to split the calculation into two steps

$$64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2$$
$$= (4)^2$$

So, $64^{\frac{2}{3}} = 16$

You must take care when expressions with *different* bases are to be combined by multiplication or division, for example $2^3 \times 4^2$. The rules such as 'multiplication means add the exponents together' are only true *when the bases are the same*. You cannot use this rule to simplify $2^3 \times 4^2$.

There is however, a rule that works when the bases are different *but* the exponents are the *same*.

Consider the following example:

$$\begin{aligned} 3^2 \times 5^2 &= 3\times3\times5\times5 \\ &= 3\times5\times3\times5 \\ &= 15\times15 \\ &= 15^2 \end{aligned}$$

This suggests the following rules:

KEY POINT 2.4

$$a^n \times b^n = (ab)^n$$

KEY POINT 2.5

$$a^n \div b^n = \left(\frac{a}{b}\right)^n$$

Exercise 2A

1. Simplify the following, leaving your answer in exponent form.

(a) (i) $6^4 \times 6^3$ (ii) $5^3 \times 5^5$

(b) (i) $a^3 \times a^5$ (ii) $x^6 \times x^3$

(c) (i) $7^{11} \times 7^{-14}$ (ii) $5^7 \times 5^{-2}$

(d) (i) $x^4 \times x^{-2}$ (ii) $x^8 \times x^{-3}$

(e) (i) $g^{-3} \times g^{-9}$ (ii) $k^{-2} \times k^{-6}$

2. Simplify the following, leaving your answer in exponent form.

(a) (i) $6^4 \div 6^3$ (ii) $5^3 \div 5^5$

(b) (i) $a^3 \div a^5$ (ii) $x^6 \div x^3$

(c) (i) $5^7 \div 5^{-2}$ (ii) $7^{11} \div 7^{-4}$

(d) (i) $x^4 \div x^{-2}$ (ii) $x^8 \div x^{-3}$

(e) (i) $2^{-5} \div 2^{-7}$ (ii) $3^{-6} \div 3^8$

(f) (i) $g^{-3} \div g^{-9}$ (ii) $k^{-2} \div k^6$

3. Express the following in the form required.

(a) (i) $(2^3)^4$ as 2^n (ii) $(3^2)^7$ as 3^n

(b) (i) $(5^{-1})^4$ as 5^n (ii) $(7^{-3})^2$ as 7^n

(c) (i) $(11^{-2})^{-1}$ as 11^n

(ii) $(13^{-3})^{-5}$ as 13^n

(d) (i) $4 \times (2^5)^3$ as 2^n

(ii) $3^{-5} \times (9^{-1})^{-4}$ as 3^n

(e) (i) $(4^2)^3 \times 3^{12}$ as 6^n

(ii) $(6^3)^2 \div (2^2)^3$ as 3^n

4. Simplify the following, leaving your answer in exponent form with a prime number as the base.

(a) (i) 4^5 (ii) 9^7

(b) (i) 8^3 (ii) 16^5

(c) (i) $4^2 \times 8^3$ (ii) $9^5 \div 27^2$

(d) (i) $4^{-3} \times 8^5$ (ii) $3^7 \div 9^{-2}$

(e) (i) $\left(\frac{1}{4}\right)^3$ (ii) $\left(\frac{1}{9}\right)^3$

(f) (i) $\left(\frac{1}{8}\right)^2 \div \left(\frac{1}{4}\right)^4$ (ii) $9^7 \times \left(\frac{1}{3}\right)^4$

5. Write the following without brackets or negative exponents:

(a) (i) $(2x^2)^3$ (ii) $(3x^4)^2$

(b) (i) $2(x^2)^3$ (ii) $3(x^4)^2$

(c) (i) $\dfrac{(3a^3)^4}{9a^2}$ (ii) $\dfrac{(4x)^4}{8(2x)^4}$

(d) (i) $(2x)^{-1}$ (ii) $\left(\dfrac{3}{y}\right)^{-2}$

(e) (i) $2x^{-1}$ (ii) $\dfrac{3}{y^{-2}}$

(f) (i) $5\div\left(\dfrac{3}{xy^2}\right)^2$ (ii) $\left(\dfrac{ab}{2}\right)^3\div\left(\dfrac{a}{b}\right)^2$

(g) (i) $\left(\dfrac{2}{q}\right)^2\div\left(\dfrac{p}{2}\right)^{-3}$ (ii) $\left(\dfrac{6}{x}\right)^4\div\left(2\times\dfrac{3^2}{x}\right)^{-3}$

6. Simplify the following:

(a) (i) $(x^6)^{\frac{1}{2}}$ (ii) $(x^9)^{\frac{4}{3}}$

(b) (i) $(4x^{10})^{0.5}$ (ii) $(8x^{12})^{-\frac{1}{3}}$

(c) (i) $\left(\dfrac{27x^9}{64}\right)^{-\frac{1}{3}}$ (ii) $\left(\dfrac{x^4}{y^8}\right)^{-1.5}$

7. Solve for x, giving your answer as a rational value:

In Section 2G you will see that there is an easier way to solve equations like this when you have a calculator and can use logarithms.

(a) (i) $8^x = 32$ (ii) $25^x = \dfrac{1}{125}$

(b) (i) $\dfrac{1}{49^x} = 7$ (ii) $\dfrac{1}{16^x} = 8$

(c) (i) $2\times 3^x = 162$ (ii) $3\div 5^x = 0.12$

(d) (i) $2\times 5^{x-1} = 250$ (ii) $5+3^{x+2} = 14$

(e) (i) $16+2^x = 2^{x+1}$ (ii) $100^{x+5} = 10^{3x-1}$

(f) (i) $6^{x+1} = 162\times 2^x$ (ii) $4^{1.5x} = 2\times 16^{x-1}$

8. Any simple computer program is able to sort n input values in $k\times n^{1.5}$ microseconds. Observations show that it sorts a million values in half a second. Find the value of k. *[3 marks]*

9. A square-ended cuboid has volume xy^2, where x and y are lengths. A cuboid for which $x = 2y$ has volume 128 cm³. Find x. *[3 marks]*

10. The volume and surface area of a family of regular solid shapes are related by the formula $V = kA^{1.5}$, where V is given in cm^3 and A in cm^2.

(a) For one such shape, $A = 81$ and $V = 243$. Find k.

(b) Hence determine the surface area of a shape with volume $\frac{64}{3}$ cm^3. *[4 marks]*

11. Prove that 2^{350} is larger than 5^{150}. *[5 marks]*

12. Given that there is more than one solution value of x to the equation $4^{ax} = b \times 8^x$, find all possible values of a and b. *[5 marks]*

2B Exponential functions

In most of the equations that you have met so far, the unknown appears as the base, for example $x^3 = 27$.

In an exponential function, the unknown appears in the exponent, leading to a fundamentally different type of function.

The general form of a simple exponential function is $f(x) = a^x$.

We will only consider situations when a is positive, because otherwise some exponents cannot be easily defined (for example, we cannot square root a negative number).

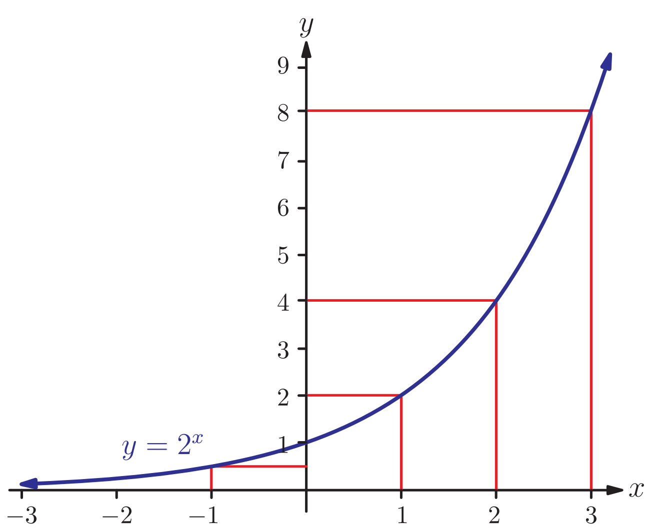
What is $(-1)^{\frac{1}{2}}$?

What about $\left((-1)^2\right)^{\frac{1}{4}}$?

What about $(-1)^{\frac{2}{4}}$?

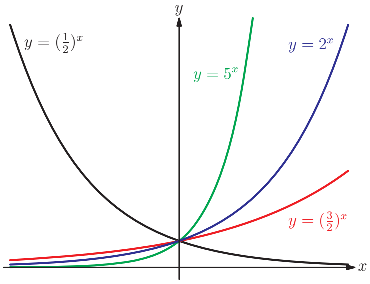
Not all mathematics is unambiguous!

Here is the graph of $y = 2^x$:



For very large positive values of x, the y-value approaches infinity. For very large negative values of x, the y-value approaches (but never reaches) zero.

A line that a function gets increasingly close to but never reaches is called an **asymptote**. In this case we would say that the x-axis is an asymptote to the graph.



If we look at the graphs of other exponential functions with different bases we can begin to make some generalisations.

For all the graphs $y = a^x$:

- The y-intercept is always (0, 1) because $a^0 = 1$.
- The graph of the function lies entirely above the x-axis since $a^x > 0$ for all values of x.
- The x-axis is an asymptote.
- If $a > 1$, then as x increases, so does $f(x)$. This is called a **positive exponential**.
- If $0 < a < 1$, then as x increases, $f(x)$ decreases. This is called a **negative exponential**.

You may observe that the blue line is a reflection of the black line. You will see why this is the case in chapter 6.

Many mathematical models use the characteristics of exponential functions: as time (t) increases by a fixed value, the value we are interested in (N) will change by a fixed *factor*, called the **growth factor**. Exponential functions can therefore be used to model many physical, financial and biological forms of **exponential growth** (positive exponential models) and **exponential decay** (negative exponential models).

To model more complex situations we may need to add more constants to our exponential equation.

We can use a function of the form $N = Ba^{\left(\frac{t}{k}\right)}$.

We can interpret the constants in the following way:

- When $t = 0$, $N = B$ so B is the **initial value** of N.
- When $t = k$, $N = Ba$ so k is the time taken for N to increase by a factor of a.
- If $k = 1$ then a is the growth factor.
- As long as k is positive when $a > 1$ the function models exponential growth.
- When $0 < a < 1$ the function models exponential decay.
- When modelling exponential decay there might sometimes be a **background level**. This means the asymptote is not $N = 0$.

We can change the asymptote to $N = c$ by adding on a constant.

The new function is then $N = Ba^{\left(\frac{t}{k}\right)} + c$.

B represents how much N starts above the background level, so the initial value is $B + c$.

KEY POINT 2.6

If $N = Ba^{\left(\frac{t}{k}\right)} + c$:

- The background level (and the asymptote) is c.
- The initial value is $B + c$.
- k is the time taken for the difference between N and the background level to increase by a factor of a.
- If $a > 1$ this models exponential growth.
- If $0 < a < 1$ this models exponential decay.

EXAM HINT

See Calculator skills sheet 2 on the CD-ROM to find out how to sketch graphs on your calculator. As asymptotes are not a part of the graph, your calculator might not show them, though you can guess approximately where they are by looking at large values of x. This is why it is important to know how to find asymptotes directly from the equation.

In many applications we are given certain information and need to find the constants in the model.

Worked example 2.3

A population of bacteria in a culture medium doubles in size every 15 minutes.

(a) Write down a model for N, the number of bacteria in terms of time, t, in hours.

(b) If there are 1000 bacterial cells at 08:00, how many cells are there at

(i) 08:15?

(ii) 09:24?

There is a constant increasing factor so use an exponential growth model

(a) Let N be the number of cells at time t hours

$$N = Ba^{\left(\frac{t}{k}\right)}$$

Every time t increases by 0.25, N doubles

Doubles every quarter hour

$a = 2, k = 0.25$

$\therefore N = B \times 2^{4t}$

All details are relative to a start time 08:00, so set $t = 0$ at that time

(b) When $t = 0, N = 1000 = B$

$N = 1000 \times 2^{4t}$

Remember to convert minutes to hours

(i) When $t = 0.25$, $N = 2000$ cells

(ii) When $t = 1.4$, $N = 48\,503$ cells

You may also be given a model and have to interpret it. Unfortunately, it might not be given in exactly the same form as in Key point 2.6 $\left(N = Ba^{\frac{t}{k}}\right)$, so you will have to use the rules of exponents to rewrite it in the correct form.

Worked example 2.4

The temperature in degrees (θ) of a cup of coffee a time t minutes after it was made is modelled using the function:

$$\theta = 70 \times 3^{-\beta t} + 22, \beta > 0$$

(a) Show that $\theta - 22$ follows an exponential decay.
(b) What was the initial temperature of the coffee?
(c) If the coffee is left for a very long time, what temperature does the model predict it will reach?
(d) Find, in terms of β, how long it takes for the temperature difference between the coffee and the room to fall by a factor of 9.
(e) Coffee made at the same time is put into a thermos flask which is much better insulated than the cup. State with a reason whether β would increase or decrease.

Use rules of exponents to rewrite the exponent in the form $N = Ba^{\left(\frac{t}{k}\right)}$

(a) $\theta - 22 = 70 \times 3^{-\beta t}$

$= 70 \times (3^{-1})^{\beta t}$

$= 70 \times \left(\frac{1}{3}\right)^{\frac{t}{\beta^{-1}}}$

which is exponential decay since the base is between 0 and 1 and $\beta > 0$

Initial means $t = 0$

(b) When $t = 0$, $\theta = 70 \times 3^0 + 22$
$= 92°$

The asymptote is given by the background level

(c) 22 degrees

Use algebra to rewrite the exponent in the form $N = Ba^{\frac{t}{k}}$

(d) $\theta - 22 = 70\left(\frac{1}{3}\right)^{\frac{t}{\beta^{-1}}}$

So the time to fall by a factor of 3 is $\frac{1}{\beta}$

To fall by a factor of 9 is a fall by a factor of 3 followed by another fall by a factor of 3. Therefore the time to fall by a factor of 9 is $2 \times \frac{1}{\beta}$ minutes.

continued . . .

(e) The time taken to fall by any given factor will be longer for a better insulated container so $\frac{1}{\beta}$ must be larger so β must be smaller.

When modelling exponential growth or decay, you may be given a percentage increase or decrease. This needs to be converted into a growth factor to be used in the exponential model.

Worked example 2.5

A car costs \$17 500. It then loses value at a rate of 18% each year.

(a) Write a model for the value of the car (V) after n years in the form $V = ka^n$.

(b) Hence or otherwise find the value of the car after 20 years.

Find the growth factor

(a) The growth factor is $1-\frac{18}{100}=0.82$

Use initial value information

When $n=0,\ V=k=17500$

$V=17500\times 0.82^n$

Substitute for n

(b) After 20 years, $V=\$330.61$

Exercise 2B

1. Using your calculator, sketch the following functions for $-5\le x\le 5$ and $0\le y\le 10$.

   Show all the axis intercepts and state the equation of the horizontal asymptote.

   (a) (i) $y=1.5^x$ (ii) $y=3^x$

   (b) (i) $y=2\times 3^x$ (ii) $y=6\times 1.4^x$

   (c) (i) $y=\left(\frac{1}{2}\right)^x$ (ii) $y=\left(\frac{2}{3}\right)^x$

   (d) (i) $y=5+2^x$ (ii) $y=8+3^x$

   (e) (i) $y=6-2^x$ (ii) $y=1-5^x$

2. An algal population on the surface of a pond grows by 10% every day, and the area it covers can be modelled by the equation $y=k\times 1.1^t$, where t is measured in days. At 09:00 on Tuesday it covered 10 m^2. What area will it cover by 09:00 on Friday? *[4 marks]*

3. The air temperature T °C around a light bulb is given by equation

$$T = A + B \times 2^{-\frac{x}{k}}$$

where x is the distance from the surface of the light bulb in millimetres. The background temperature in the room is a constant 25 °C, and the temperature on the surface of the light bulb is 125 °C.

(a) You find that the air temperature 3 mm from the surface of the bulb is only 75 °C. Find the integer values of A, B and k.

(b) Determine the air temperature 2 cm from the surface of the bulb.

(c) Sketch a graph of air temperature against distance.

[10 marks]

4. A tree branch is observed to bend as the fruit growing on it increases in size. By estimating the mass of the developing fruit, and plotting the data over time, a student finds that the height in metres of the branch tip above the ground closely follows the graph of:

$$h = 2 - 0.2 \times 1.6^{0.2m}$$

where m is the estimated mass, in kilograms, of fruit on the branch.

(a) Plot a graph of h against m.

(b) What height above ground level is a branch without fruit?

(c) The total mass of fruit on the branch at harvest was 7.5 kg. Find the height of the branch immediately prior to harvest.

(d) The student wishes to estimate what mass of fruit would cause the branch tip to touch the ground. Why might his model not be suitable to assess this? *[10 marks]*

5. (a) Sketch the graph of $y = 1 + 16^{1-x^2}$.

Label clearly the horizontal asymptote and maximum value.

(b) Find all values of x for which $y = 3$. *[6 marks]*

6. A bowl of soup is served at a temperature of 55 °C in a room with a constant air temperature of 20 °C. Every 5 minutes, the temperature difference between the soup and the room air decreases by 30%. Assuming the room air temperature is constant, at what temperature will the soup be seven minutes after serving? *[7 marks]*

7. The speed (V metres per second) of a parachutist t seconds after jumping from an aeroplane is modelled by the expression:

$$V = 40(1 - 3^{-0.1t})$$

(a) Find his initial speed.
(b) What speed does the model predict that he will eventually reach? *[6 marks]*

2C The value e

In this section we introduce a mathematical constant, e, which will be used extensively in the rest of this chapter and in many other chapters throughout the course.

Consider the following different situations, each of which is typical of early population growth of a cell culture.

(a) There is a 100% increase every 100 seconds.
(b) There is a 50% increase every 50 seconds.
(c) There is a 25% increase every 25 seconds.

Although these may at first appear to be equivalent statements, they are subtly different because of the compounding nature of percentage increases.

If we begin with a population of size P, then after 100 seconds, we have

(a) $P \times (1+1) = 2P$

(b) $P \times \left(1 + \frac{1}{2}\right)^2 = 2.25P$

(c) $P \times \left(1 + \frac{1}{4}\right)^4 = 2.44P$

To generalise the situation, if we considered an increase of $\frac{1}{n}\%$ which occurred n times every 100 seconds, the population after 100 seconds would be given by $P \times \left(1 + \frac{1}{n}\right)^n$.

It may seem from the above that as n increases, the overall increase over 100 seconds will keep on getting larger, and this is indeed the case, but not without limit.

In fact, as can be seen by taking greater and greater values of n, the resultant increase factor $\left(1+\frac{1}{n}\right)^n$ tends towards a value of approximately 2.71828182849 which, much like π, is such an important part of mathematical studies that it has been given its own letter, e.

π and e have many similar properties. Both are irrational, meaning that they cannot be written as a ratio of two whole numbers and both are transcendental, meaning that they cannot be written as the solution to a polynomial equation. The proof of these facts is intricate but beautiful.

KEY POINT 2.7

$e = 2.71828182849\ldots$

Although e has many important properties in mathematics, it is still just a number, and so all the standard rules of arithmetic and exponents apply. You are not likely to find an exam question just on e, but the number e *is* likely to appear in questions on many other topics.

In chapter 16 you will see that e plays a major role in studying rates of change.

EXAM HINT

In questions involving the number e you may be asked to either give an exact answer (such as e^2) or to use your calculator, in which case you should usually round the answer to 3 significant figures.

Exercise 2C

1. Find the values of the following to 3 significant figures:

(a) (i) $e+1$ (ii) $e-4$

(b) (i) $3e$ (ii) $\frac{e}{2}$

(c) (i) e^2 (ii) e^{-3}

(d) (i) $5e^{0.5}$ (ii) $\frac{3}{e^7}$

2. Evaluate $\sqrt[6]{(\pi^4+\pi^5)}$. What do you notice about this result?

2D Introducing logarithms

In this section we shall look at a new operation which reverses the effect of exponentiating (that is, raising a number to an exponent).

See Supplementary sheet 2 'Logarithmic scales and log-log graphs' on the CD-ROM if you would like to discover more about logarithms for yourself.

If you are asked to solve

$$x^2 = 3, x \geq 0$$

you can either find a decimal approximation (using a calculator or trial and improvement) or you can use the square root symbol: $x = \sqrt{3}$.

This restates the problem as 'x is the positive value which when squared gives 3'.

Similarly, if asked to solve:

$$10^x = 50$$

you could use trial and improvement to seek a decimal value:

$$10^1 = 10$$

$$10^2 = 100$$

So x is between 1 and 2:

$$10^{1.5} = 31.6$$

$$10^{1.6} = 39.8$$

$$10^{1.7} = 50.1$$

So the answer is around 1.7.

However, just as with squares and square roots there is also a function to answer the question 'What is the number, which when put as the exponent of 10, gives this value?'

The function is called a base-10 **logarithm**, written $\log_{10}$.

So in the above example: $10^x = 50$ so $x = \log_{10} 50$.

This means that $y = 10^x$ may be re-expressed as $x = \log_{10}(y)$.

In fact, the base involved need not be 10, but could be any positive value other than 1.

KEY POINT 2.8

$$b = a^x \Leftrightarrow x = \log_a b$$

The symbol $\Leftrightarrow$ has a very specific meaning (implies and is implied by) in mathematical logic. It means that if the left-hand side is true then so is the right-hand side, and also if the right-hand side is true then so is the left-hand side.

This is the form written in the Formula booklet; you only need to remember that it means you can switch between them.

It is worth noting that the two most common bases have abbreviations for their logarithms.

Since we use a decimal system of counting, base 10 is the default base for a logarithm, so that $\log_{10} x$ is generally written more simply as just $\log x$, called the **common logarithm**. Also e, encountered in Section 2C, is considered the 'natural' base, and its counterpart the **natural logarithm** is denoted by $\ln x$.

KEY POINT 2.9

$\log_{10} x$ is often written as $\log x$.

$\log_e x$ is often written as $\ln x$.

See Key point 2.18.

Since taking a logarithm reverses the process of exponentiation, it follows that:

KEY POINT 2.10

$$\log_a(a^x) = x$$

KEY POINT 2.11

$$a^{\log_a x} = x$$

These are sometimes referred to as the cancellation principles. This sort of 'cancellation', similar to stating that (for positive x) $\sqrt[n]{x^n} = x = \left(\sqrt[n]{x}\right)^n$, is frequently useful when simplifying logarithm expressions, but you can only apply it when the base of the logarithm and the base of the exponential match.

Worked example 2.6

Evaluate:

(a) $\log_5 625$ (b) $\log_8 16$

Express the argument of the logarithm in exponent form with the same base	(a) $\log_5 625 = \log_5 5^4$
Apply the cancellation principle $\log_a(a^x) = x$	$= 4$
16 is not a power of 8, but they are both powers of 2	(b) $\log_8(16) = \log_8(2^4)$
We need to convert 2^4 to an exponent of $8 = 2^3$. Rewrite 4 as $3 \times \frac{4}{3}$ and use $a^{mn} = (a^m)^n$	$= \log_8\left(2^{3\times\frac{4}{3}}\right) = \log_8\left(8^{\frac{4}{3}}\right)$
Apply the cancellation principle $\log_a(a^x) = x$	$= \frac{4}{3}$

Actually, you can find the logarithm of a negative number, but the answer turns out not to be real; it is a complex number, which is a new type of number you will meet in chapter 15.

Whenever you raise a positive number to a power, positive or negative, the answer is always positive. Therefore we currently have no answer to a question such as $10^x = -3$.

This means:

KEY POINT 2.12

The logarithm of a negative number or zero has no real value.

Exercise 2D

1. Evaluate the following:

(a) (i) $\log_3 27$ (ii) $\log_4 16$

(b) (i) $\log_5 5$ (ii) $\log_3 3$

(c) (i) $\log_{12} 1$ (ii) $\log_{15} 1$

(d) (i) $\log_3 \frac{1}{3}$ (ii) $\log_4 \frac{1}{64}$

(e) (i) $\log_4 2$ (ii) $\log_{27} 3$

(f) (i) $\log_8 \sqrt{8}$ (ii) $\log_2 \sqrt{2}$

(g) (i) $\log_8 4$ (ii) $\log_{81} 27$

(h) (i) $\log_{25} 125$ (ii) $\log_{16} 32$

(i) (i) $\log_4 2\sqrt{2}$ (ii) $\log_9 81\sqrt{3}$

(j) (i) $\log_{25} 0.2$ (ii) $\log_4 0.5$

2. Use a calculator to evaluate each of the following, giving your answer correct to 3 significant figures:

(a) (i) $\log 50$ (ii) $\log\left(\frac{1}{4}\right)$

(b) (i) $\ln 0.1$ (ii) $\ln 10$

EXAM HINT

On most calculators log x is denoted with a [log] button, and ln x by a [ln] button.

3. Simplify the following expressions:

(a) (i) $7\log x - 2\log x$ (ii) $2\log x + 3\log x$

(b) (i) $(\log x - 1)(\log y + 3)$ (ii) $(\log x + 2)^2$

(c) (i) $\frac{\log a + \log b}{\log a \log b}$ (ii) $\frac{(\log a)^2 - 1}{\log a - 1}$

EXAM HINT

Remember that 'log x' is just a number so can be treated the same way as any variable.

4. Make x the subject of the following:

(a) (i) $\log_3 x = y$ (ii) $\log_4 x = 2y$

(b) (i) $\log_a x = 1 + y$ (ii) $\log_a x = y^2$

(c) (i) $\log_x 3y = 3$ (ii) $\log_x y = 2$

5. Find the value of x in each of the following:

(a) (i) $\log_2 x = 32$ (ii) $\log_2 x = 4$

(b) (i) $\log_5 25 = 5x$ (ii) $\log_{49} 7 = 2x$

(c) (i) $\log_x 36 = 2$ (ii) $\log_x 10 = \frac{1}{2}$

6. Solve the equation $\log_{10}(9x+1)=3$. *[4 marks]*

7. Solve the equation $\log_8 \sqrt{1-x}=\frac{1}{3}$. *[4 marks]*

8. Find the exact solution to the equation $\ln(3x-1)=2$. *[5 marks]*

9. Find all values of x which satisfy $(\log_3 x)^2=4$. *[5 marks]*

10. Solve the simultaneous equations:

$$\log_3 x+\log_5 y=6$$
$$\log_3 x-\log_5 y=2$$

[6 marks]

11. Solve the equation $3(1+\log x)=6+\log x$. *[5 marks]*

12. Solve the equation $\log_x 4=9$. *[4 marks]*

13. The Richter scale is a way of measuring the strength of earthquakes. An increase of one unit on the Richter scale corresponds to an increase by a factor of 10 in the strength of the earthquake. What would be the Richter level of an earthquake which is twice as strong as a level 5.2 earthquake? *[5 marks]*

2E Laws of logarithms

Just as there are rules which hold when working with exponents, so there are corresponding rules which apply to logarithms. These are derived from the laws of exponents and can be found on the Fill-in proof sheet 18. 'Differentiating logarithmic functions graphically' on the CD-ROM.

- The logarithm of a *product* is the *sum* of the logarithms.

KEY POINT 2.13

$\log_a xy=\log_a x+\log_a y$ for $x, y>0$

- The logarithm of a *quotient* is the *difference* of the logarithms.

KEY POINT 2.14

$$\log_a \frac{x}{y} = \log_a x - \log_a y \text{ for } x, y > 0$$

EXAM HINT

These formulae are *not* in the Formula booklet.

- The logarithm of a *reciprocal* is the *negative* of the logarithm.

KEY POINT 2.15

$$\log_a \frac{1}{x} = -\log_a x \text{ for } x > 0$$

EXAM HINT

It is important to know what you *cannot* do with logarithms. One very common mistake is to try to simplify $\log(x + y)$ into either $\log x + \log y$ or $\log x \log y$.

- The logarithm of a *exponent* is the *multiple* of the logarithm.

KEY POINT 2.16

$$\log_a x^p = p \log_a x \text{ for } x > 0$$

- The logarithm of 1 is always 0, irrespective of the base.

KEY POINT 2.17

$$\log_a 1 = 0$$

The rules of logarithms can be used to manipulate expressions and solve equations involving logarithms.

Worked example 2.7

If $x = \log_{10} a$ and $y = \log_{10} b$, express $\log_{10} \frac{100a^2}{b}$ in terms of x, y and integers.

Use laws of logs to isolate $\log_{10} a$ and $\log_{10} b$ First, use $\log \frac{x}{y} = \log x - \log y$	$\log_{10} \frac{100a^2}{b} = \log_{10}(100a^2) - \log_{10} b$
...then $\log xy = \log x + \log y$...	$= \log_{10} 100 + \log_{10} a^2 - \log_{10} b$
... then $\log x^p = p \log x$...	$= \log_{10} 100 + 2\log_{10} a - \log_{10} b$
... then calculate $\log_{10} 100$ ... then write in terms of x and y	$= 2 + 2\log_{10} a - \log_{10} b$ $= 2 + 2x - y$

Worked example 2.8

Solve the equation $\log_2 x + \log_2(x+4) = 5$.

Rewrite one side as a single logarithm using $\log x + \log y = \log xy$

$\log_2 x + \log_2(x+4) = 5$

$\Leftrightarrow \log_2(x(x+4)) = 5$

Undo the logarithm by exponentiating both sides with base 2 (to balance, you must also exponentiate 5 by base 2) …

$\Leftrightarrow 2^{\log_2(x(x+4))} = 2^5$

and use the cancellation principle

$\Leftrightarrow x^2 + 4x = 32$

Use standard methods for quadratic equations

$\Leftrightarrow x^2 + 4x - 32 = 0$

$\Leftrightarrow (x+8)(x-4) = 0$

$\Rightarrow x = -8$ or $x = 4$

Check your solution in the original equation

When $x = -8$:

LHS: $\log_2(-8)$ and $\log_2(-4)$ are not real so this solution does not work

When $x = 4$:

LHS: $\log_2 4 + \log_2 8 = 2 + 3 = 5$

$= \text{RHS}$

EXAM HINT

Checking your solutions is more than just looking for an arithmetic error – as you can see from the example above, false solutions can occur through *correct* algebraic manipulation.

You will notice that although we have discussed logarithms for a general base, *a*, your calculator may only have buttons for the common logarithm and the natural logarithm ($\log x$ and $\ln x$).

To use a calculator to evaluate, for example, $\log_5 20$ you can use the **change of base rule** of logarithms.

KEY POINT 2.18

Change of base rule for logarithms:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

So, we can calculate $\log_5 20$ using the logarithm to the base 10:

$$\log_5 20 = \frac{\log 20}{\log 5} = 1.86 \text{ (3 SF)}$$

The change of base rule is useful for more than just evaluating logarithms.

For an insight into what mathematics was like before calculators, have a look at Supplementary sheet 3, 'A history of logarithms' on the CD-ROM.

Worked example 2.9

Solve the equation $\log_3 x + \log_9 x = 2$.

We want to have logarithms involving just one base, so use the change of base rule to turn logs with base 9 into logs with base 3

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{2}$$

Therefore:

$$\log_3 x + \log_9 x = 2$$

$$\Leftrightarrow \log_3 x + \frac{\log_3 x}{2} = 2$$

Collecting the logs together

$$\Leftrightarrow \frac{3}{2}\log_3 x = 2$$

$$\Leftrightarrow \log_3 x = \frac{4}{3}$$

Exponentiate both sides with base 3

$$\Leftrightarrow x = 3^{\frac{4}{3}} = 4.33 \text{ (3SF)}$$

Equations combining logarithms and quadratics can be found in Section 4B.

Exercise 2E

1. Given $b > 0$, simplify each of the following:

   (a) (i) $\log_b b^4$ (ii) $\log_b \sqrt{b}$

   (b) (i) $\log_{\sqrt{b}} b^3$ (ii) $\log_b b^2 - \log_{b^2} b$

2. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z:

   (a) (i) $\log b^7$ (ii) $\log a^2 b$

   (b) (i) $\log\left(\frac{ab^2}{c}\right)$ (ii) $\log\left(\frac{a^2}{bc^3}\right)$

(c) (i) $\log\left(\frac{100}{bc^5}\right)$ (ii) $\log(5b)+\log(2c^2)$

(d) (i) $\log a^3 - 2\log ab^2$ (ii) $\log(4b)+2\log(5ac)$

(e) (i) $\log_a a^2b$ (ii) $\log_b\left(\frac{a}{bc}\right)$

(f) (i) $\log_{ab}(b^a)$ (ii) $\log_{ab} ac^2$

3. Solve the following for x:

(a) (i) $\log_3\left(\frac{2+x}{2-x}\right)=3$ (ii) $\log_2(7x+4)=5$

(b) (i) $\log_3 x - \log_3(x-6)=1$ (ii) $\log_8 x - 2\log_8\left(\frac{1}{x}\right)=1$

(c) (i) $\log_2 x + 1 = \log_4 x$ (ii) $\log_8 x + \log_2 x = 4$

(d) (i) $\log_4 x + \log_8 x = 2$ (ii) $\log_{16} x - \log_{32} x = 0.5$

(e) (i) $\log_3(x-7)+\log_3(x+1)=2$

(ii) $2\log(x-2)-\log x = 0$

(f) (i) $\log(x^2+1)=1+2\log x$

(ii) $\log(3x+6)=\log 3 + 1$

4. Find the exact solution to the equation $2\ln x + \ln 9 = 3$, giving your answer in the form Ae^B where A and B are rational numbers. *[5 marks]*

5. If $a=\ln 2$ and $b=\ln 5$, find in terms of a and b: *[6 marks]*

(a) $\ln 50$ (b) $\ln 0.16$

6. Solve $\log_2 x = \log_x 2$. *[5 marks]*

7. Prove that if $a^x = b^y = (ab)^{xy}$ where $a, b > 1$ then $x+y=1$ or $x=y=0$. *[5 marks]*

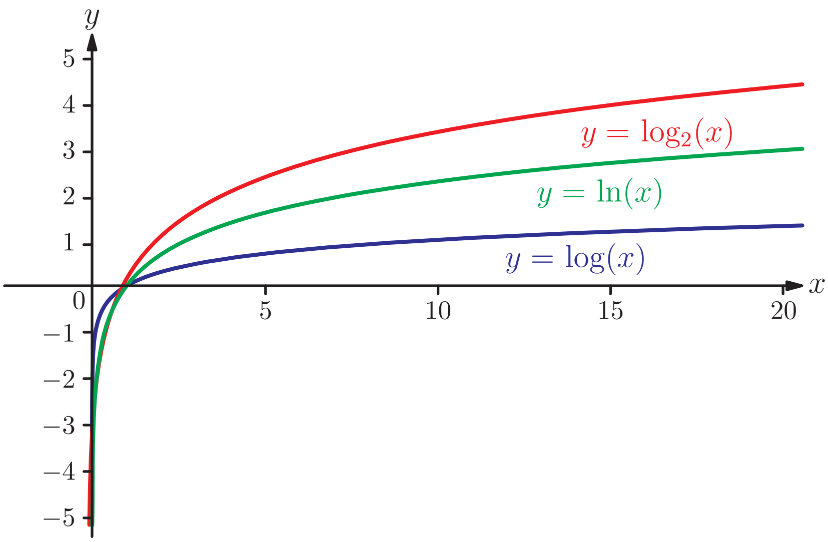
8. Evaluate $\log\frac{1}{2}+\log\frac{2}{3}+\log\frac{3}{4}+\log\frac{4}{5}\ldots+\log\frac{8}{9}+\log\frac{9}{10}$. *[4 marks]*

9. Given that $\log_a b = \log_b a$, and that $a,b\neq 1$ and $a\neq b$, find b in terms of a. *[5 marks]*

2F Graphs of logarithms

It is important also to know the graph of the logarithm function, and the various properties of logarithms which we can deduce from it.

Below are the graphs of $y = \log x$, $y = \log_2 x$ and $y = \ln x$:



These three curves all have a similar shape.

The change of base rule (Key point 2.18) states that $\log_b a = \dfrac{\log_c a}{\log_b a}$.

So, $\log_2 x = \dfrac{\log x}{\log 2}$ and $\ln x = \dfrac{\log x}{\log \mathrm{e}}$ and all other logarithm functions are simply multiples of the common logarithm function $y = \log x$.

In chapter 6 you will see how this type of change causes a vertical stretch of the graph.

The graph shows the following important facts about the logarithm function (Key point 2.19).

EXAM HINT

Vertical asymptotes are even harder to detect accurately from a calculator display than the horizontal ones. This is why it is important that you know how to find an asymptote of a logarithmic graph.

KEY POINT 2.19

If $y = \log_a x$ then:

- The graph of y against x crosses the x-axis at (1, 0), because $\log_a 1 = 0$ (for any positive value of a).
- Log x is negative for $0 < x < 1$ and positive for $x > 1$.
- The graph of the function lies entirely to the right of the y-axis, since the logarithm of a negative value does not produce a real solution.
- The logarithm graph increases throughout: as x tends to infinity so does y.
- The y-axis is an asymptote to the curve.

You may observe that a logarithm graph is the reflection of an exponential graph. You will see why this is the case in chapter 5.

Exercise 2F

It is unlikely you will find exam questions testing *just* this topic, but you may be required to sketch a graph involving a logarithm as a part of another question.

1. Sketch the following graphs, labelling clearly the vertical asymptote and all axis intercepts.

   (a) (i) $y = \log(x^2)$ (ii) $y = \log(x^3)$

   (b) (i) $y = \log 4x$ (ii) $y = \log 2x$

   (c) (i) $y = \log(x-2)$ (ii) $y = \log(x+1)$

2. Why are the graphs of $y = \log(x^2)$ and $y = 2\log(x)$ different?

2G Solving exponential equations

One of the main uses of logarithms is to solve equations with the unknown in the exponent. By taking logarithms, the unknown becomes a factor, which is easier to deal with.

Worked example 2.10

Solve the equation $3\times 2^x = 5^{x-1}$, giving your answer in the form $\frac{\log p}{\log q}$ where p and q are rational numbers.

Since the unknown is in the exponent, taking logarithms is a good idea

$$\log(3\times 2^x) = \log(5^{x-1})$$

Use the rules of logarithms to simplify the expression

$$\log 3 + \log 2^x = \log(5^{x-1})$$
$$\log 3 + x\log 2 = (x-1)\log 5$$

Expand the brackets and get all of the x's on one side and everything else on the other side

$$\log 3 + x\log 2 = x\log 5 - \log 5$$
$$\log 3 + \log 5 = x\log 5 - x\log 2$$

Factorise and divide to find x

$$x = \frac{\log 3 + \log 5}{\log 5 - \log 2}$$

Use the rules of logarithms to write it in the correct form

$$x = \frac{\log 15}{\log\left(\frac{5}{2}\right)}$$

EXAM HINT

A common mistake is to say that $\log(3 \times 2^x)$ is $\log 3 \times \log 2^x$ Make sure you learn the rules of logarithms carefully.

Logarithmic scales make it easy to compare values which are very different. Some of these scales are explored in Supplementary sheet 2 on the CD-ROM.

You can use similar ideas to solve inequalities with the unknown in the exponent:

Worked example 2.11

The number of bacteria in a culture medium is given by $N = 1000 \times 2^{4t}$, where t is the number of hours elapsed since 08:00. At what time will the population first exceed one million?

Simplify the equation where possible

$1000 \times 2^{4t} \geq 1\,000\,000$

$\Leftrightarrow 2^{4t} \geq 1000$

Take logarithms of each side

$\Leftrightarrow \log(2^{4t}) \geq \log 1000$

Use the rule $\log_a x^p = p \log_a x$

$\Leftrightarrow 4t \log 2 \geq 3$

Note that $\log 2 > 0$, so the inequality remains in the same orientation when we divide

$\Leftrightarrow t \geq \dfrac{3}{4\log 2} = 2.49(3SF)$

To answer the question, convert 2.49 to hours and minutes, then add it to 08:00

10:29

You know that you can always apply the same operation to both sides of an equation, but in this worked example we took logarithms of both sides of an inequality. You might like to investigate for which operations this is valid.

EXAM HINT

Whenever dividing an inequality by a logarithm it is important to remember to check if it is positive or negative.

There is another type of exponential equation which you will meet – a disguised quadratic equation. These are explored in Section 4B.

Exercise 2G

1. Solve for x, giving your answer correct to 3 significant figures.

   (a) (i) $3\times4^{x}=90$ (ii) $1000\times1.02^{x}=10\,000$

   (b) (i) $6\times7^{3x+1}=1.2$ (ii) $5\times2^{2x-5}=94$

   (c) (i) $3^{2x}=4^{x-1}$ (ii) $5^{x}=6^{1-x}$

   (d) (i) $3\times2^{3x}=7\times3^{3x-2}$ (ii) $4\times8^{x-1}=3\times5^{2x+1}$

2. In a yeast culture, cell numbers are given by $N=100\mathrm{e}^{1.03t}$, where t is measured in hours after the cells are introduced to the culture.

   (a) What is the initial number of cells? *[1 mark]*

   (b) How many cells will be present after 6 hours? *[1 mark]*

   (c) How long will it take for the population to exceed one thousand? *[2 marks]*

3. A rumour spreads exponentially through a school. When school begins (at 9 a.m.) 18 people know it. By 10 a.m. 42 people know it.

   (a) How many people know it at 10.30? *[3 marks]*

   (b) There are 1200 people in the school. According to the exponential model, at what time will everyone know the rumour? *[2 marks]*

4. In an experimental laboratory, a scientist sets up a positive feedback loop for a fission reaction and extracts heat to control the experiment and produce power. When the reaction is established, and while sufficient fuel is present, the power he can siphon off is given by $P=32\left(\mathrm{e}^{0.0012t}-1\right)$, where P is measured in units of energy per second and t in seconds.

   (a) How much energy is being produced after 2 minutes? *[1 mark]*

   (b) The equipment reaches a dangerous temperature when P exceeds 7×10^{5}. For how long can the experiment safely be run? *[2 marks]*

5. The weight of a block of salt W in a salt solution after t seconds is given by:

$$W=k\mathrm{e}^{-0.01t}$$

   (a) Sketch the graph of W against t. [2 marks]

   (b) How long will it take to reach 25% of its original weight? [2 marks]

6. Solve the equation $5 \times 4^{x-1} = \frac{1}{3^{2x}}$, giving your answer in the form $x = \frac{\ln p}{\ln q}$, where p and q are rational numbers. *[5 marks]*

7. Solve the equation $\frac{1}{7^x} = 3 \times 49^{5-x}$, giving your answer in the form $a + \log_7 b$ where $a, b \in \mathbb{Z}$.

8. A cup of tea is poured at 98 °C. After two minutes it has reached 94 °C. The difference between the temperature of the tea and the room temperature (22 °C) falls exponentially. Find the time it takes for the tea to cool to 78 °C. *[5 marks]*

9. (a) Show that the equation $3^x = 3 - x$ has only one solution. *[2 marks]*

(b) Find the solution, giving your answer to 3 significant figures. *[4 marks]*

Summary

- In this chapter, we revisited the rules for **exponents**.

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$a^n \times b^n = (ab)^n$$
$$a^n \div b^n = \left(\frac{a}{b}\right)^n$$
$$(a^m)^n = a^{m \times n}$$
$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

- Exponential equations can be used to model growth and decay of some simple real-life systems, taking as a general form the function:

$$N = Ba^{\left(\frac{t}{k}\right)} + c$$

- $\log_a b$ asks the question: 'What exponent do I have to raise a to in order to get b?'

$$b = a^x \Leftrightarrow x = \log_a b$$

- e is the mathematical constant, (Euler's number): e = 2.71828182849…
- $\log_{10} x$ (common logarithm) is often written as $\log x$.

- $\log_e x$ (natural logarithm) is often written as $\ln x$.
- **Logarithms** undo the effect of exponentiating and vice versa:

$$\log_a(a^x) = x \Leftrightarrow x = a^{\log_a x}$$

- Logarithms obey these rules, which are *not* covered in the Formula booklet.

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$
$$\log_a x^p = p\log_a x$$
$$\log_a 1 = 0$$

- Taking the logarithm of a negative number or zero does not give a real value.
- There is also the change of base formula (not in the Formula booklet): $\log_b a = \dfrac{\log_c a}{\log_c b}$ and a related rule for exponents (*is* in the Formula booklet): $a^x = e^{x\ln a}$
- The following properties of the logarithm function can be deduced from its graph, if $y = \log_a x$ then:
  – The graph of y against x crosses the x-axis at (1,0)
  – Log x is negative for $0 < x < 1$ and positive for $x > 1$
  – The graph lies to the right of the y-axis; the y-axis is an asymptote to the curve
  – The logarithm graph increases throughout; as x tends to infinity so does y.
- Logarithms are used to solve many exponential equations.

Introductory problem revisited

A radioactive substance has a half-life of 72 years. A 1 kg block of the substance is found to have a radioactivity of 25 million Becquerel (Bq). How long, to the nearest 10 years, would it take for the radioactivity to have fallen below 10 000 Bq?

Establish the exponential equation	Let R be the radioactivity after t years $R = Ba^{\frac{t}{k}}$
Initial condition gives B	When $t = 0$, $R = 25 \times 10^6 = B$
Every time t increases by 72, R falls by 50%	$a = 0.5, k = 72$ $C = 25 \times 10^6 \times 0.5^{\frac{t}{72}}$

continued...

The unknown is in the exponent so take logarithms

Now rearrange to find t
Note that $\log(0.5) < 0$, so the inequality is reversed when dividing through

Require $R \leq 10^4$

$$25 \times 10^6 \times 0.5^{\frac{t}{72}} \leq 10^4$$

$$\Leftrightarrow 0.5^{\frac{t}{72}} \leq 0.0004$$

$$\Leftrightarrow \log\left(0.5^{\frac{t}{72}}\right) \leq \log(0.0004)$$

$$\frac{t}{72}\log(0.5) \leq \log(0.0004)$$

$$\Leftrightarrow t \geq \frac{72\log(0.0004)}{\log(0.5)} = 812.7$$

It takes around 810 years for the radioactivity to fall below 10 000 Bq.

Mixed examination practice 2

Short questions

1. Solve $\log_5\left(\sqrt{x^2+49}\right)=2$. *[4 marks]*

2. If $a=\log x$, $b=\log y$ and $c=\log z$ (all logs base 10) find in terms of a, b, c and integers:

(a) $\log\dfrac{x^2\sqrt{y}}{z}$ (b) $\log\sqrt{0.1x}$ (c) $\log_{100}\left(\dfrac{y}{z}\right)$ *[6 marks]*

3. Solve the simultaneous equations:

$$\ln x+\ln y^2=8$$

$$\ln x^2+\ln y=6$$ *[6 marks]*

4. If $y=\ln x-\ln(x+2)+\ln(4-x^2)$, express x in terms of y. *[6 marks]*

5. Find the exact value of x satisfying the equation

$$2^{3x-2}\times 3^{2x-3}=36^{x-1}$$

giving your answer in simplified form $\dfrac{\ln p}{\ln q}$, where $p,q\in\mathbb{Z}$. *[5 marks]*

6. Given $\log_a b^2=c$ and $\log_b a=c-1$ for some value c, where $0<a<b$, express a in terms of b. *[6 marks]*

7. Solve the equation $9\log_5 x=25\log_x 5$, expressing your answers in the form $5^{\frac{p}{q}}$, where $p,q\in\mathbb{Z}$. *[6 marks]*

8. Find the exact solution to the equation $\ln x=4\log_x \mathrm{e}$. *[5 marks]*

Long questions

1. The speed of a parachutist (V) in metres per second, t seconds after jumping is modelled by the expression:

$$V=42(1-\mathrm{e}^{-0.2t})$$

(a) Sketch a graph of V against t.

(b) What is the initial speed?

(c) What is the maximum speed that the parachutist could reach?

When the parachutist reaches 22 ms^{-1} he opens the parachute.

(d) How long is he falling before he opens his parachute? *[9 marks]*

2. Scientists think that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37 000 tigers but by 1980 the number had dropped to 22 000.

(a) Form a model of the form $T = ka^n$ connecting the number of tigers (T) with the number of years after 1970 (n).

(b) What does the model predict that the population will be in 2020?

(c) When the population reaches 1000 the tiger population will be described as 'near extinction'. In which year will this happen?

In the year 2000 a worldwide ban on the sale of tiger products was implemented, and it is believed that by 2010 the population of tigers had recovered to 10 000.

(d) If the recovery has been exponential find a model of the form $T = ka^m$ connecting the number of tigers (T) with the number of years after 2000 (m).

(e) If in each year since 2000 the rate of growth has been the same, find the percentage increase each year. *[12 marks]*

3. (a) If $\ln y = 2\ln x + \ln 3$ find y in terms of x.

(b) If the graph of $\ln y$ against $\ln x$ is a straight line with gradient 4 and y-intercept 6, find the relationship between x and y.

(c) If the graph of $\ln y$ against x is a straight line with gradient 3 and it passes through the point (1, 2), express y in terms of x.

(d) If the graph of e^y against x^2 is a straight line through the origin with gradient 4, find the gradient of the graph of y against $\ln x$. *[10 marks]*

In this chapter you will learn:

- the formal definition of a polynomial
- operations with polynomials
- a trick for factorising polynomials and finding remainders
- how to sketch the graphs of polynomials
- how to identify the number of solutions of a quadratic equation.

3 Polynomials

Introductory problem

Without using your calculator, solve the cubic equation

$$x^3 - 13x^2 + 47x - 35 = 0$$

You may think that your calculator can 'magically' do any calculation, such as $\sin 45°$ or e^{-2} or $\ln 2$. However, like any computer, it is built from a device called a logic circuit which can only really do addition. If you repeatedly do addition you get multiplication and if you repeatedly do multiplication you raise to a power. Any expression which only uses these operations is called a polynomial. What is particularly surprising is that all the other operations your calculator does can be approximated very accurately using these polynomials.

EXAM HINT

The order of a polynomial is sometimes also called its degree.

3A Working with polynomials

The **polynomial functions** of x make up a family of functions, each of which can be written as a sum of non-negative integer powers of x. Polynomial functions are classified according to the highest power of x occurring in the function, called the **order of the polynomial**.

General form of the polynomial	Order	Classification	Example
a	0	Constant polynomial	$y = 5$
$ax + b$	1	Linear polynomial	$y = x + 7$
$ax^2 + bx + c$	2	Quadratic polynomial	$y = -3x^2 + 4x - 1$
$ax^3 + bx^2 + cx + d$	3	Cubic polynomial	$y = 2x^3 + 7x$
$ax^4 + bx^3 + cx^2 + dx + e$	4	Quartic polynomial	$y = x^4 - x^3 + 2x + \frac{1}{2}$

The letters a, b, c, … are called the **coefficients** of the powers of x, and the coefficient of the highest power of x in the function (a in the table above) is called the **lead coefficient** and the term containing it is the **leading order term**.

Coefficients can take any value, with the restriction that the lead coefficient cannot equal zero; a polynomial of order n which had a lead coefficient 0 could be more simply written as a polynomial of order $(n-1)$. The sign of the lead coefficient dictates whether the polynomial is a **positive polynomial** $(a > 0)$ or a **negative polynomial** $(a < 0)$.

Adding and subtracting two polynomials is straightforward – it is just collecting like terms. For example:

$$(x^4 + 3x^2 - 1) - (2x^3 - x^2 + 2) = x^4 - 2x^3 + 4x^2 - 3.$$

Multiplying is a little more difficult. Below is one suggested way of setting out polynomial multiplication to ensure that you include all of the terms.

The Greeks knew how to solve quadratic equations, and general cubics and quartics were first solved in 14th Century Italy. For over three hundred years nobody was able to find a general solution to the quintic equation, until in 1821 Niels Abel used a branch of mathematics called group theory to prove that there could never be a 'quintic formula'.

Worked example 3.1

Expand $(x^3 + 3x^2 - 2)(x^2 - 5x + 4)$.

Multiply each term in the 1st bracket by the whole of the 2nd bracket

$$\begin{aligned}
&(x^3 + 3x^2 - 2)(x^2 - 5x + 4)\\
&= x^3(x^2 - 5x + 4) + 3x^2(x^2 - 5x + 4) - 2(x^2 - 5x + 4)\\
&= x^5 - 5x^4 + 4x^3\\
&\quad + 3x^4 - 15x^3 + 12x^2\\
&\quad - 2x^2 + 10x - 8\\
&= x^5 - 2x^4 - 11x^3 + 10x^2 + 10x - 8
\end{aligned}$$

It is important to be able to decide when two polynomials are 'equal'. x^2 and $3x-2$ may take the same value when $x=1$ or 2 but when $x=3$ they do not. However, $3x-2$ and $4x-(x+2)$ are always equal, no matter what value of x you choose.

You will find that this type of equality is called an identity and is looked at in more detail in Section 4H.

This leads to what seems like a very obvious definition of equality of polynomials:

KEY POINT 3.1

Two polynomials being equal means that they have the same order and all of their coefficients are equal.

Ideas similar to comparing coefficients will be applied to vectors (chapter 13) and complex numbers (chapter 15).

However, this definition can be used to solve some quite tricky problems. We tend to use it in questions which start from telling us that two polynomials are equal; we can then **compare coefficients**.

Worked example 3.2

If $(ax)^2 + bx + 3(ax^2 + x) = 4x - 2x^2$ find the values of a and b.

Rearrange each side to make the coefficients clear

$$\begin{aligned} \text{LHS: } & a^2x^2 + bx + 3ax^2 + 3x \\ &= a^2x^2 + 3ax^2 + bx + 3x \\ &= x^2(a^2 + 3a) + x(b+3) \end{aligned}$$

Compare coefficients and solve the resulting equations

$$\begin{aligned} x^2: \quad & (a^2 + 3a) = -2 \\ \Leftrightarrow\ & a^2 + 3a + 2 = 0 \\ \Leftrightarrow\ & (a+1)(a+2) = 0 \\ \Leftrightarrow\ & a = -1 \text{ or } a = -2 \\ x: \quad & b + 3 = 4 \\ \Leftrightarrow\ & b = 1 \end{aligned}$$

A very important use of this technique is factorising a polynomial if one factor is already known. To do this we need to know what the remaining factor looks like. For example, if a cubic has one linear factor, the other factor must be quadratic.

Worked example 3.3

$(x-1)$ is a factor of $x^4 + 3x^2 + 2x - 6$. Find the remaining factor.

The remaining factor must be a cubic. We can write the original function as a product of $(x - 1)$ and a general cubic

$$x^4 + 3x^2 + 2x - 6 = (x-1)(ax^3 + bx^2 + cx + d)$$

Multiplying out the right hand side and grouping like terms allows us to compare coefficients

$$\begin{aligned} &= ax^4 + bx^3 + cx^2 + dx \\ &\quad - ax^3 - bx^2 - cx - d \\ &= ax^4 + x^3(b-a) + x^2(c-b) + x(d-c) - d \end{aligned}$$

continued . . .

Remember that the coefficient of x^3 in the original expression is zero!

Compare x^4:

$a = 1$

Compare x^3:

$b - a = 0$

$b = 1$

Compare x^2:

$c - b = 3$

$c = 4$

Compare x:

$d - c = 2$

$d = 6$

Answer the question

The remaining factor is $x^3 + x^2 + 4x + 6$

'Finding the remaining factor' is another way of asking you to divide. We can conclude from this example that:

$$\frac{x^4 + 3x^2 + 2x - 6}{x - 1} = x^3 + x^2 + 4x + 6$$

EXAM HINT

Notice in the previous example we could check using the constant term that $d = 6$.

Exercise 3A

1. Decide whether each of the following expressions are polynomials. For those that are, give the order and the lead coefficient.

(a) $3x^3 - 3x^2 + 2x$

(b) $1 - 3x - x^5$

(c) $5x^2 - x^{-3}$

(d) $9x^4 - \frac{5}{x}$

(e) $4e^x + 3e^{2x}$

(f) $x^4 + 5x^2 - 3\sqrt{x}$

(g) $4x^5 - 3x^3 + 2x^7 - 4$

(h) 1

2. Expand the brackets for the following expressions:

(a) (i) $(3x - 2)(2x^2 + 4x - 7)$

(ii) $(3x + 1)(x^2 + 5x + 6)$

(b) (i) $(2x + 1)(x^3 - 8x^2 + 6x - 1)$

(ii) $(2x + 5)(x^3 - 6x^2 + 3)$

(c) (i) $(b^2+3b-1)(b^2-2b+4)$

(ii) $(r^2-3r+7)(r^2-8r+2)$

(d) (i) $(5-x^2)(x^4-2x^3+1)$

(ii) $(x-x^3)(x^3-x-1)$

3. Find the remaining factor if

(a) (i) $x^3+3x^2-11x+2$ has a factor of $x-2$

(ii) $x^3+4x^2-3x-18$ has a factor of $x+3$

(b) (i) $6x^3+13x^2+2x-6$ has a factor of $2x+3$

(ii) $25x^3+5x^2-10x-2$ has a factor of $5x+1$

(c) (i) $x^4-5x^3+9x^2-2x-21$ has a factor of $x-3$

(ii) $x^4-5x^3+5x^2+3x-28$ has a factor of $x-4$

(d) (i) $x^4-3x^3+12x^2-15x+35$ has a factor of x^2-3x+7

(ii) x^3-2x^2+3x-6 has a factor of x^2+3

4. Given that the result of each division is a polynomial, simplify each expression.

(a) (i) $\dfrac{x^4-x^3-2x^2+3x-6}{x-2}$ (ii) $\dfrac{x^4+3x^3+2x^2+2x+4}{x+2}$

(b) (i) $\dfrac{2x^4+27x^2+36}{x^2+12}$ (ii) $\dfrac{2x^3-3x^2-27}{2x^2+3x+9}$

5. Find the unknown constants a and b in these identities.

(a) (i) $ax^2+bx=4x^2-6x$ (ii) $ax^2+4=3x^2+4b$

(b) (i) $ax^2+bx=4x+bx^2-ax$ (ii) $ax^2+2bx+6x=0$

(c) (i) $(ax+1)^2+3bx=2ax^2-2x+1$

(ii) $(x+a)^2+b=x^2+4x+9$

(d) (i) $ax^2+bx-2ax=2x-4x^2$

(ii) $ax^2-3bx^2+bx+4x=x^2+7x$

(e) (i) $(ax)^2-(bx)^2+bx=2x$

(ii) $(ax+b)^2=4x^2-20x+25$

6. In what circumstances might you want to expand brackets? In what circumstances is the factorised form better?

7. (a) Is it always true that the sum of a polynomial of order n and a polynomial of order $n-1$ has order n?

(b) Is it always true that the sum of a polynomial of order n and a polynomial of order n has order n?

3B Remainder and factor theorems

We saw in the last section that we can factorise polynomials by comparing coefficients. For example, if we know that $(x+2)$ is one factor of x^3+2x^2+x+2, we can write $x^3+2x^2+x+2=(x+2)(ax^2+bx+c)$ and compare coefficients to find that the other factor is (x^2+1).

If we try to factorise x^3+2x^2+x+5 using $(x+2)$ as one factor, we find that it is not possible; $(x+2)$ is not a factor of x^3+2x^2+x+5. However, using the factorisation of x^3+2x^2+x+2 we can write:

$$x^3+2x^2+x+5=(x+2)(x^2+1)+3$$

The number 3 is the **remainder** – it is what is left over when we try to write x^3+2x^2+x+5 as a multiple of $(x+2)$. In the last section we saw that factorising is related to division. In this case, we could say that:

$$\frac{x^3+2x^2+x+5}{x+2}=(x^2+1) \text{ with remainder } 3$$

This is similar to the concept of a remainder when dividing numbers: for example, $25=3\times 7+4$, so we would say that 4 is the remainder when 25 is divided by 7.

We can find the reminder by including it as another unknown coefficient. For example, to find the reminder when x^3+2x^2+x+5 is divided by $(x+2)$, we could write

$$x^3+2x^2+x+5=(x+2)(ax^2+bx+c)+R$$

then expand and compare coefficients. This is not a quick task. Luckily there is a shortcut which can help us find the reminder without finding all the other coefficients. If we substitute in a value of x that makes the first bracket equal to zero, in this case $x=-2$, into the above equation, it becomes

$$3=(0)(ax^2+bx+c)+R$$

so $R=3$. This means that R is the value we get when we substitute $x=-2$ into the polynomial expression on the left. Fill-in proof sheet 6 on the CD-ROM, 'Remainder theorem', shows you that the same reasoning can be applied when dividing any polynomial by a linear factor. This leads us to the **Remainder theorem**.

EXAM HINT

Notice that $x = \frac{b}{a}$ is the value which makes $ax - b = 0$.

KEY POINT 3.2

The remainder theorem

The remainder when a polynomial expression is divided by $(ax - b)$ is the value of the expression when $x = \frac{b}{a}$.

Worked example 3.4

Find the remainder when $x^3 + 2x + 7$ is divided by $x + 2$.

Use the remainder theorem by rewriting the divisor in the form $(ax - b)$...

$(x+2) = (x-(-2))$

... then substitute the value of x (obtained from $x = \frac{b}{a}$) into the expression when $x = \frac{b}{a}$

When $x = -2: (-2)^3 + 2 \times (-2) + 7 = -5$

So the remainder is -5

If the remainder is zero then $(ax - b)$ is a factor. This is summarised by the **factor theorem**.

KEY POINT 3.3

The factor theorem

If the value of a polynomial expression is zero when $x = \frac{b}{a}$, then $(ax - b)$ is a factor of the expression.

Worked example 3.5

Show that $2x - 3$ is a factor of $2x^3 - 13x^2 + 19x - 6$.

To use the factor theorem we need to substitute in $x = \frac{3}{2}$

When $x = \frac{3}{2}$:

$$2\times\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right)^2 + 19\times\left(\frac{3}{2}\right) - 6$$

$$= \frac{27}{4} - \frac{117}{4} + \frac{57}{2} - 6 = 0$$

continued . . .

Therefore, by the factor theorem, $(2x-3)$ is a factor of $2x^3-13x^2+19x-6$.

We can also use the factor theorem to identify a factor of an expression, by trying several different numbers. Once one factor has been found, then comparing coefficients can be used to find the remaining factors. This is the recommended method for factorising cubic expressions on the non-calculator paper.

Worked example 3.6

Fully factorise $x^3+3x^2-33x-35$.

When factorising a cubic with no obvious factors we must put in some numbers and hope that we can apply the factor theorem

When $x=1$ the expression is -64
When $x=2$ the expression is -81
When $x=-1$ the expression is 0
Therefore $x+1$ is a factor.

We can rewrite the expression as $(x+1)\times$ general quadratic and compare coefficients

$$x^3+3x^2-33x-35$$
$$=(x+1)(ax^2+bx+c)$$
$$=ax^3+(a+b)x^2+(b+c)x+c$$
$a=1, b=2, c=-35$
$$x^3+3x^2-33x-35=(x+1)(x^2+2x-35)$$

The remaining quadratic also factorises

$$=(x+1)(x+7)(x-5)$$

EXAM HINT

If the expression is going to factorise easily then you only need to try numbers which are factors of the constant term.

A very common type of question asks you to find unknown coefficients in an expression if factors or remainders are given.

Worked example 3.7

$x^3 + 4x^2 + ax + b$ has a factor of $(x - 1)$ and leaves a remainder of 17 when divided by $(x - 2)$. Find the constants a and b.

Apply factor theorem

when $x = 1$:

$1 + 4 + a + b = 0$

$\Leftrightarrow a + b = -5$ (1)

Apply remainder theorem

when $x = 2$:

$8 + 16 + 2a + b = 17$

$\Leftrightarrow 2a + b = -7$ (2)

Two equations with two unknowns can be solved simultaneously

(2) − (1)

$a = -2$

$b = -3$

Exercise 3B

1. Use the remainder theorem to find the remainder when:
   (a) (i) $x^2 + 3x + 5$ is divided by $x + 1$
   (ii) $x^2 + x - 4$ is divided by $x + 2$
   (b) (i) $x^3 - 6x^2 + 4x + 8$ is divided by $x - 3$
   (ii) $x^3 - 7x^2 + 11x$ is divided by $x - 1$
   (c) (i) $6x^4 + 7x^3 - 5x^2 + 5x + 10$ is divided by $2x + 3$
   (ii) $12x^4 - 10x^3 + 11x^2 - 5$ is divided by $3x - 1$
   (d) (i) x^3 is divided by $x + 2$
   (ii) $3x^4$ is divided by $x - 1$

2. Decide whether each of the following expressions are factors of $2x^3 - 73 - 3x + 2$.
   (a) x
   (b) $x - 1$
   (c) $x + 1$
   (d) $x - 2$
   (e) $x + 2$
   (f) $x - \frac{1}{2}$
   (g) $x + \frac{1}{2}$
   (h) $2x - 1$
   (i) $2x + 1$
   (j) $3x - 1$

3. Fully factorise the following expressions:

(a) (i) $x^3 + 2x^2 - x - 2$ (ii) $x^3 + x^2 - 4x - 4$

(b) (i) $x^3 - 7x^2 + 16x - 12$ (ii) $x^3 + 6x^2 + 12x + 8$

(c) (i) $x^3 - 3x^2 + 12x - 10$ (ii) $x^3 - 2x^2 + 2x - 15$

(d) (i) $6x^3 - 11x^2 + 6x - 1$ (ii) $12x^3 + 13x^2 - 37x - 30$

4. $6x^3 + ax^2 + bx + 8$ has a factor $(x + 2)$ and leaves a remainder of -3 when divided by $(x - 1)$.
Find a and b. *[5 marks]*

5. $x^3 + 8x^2 + ax + b$ has a factor of $(x - 2)$ and leaves a remainder of 15 when divided by $(x - 3)$.
Find a and b. *[5 marks]*

6. The polynomial $x^2 + kx - 8k$ has a factor $(x - k)$. Find the possible values of k. *[5 marks]*

7. The polynomial $x^2 - (k+1)x - 3$ has a factor $(x - k + 1)$. Find k. *[6 marks]*

8. $x^3 - ax^2 - bx + 168$ has factors $(x - 7)$ and $(x - 3)$.

(a) Find a and b.

(b) Find the remaining factor of the expression. *[6 marks]*

9. $x^3 + ax^2 + 9x + b$ has a factor of $(x - 11)$ and leaves a remainder of -52 when divided by $(x + 2)$.

(a) Find a and b.

(b) Find the remainder when $x^3 + ax^2 + 9x + b$ is divided by $(x - 2)$. *[6 marks]*

10. $f(x) = x^3 + ax^2 + 3x + b$.
The remainder when $f(x)$ is divided by $(x + 1)$ is 6. Find the remainder when $f(x)$ is divided by $(x - 1)$. *[5 marks]*

$f(x)$ is just a name given to the expression. You will learn more about this notation in chapter 5.

11. The polynomial $x^2 - 5x + 6$ is a factor of $2x^3 - 15x^2 + ax + b$. Find the values of a and b. *[6 marks]*

3C Sketching polynomial functions

The word quadratic indicates that the term with the highest power in the equation is x^2. It comes from the Latin *quadratus*, meaning square.

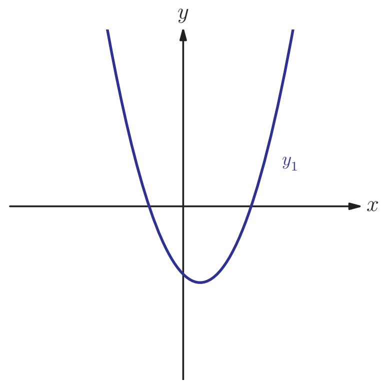
The simplest polynomial with a curved graph is a quadratic.

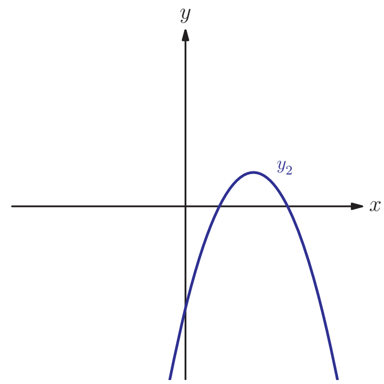
Let us look at two examples of quadratic functions:

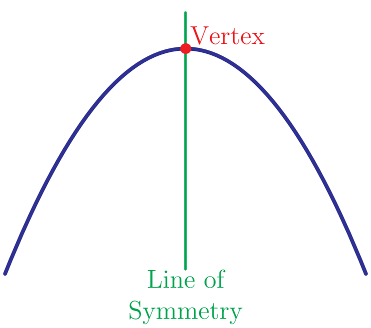
$y_1 = 2x^2 + 2x - 4$ and $y_2 = -x^2 + 4x - 3$ $(x \in \mathbb{R})$

$x \in \mathbb{R}$ means that x can be any number. See Prior learning Section G on the CD-ROM for the meaning of other similar statements.

You can use your calculator to plot the two graphs. See Calculator sheet 2 on the CD-ROM to see how to sketch graphs on your GDC.

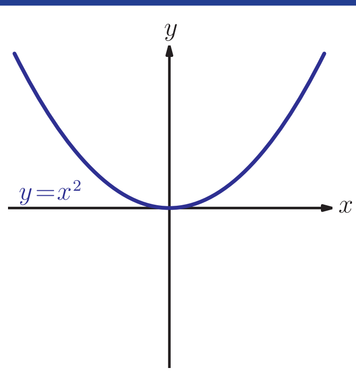
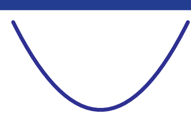
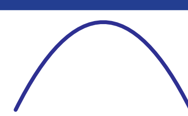


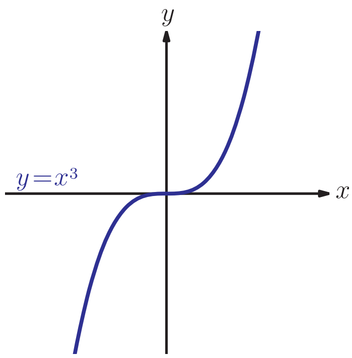
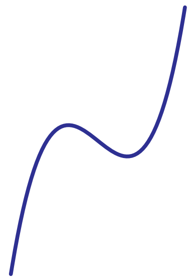
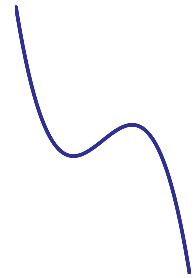
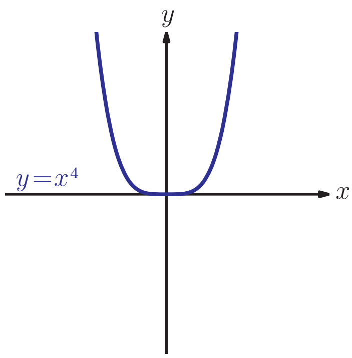
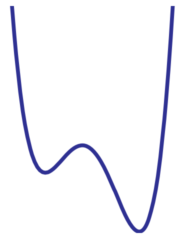
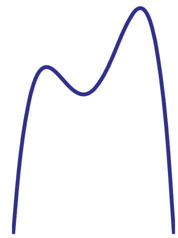




These two graphs have a similar shape, called a **parabola.** A parabola has a single turning point (also called the **vertex**) and a vertical line of symmetry. The most obvious difference is that y_1 has a minimum point, whereas y_2 has a maximum point. This is because of the different signs of the x^2 term.

The graphs of other polynomial functions are also smooth curves. You need to know the shapes to expect for these graphs.

n	$y = x^n$	Positive degree n polynomial	Negative degree n polynomial	Number of times graph can cross or touch the x-axis	Number of turning points
2				0, 1 or 2	1

3				1, 2, or 3	0 or 2
4				0, 1, 2 or 3	1 or 3

The constant term in the polynomial expression gives the position of the y-intercept of the graph (where the curve crosses the y-axis). This is because all other terms contain x, so when $x = 0$ the only non-zero term is the constant term. For example, $2x^3 - 3x + 5 = 5$ when $x = 0$.

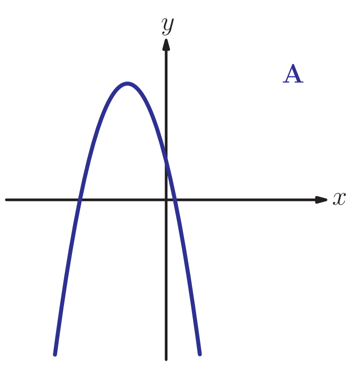
Worked example 3.8

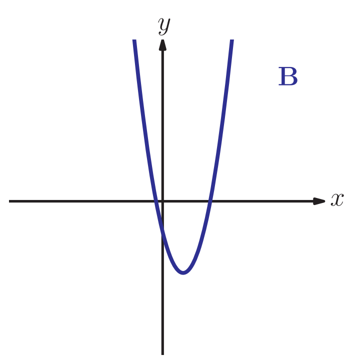
Match each equation to the corresponding graph, explaining your reasons.

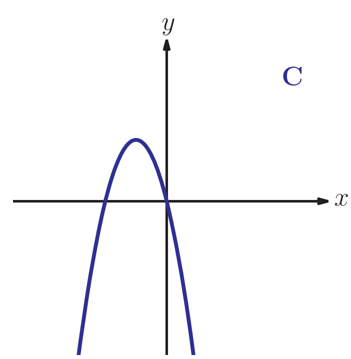
(a) $y = 3x^2 - 4x - 1$

(b) $y = -2x^2 - 4x$

(c) $y = -x^2 - 4x + 2$







continued . . .

Graph B is the only positive quadratic

Graph B shows a positive quadratic, so graph B corresponds to equation (a).

We can distinguish between the other two graphs based on their y-intercept

Graph A has a positive y-intercept, so graph A corresponds to equation (c).
Graph C corresponds to equation (b).

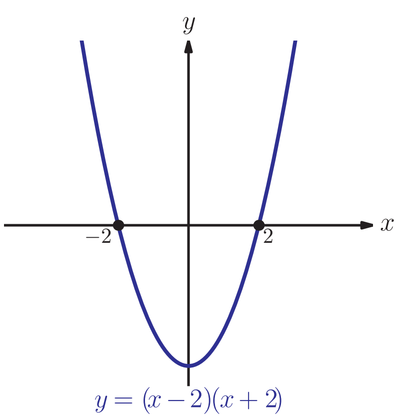
Factorising is covered in Prior learning Section N.

To sketch the graph of a polynomial it is also useful to know its x-intercepts. These are the points where the graph crosses the x-axis, so at those points $y = 0$. For this reason, they are called **zeros** of the polynomial. For example, the quadratic polynomial $x^2 - 5x + 6$ has zeros at $x = 2$ and $x = 3$. They are the **roots** (solutions) of the equation $x^2 - 5x + 6 = 0$ and can be found from the factorised form of the polynomial: $(x-2)(x-3)$.

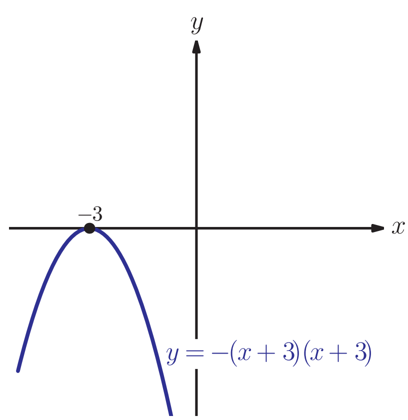
KEY POINT 3.4

The polynomial $a(x-p_1)(x-p_2)(x-p_3)\ldots$ has zeros $p_1, p_2, p_3, \ldots$

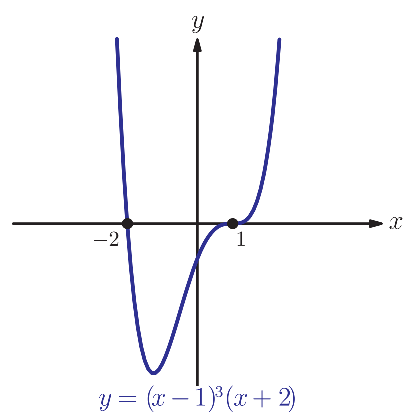
If some of the zeros are the same, we say that the polynomial has a repeated root. For example, the equation $(x-1)^2(x+3)$ has a repeated (double) root $x = 1$ and a single root $x = -3$. Repeated roots tell us how the graph meets the x-axis.



If a polynomial has a factor $(x - a)$ then the curve passes straight through the x-axis at a.



If a polynomial has a double factor $(x - a)^2$ then the curve touches the x-axis at a.



If a polynomial has a triple factor $(x - a)^3$ then the curve passes through the x-axis at a, flattening as it does so.

KEY POINT 3.5

The stages of sketching graphs of polynomials are:

- classify the order of the polynomial and whether it is positive or negative to deduce the basic shape
- set $x = 0$ to find the y-intercept
- write in factorised form
- find x-intercepts
- decide on how the curve meets the x-axis at each intercept
- connect all this information to make a smooth curve.

Worked example 3.9

Sketch the graph of $y = (1 - x)(x - 2)^2$.

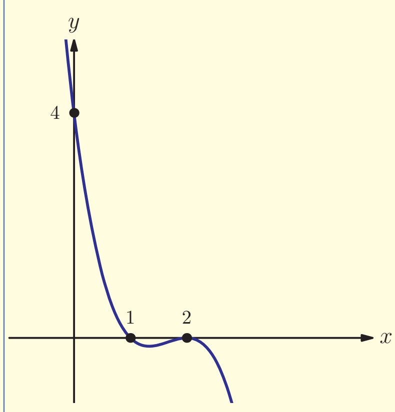
Classify the basic shape — This is a negative cubic

Find y-intercept — When $x = 0$ $\quad y = 1 \times (-2)^2 = 4$

Find x-intercepts — When $y = 0$, $x = 1$ or $x = 2$

Decide on shape of curve at x-intercepts — At $x = 1$ curve passes through the axis. At $x = 2$ curve just touches the axis

Sketch the curve —



EXAM HINT

A sketch does not have to be accurate or to scale. It must have approximately the correct shape and all important points, such as axis intercepts, must be clearly labelled.

Sometimes we need to deduce possible equations from a given curve.

We will examine the Fundamental theorem of algebra in more detail in Section 15D.

The first thing we should ask is what order polynomial we need to use. There is a result called the Fundamental Theorem of Algebra, which states that a polynomial of order n can have at most n real roots. So, for example, if the given curve has three x-intercepts, we know that the corresponding polynomial must have degree at least 3. We can then use those intercepts to write down the factors of the polynomial.

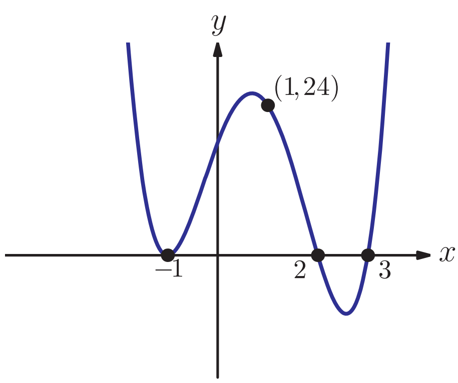
KEY POINT 3.6

To find the equation of a polynomial from its graph:

- use the shape and position of the x-intercepts to write down the factors of the polynomial
- use any other point to find the lead coefficient.

Worked example 3.10

Find a possible equation for this graph.



Describe x-intercepts

Single root at $x = 2$ and $x = 3$
Double root at $x = -1$

Convert this to a factorised form

$$\therefore y = k(x+1)^2(x-2)(x-3)$$

Use the fact that when $x = 1$, $y = 24$

$$24 = k \times (2)^2 \times (-1) \times (-2)$$
$$\Leftrightarrow 24 = 8k$$
$$\Leftrightarrow k = 3$$

So the equation is $y = 3(x+1)^2(x-2)(x-3)$

Exercise 3C

1. Sketch the following graphs, labelling all axis intercepts.

(a) (i) $y = 2(x-2)(x-3)$

(ii) $y = 7(x-5)(x+1)$

(b) (i) $y = 4(5-x)(x-3)(x-3)$

(ii) $y = 2(x-1)(2-x)(x-3)$

(c) (i) $y = -(x-4)^2$ (ii) $y = (x-2)^2$

(d) (i) $y = x(x^2+4)$ (ii) $y = (x+1)(x^2-3x+7)$

(e) (i) $y = (1-x)^2(1+x)$ (ii) $y = (2-x)(3-x)^2$

2. Sketch the following graphs, labelling all axis intercepts.

(a) (i) $y = x(x-1)(x-2)(2x-3)$

(ii) $y = (x+2)(x+3)(x-2)(x-3)$

(b) (i) $y = -4(x-3)(x-2)(x+1)(x+3)$

(ii) $y = -5x(x+2)(x-3)(x-4)$

(c) (i) $y = (x-3)^2(x-2)(x-4)$

(ii) $y = -x^2(x-1)(x+2)$

(d) (i) $y = 2(x+1)^3(x-3)$

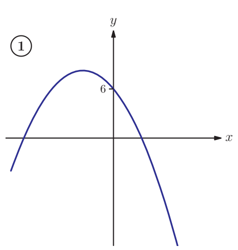
(ii) $y = -x^3(x-4)$

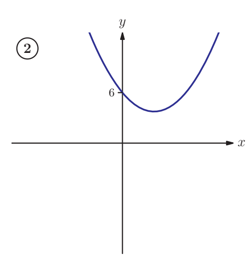
(e) (i) $y = (x^2+3x+12)(x+1)(3x-1)$

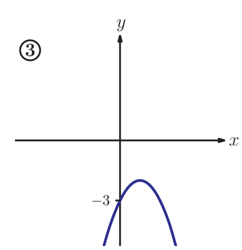
(ii) $y = (x+2)^2(x^2+4)$

3. Match equations and corresponding graphs.

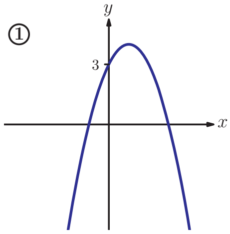
(i) A: $y = -x^2-3x+6$ B: $y = 2x^2-3x+3$ C: $y = x^2-3x+6$

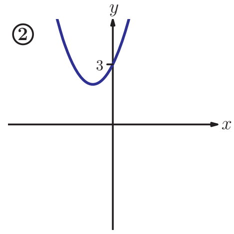


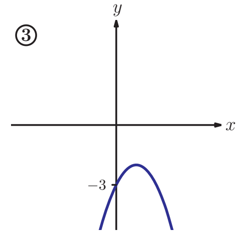




(ii) A: $y = -x^2 + 2x - 3$ B: $y = -x^2 + 2x + 3$ C: $y = x^2 + 2x + 3$

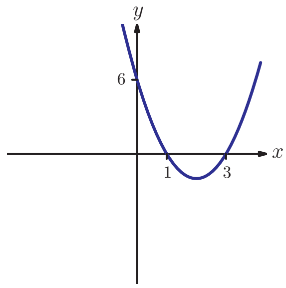




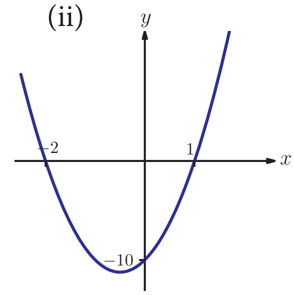


4. The diagrams show graphs of quadratic functions of the form $y = ax^2 + bx + c$. Write down the value of c, and then find the values of a and b.

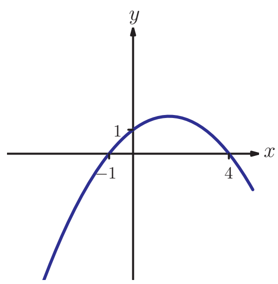
(a) (i)



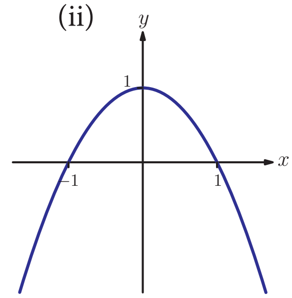
(ii)



(b) (i)

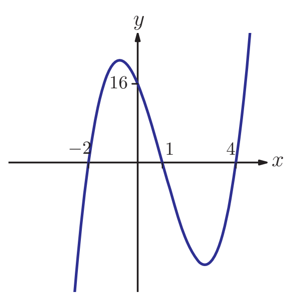


(ii)

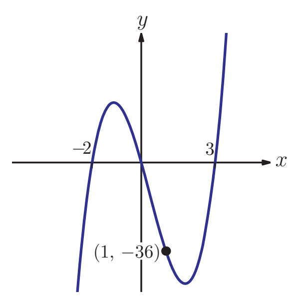


5. Find lowest order polynomial equation for each of these graphs.

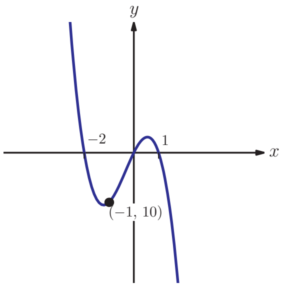
(a) (i)



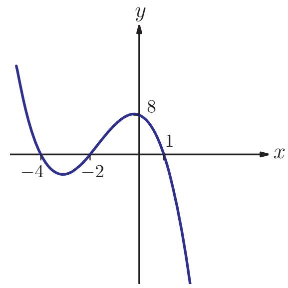
(ii)



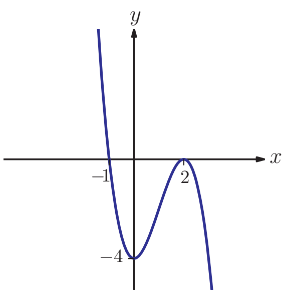
(b) (i)



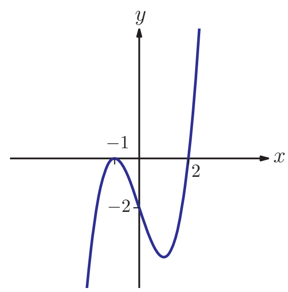
(ii)



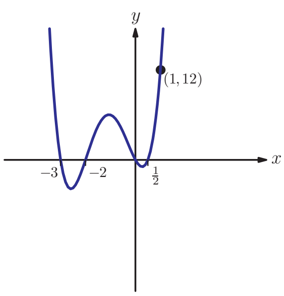
(c) (i)



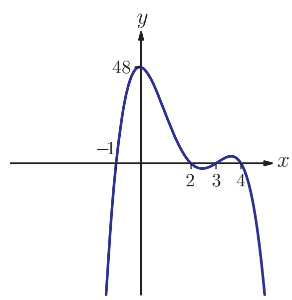
(ii)



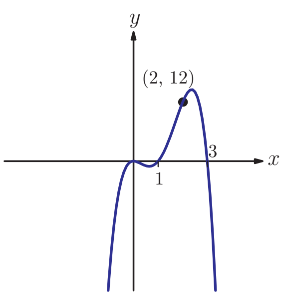
(d) (i)



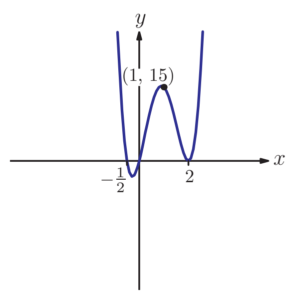
(ii)



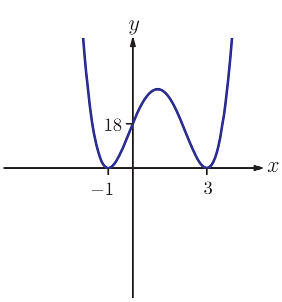
(e) (i)



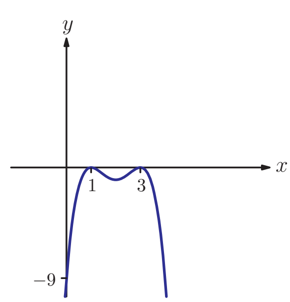
(ii)



(f) (i)



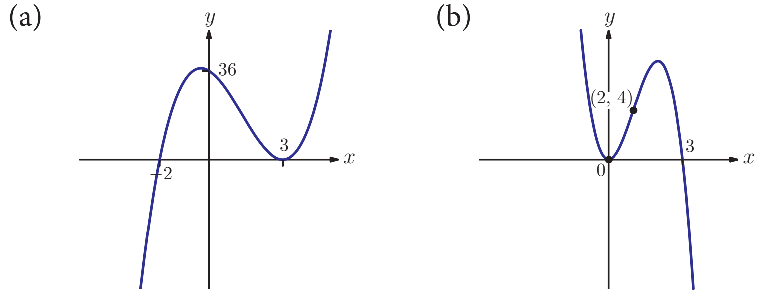
(ii)



6. (a) Show that $(x-2)$ is a factor of $f(x) = 2x^3 - 5x^2 + x + 2$.
   (b) Factorise $f(x)$.
   (c) Sketch the graph $y = f(x)$.

7. Sketch the graph of $y = 2(x+2)^2(3-x)$, labelling clearly any axes intercepts. *[5 marks]*

8. The two graphs below each have equations of the form $y = px^3 + qx^2 + rx + s$. Find the values of p, q, r and s for each graph.



[10 marks]

9. (a) Factorise fully $x^4 - q^4$ where q is a positive constant.
   (b) Hence or otherwise sketch the graph $y = x^4 - q^4$, labelling any points where the graph meets an axis. *[5 marks]*

10. (a) Sketch the graph of $y = (x-p)^2(x-q)$ where $p < q$.
   (b) How many solutions does the equation $(x-p)^2(x-q) = k$ have when $k > 0$? *[5 marks]*

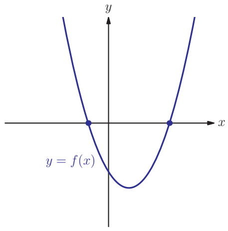
EXAM HINT

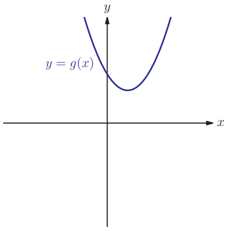
Don't spend too long trying to factorise a quadratic – use the formula in Key point 3.7 if you are asked to find exact solutions and use a calculator (graph or equation solver) otherwise.

See Calculator sheets 4 and 6 on the CD-ROM.

3D The quadratic formula and the discriminant

It is not always possible to find zeros of a polynomial using factorising. Try factorising the two quadratic expressions $f(x) = x^2 - 3x - 3$ and $g(x) = x^2 - 3x + 3$. It appears that neither of the expressions can be factorised, but sketching the graphs reveals that y_1 has two zeros, while y_2 has no zeros.





In the case where the polynomial is a quadratic we have another option – using the quadratic formula to find the zeros (roots):

KEY POINT 3.7

The zeros of $ax^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can find the proof of this formula in Fill-in proof 3 'Proving quadratic formula'.

The solution to quadratic equations was known to the Greeks, although they did everything without algebra, using geometry instead. They would have described the quadratic equation $x^2 + 3x = 10$ as 'the area of a square plus three times its length measures ten units'.

Worked example 3.11

Use the quadratic formula to find the zeros of $x^2 - 5x - 3$.

It is not obvious how to factorise the quadratic expression, so use the quadratic formula

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times (-3)}}{2}$$

$$= \frac{5 \pm \sqrt{37}}{2}$$

The roots are

$$\frac{5 + \sqrt{37}}{2} = 5.54 \text{ and } \frac{5 - \sqrt{37}}{2} = -0.541 \text{ (3SF)}$$

EXAM HINT

In International Baccalaureate® exams you should either give exact answers (such as $\frac{5 + \sqrt{37}}{2}$) or round your answers to 3 significant figures, unless you are told otherwise. See Prior learning Section B on the CD-ROM for rules of rounding.

Let us examine what happens if we try to apply the quadratic formula to find the zeros of $x^2 - 3x + 3$:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 3}}{2} = \frac{3 \pm \sqrt{-3}}{2}$$

In chapter 15 you will meet imaginary numbers, a new type of number which makes it possible to find zeros of functions like this.

As the square root of a negative number is not a real number, it follows that the expression has no real zeros.

Looking more closely at the quadratic formula, we see that it can be separated into two parts:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The line of symmetry of the parabola lies halfway between the two roots:

KEY POINT 3.8

The line of symmetry of $y = ax^2 + bx + c$ is

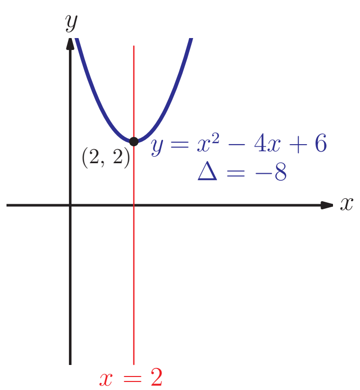
$$x = -\frac{b}{2a}$$

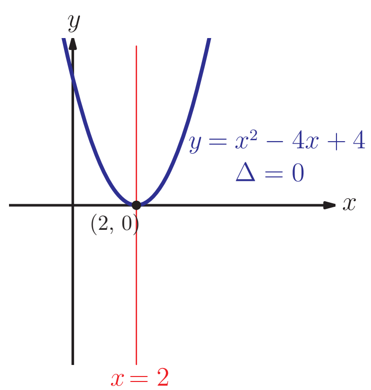
The second part of the formula involves a root expression: $\sqrt{b^2 - 4ac}$.

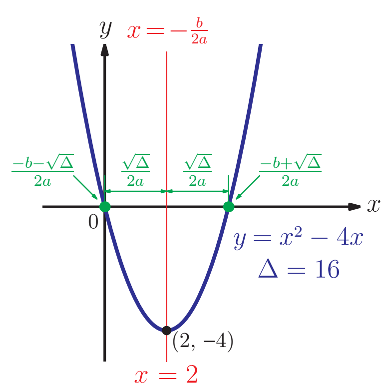
The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic (often symbolised by the Greek letter Δ) and $\frac{\sqrt{b^2-4ac}}{2a}$ is the distance of the zeros from the line of symmetry $x = -\frac{b}{2a}$.

The square root of a negative number is not a real number, so if the discriminant is negative, there can be no real zeros and the graph will not cross the x-axis. If the discriminant is zero, the graph is tangent to the x-axis at a point which lies on the line of symmetry (it touches the x-axis rather than crossing it).

The graphs below are examples of the three possible situations:







KEY POINT 3.9

For a quadratic expression $ax^2 + bx + c$, the discriminant is: $\Delta = b^2 - 4ac$.

- If $\Delta < 0$ the expression has no real zeros
- If $\Delta = 0$ the expression has one (repeated) zero
- If $\Delta > 0$ the expression has two distinct real zeros

There is also a formula for solving cubic equations called 'Cardano's Formula'. It too has a discriminant which can be used to decide how many solutions there will be:

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

This gets a lot more complicated than the quadratic version!

Worked example 3.12

Find the exact values of k for which the quadratic equation $kx^2 - (k+2)x + 3 = 0$ has a repeated root.

Repeated root means that $\Delta = b^2 - 4ac = 0$

$$b^2 - 4ac = 0$$

We can form an equation in k using $a = k$ and $b = -(k + 2)$

$$(k+2)^2 - 4(k)(3) = 0$$

This is a quadratic equation

$$k^2 + 4k + 4 - 12k = 0$$
$$k^2 - 8k + 4 = 0$$

It doesn't appear to factorise, so use the quadratic formula to find k

$$k = \frac{8 \pm \sqrt{8^2 - 4 \times 4}}{2}$$
$$= \frac{8 \pm \sqrt{48}}{2}$$
$$= \frac{(8 \pm 4\sqrt{3})}{2}$$
$$= 4 \pm 2\sqrt{3}$$

When $\Delta < 0$, the graph does not intersect the x-axis, so it is either entirely above or entirely below it. The two cases are distinguished by the value of a.

Questions involving discriminants often lead to quadratic inequalities, which are covered in Prior learning Section Z.

KEY POINT 3.10

For a quadratic function with $\Delta < 0$:

if $a > 0$ then $y > 0$ for all x

if $a < 0$ then $y < 0$ for all x

Worked example 3.13

Let $y = -3x^2 + kx - 12$.
Find the values of k for which $y < 0$ for all x.

y is a negative quadratic. $y < 0$ means that the graph is entirely below the x-axis. This will happen when $y = 0$ has no real roots

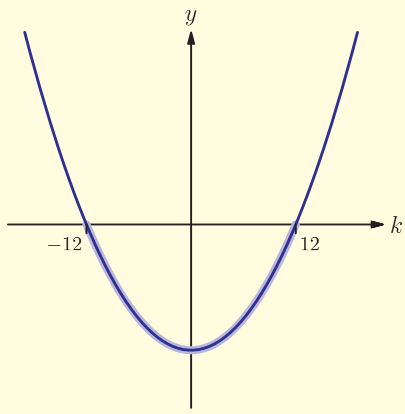
No real roots, $\therefore \Delta < 0$

$k^2 - 4(-3)(-12) < 0$

$k^2 < 144$

This is a quadratic inequality. A sketch of the graph will help

$\Rightarrow k^2 - 144 < 0$



$\therefore -12 < k < 12$

Exercise 3D

1. Evaluate the discriminant of these quadratic expressions.

(a) (i) $x^2 + 4x - 5$ (ii) $x^2 - 6x - 8$

(b) (i) $2x^2 + x + 6$ (ii) $3x^2 - x + 10$

(c) (i) $3x^2 - 6x + 3$ (ii) $9x^2 - 6x + 1$

(d) (i) $12 - x - x^2$ (ii) $-x^2 - 3x + 10$

2. State the number of zeros for each expression in Question 1.

3. Find the exact solutions of these equations.

(a) (i) $x^2 - 3x + 1 = 0$ (ii) $x^2 - x - 1 = 0$

(b) (i) $3x^2 + x - 2 = 0$ (ii) $2x^2 - 6x + 1 = 0$

(c) (i) $4 + x - 3x^2 = 0$ (ii) $1 - x - 2x^2 = 0$

(d) (i) $x^2 - 3 = 4x$ (ii) $3 - x = 2x^2$

Some of the many applications of quadratic equations are explored in Supplementary sheet 1 'The many applications of quadratic equations'.

4. Find the values of k for which:

(a) (i) the equation $x^2 - x + k =$ has two distinct real roots

(ii) the equation $3x^2 - 5x + k = 0$ has two distinct real roots

(b) (i) the equation $5x^2 - 2x + (2k - 1) = 0$ has equal roots

(ii) the equation $2x^2 + 3x - (3k + 1) = 0$ has equal roots

(c) (i) the equation $-x^2 + 3x + (k - 1) = 0$ has real roots

(ii) the equation $-2x^2 + 3x - (2k + 1) = 0$ has real roots

(d) (i) the equation $3kx^2 - 3x + 2 = 0$ has no real solutions

(ii) the equation $kx^2 + 5x + 3 = 0$ has no real solutions

(e) (i) the quadratic expression $(k - 2)x^2 + 3x + 1$ has a repeated zero

(ii) the quadratic expression $-4x^2 + 5x + (2k - 5)$ has a repeated zero

(f) (i) the graph of $y = x^2 - 4x + (3k + 1)$ is tangent to the x-axis

(ii) the graph of $y = -2kx^2 + x - 4$ is tangent to the x-axis

(g) (i) the expression $-3x^2 + 5k$ has no real zeros

(ii) the expression $2kx^2 - 3$ has no real zeros

5. Find the values of parameter m for which the quadratic equation $mx^2 - 4x + 2m = 0$ has equal roots. *[5 marks]*

6. Find the exact values of k such that the equation $-3x^2 + (2k + 1)x - 4k = 0$ has a repeated root. *[6 marks]*

7. Find the range of values of the parameter c such that $2x^2 - 3x + (2c - 1) \geq 0$ for all x. *[6 marks]*

8. Find the set of values of k for which the equation $x^2 - 2kx + 6k = 0$ has no real solutions. *[6 marks]*

9. Find the range of value of k for which the quadratic equation $kx^2 - (k + 3)x - 1 = 0$ has no real roots. *[6 marks]*

10. Find the range of values of m for which the equation $mx^2 + mx - 2 = 0$ has one or two real roots. *[6 marks]*

11. Find the possible values of m such that $mx^2 + 3x - 4 < 0$ for all x. *[6 marks]*

12. The positive difference between the zeros of the quadratic expression $x^2 + kx + 3$ is $\sqrt{69}$. Find the possible values of k. *[5 marks]*

Summary

- **Polynomials** are expressions involving only addition, multiplication and raising to a power.
- If two polynomials are equal we can **compare coefficients**. This can be used to divide two polynomials.
- The **remainder theorem** says that the remainder when a polynomial is divided by $ax - b$ is the value of the polynomial expression when $x = \frac{b}{a}$.
- The **factor theorem** says that if a polynomial expression is zero when $x = \frac{b}{a}$ then $ax - b$ is a factor of the expression.
- The graphs of polynomial are best sketched using their factorised form. Look for repeated factors and find where the graph crosses the axes.
- You can use the shape and position of the x-intercepts to find the equation of a polynomial graph.
- The number of solutions of a quadratic equation depends on the value of the **discriminant**, $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$ there are two distinct solutions.
  - If $\Delta = 0$ there is one solution.
  - If $\Delta < 0$ there is no solution.

Introductory problem revisited

Without using your calculator, solve the cubic equation

$$x^3 - 13x^2 + 47x - 35 = 0$$

If we put $x = 1$ into this expression we get 0, so we can conclude that $x - 1$ is a factor. The remaining quadratic factor can be found by comparing coefficients; it is $x^2 - 12x + 35$, which factorises to give $(x - 5)(x - 7)$.

The equation therefore becomes $(x - 1)(x - 5)(x - 7) = 0$ which has solutions $x = 1, 5$, or 7.

Mixed examination practice 3

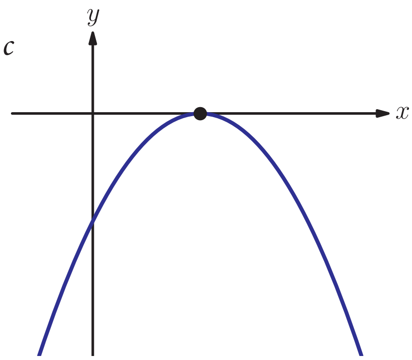
Short questions

1. A quadratic graph passes through the points $(k, 0)$ and $(k + 4, 0)$. Find in terms of k the x-coordinates of the turning point. *[4 marks]*

2. The diagram shows the graph of $y = ax^2 + bx + c$

Complete the table to show whether each expression is positive, negative or zero.

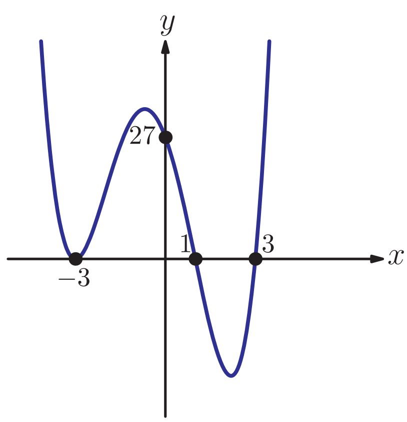
expression	positive	negative	zero
a			
c			
$b^2 - 4ac$			
b			



[6 marks]

3. The diagram shows the graph with equation $y = ax^4 + bx^3 + cx^2 + dx + e$.

Find the values of a, b, c, d and e. *[6 marks]*



4. The remainder when $(ax + b)^3$ is divided by $(x - 2)$ is 8 and the remainder when it is divided by $(x + 3)$ is -27. Find the values of a and b. *[5 marks]*

5. (a) Show that $(x - 2)$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$.

(b) Factorise $f(x)$.

(c) Sketch the graph of $y = f(x)$. *[7 marks]*

6. The remainder when $(ax + b)^4$ is divided by $(x - 2)$ is 16 and the remainder when it is divided by $(x + 1)$ is 81. Find the possible values of a and b. *[6 marks]*

7. Sketch the graph of $y = (x-a)^2(x-b)(x-c)$ where $b < 0 < a < c$. *[5 marks]*

8. Find the exact values of k for which the equation $2kx^2 + (k+1)x + 1 = 0$ has equal roots. *[5 marks]*

9. Find the set of values of k for which the equation $2x^2 + kx + 6 = 0$ has no real roots. *[6 marks]*

10. Find the range of values of k for which the quadratic function $x^2 - (2k+1)x + 5$ has at least one real zero. *[6 marks]*

11. The polynomial $x^2 - 4x + 3$ is a factor of the polynomial $x^3 + ax^2 + 27x + b$. Find the values of a and b. *[6 marks]*

12. Let α and β denote the roots of the quadratic equation

$x^2 - kx + (k-1) = 0$.

(a) Express α and β in terms of the real parameter k.

(b) Given that $\alpha^2 + \beta^2 = 17$, find the possible values of k. *[7 marks]*

13. Let $q(x) = kx^2 + (k-2)x - 2$. Show that the equation $q(x) = 0$ has real roots for all values of k. *[7 marks]*

14. Find the range of values of k such that for all x, $kx - 2 \le x^2$. *[7 marks]*

Long questions

1. (a) Find the coordinates of the point where the curve $y = x^2 + bx - a$ crosses the y-axis, giving your answer in terms of a and/or b.

(b) State the equation of the axis of symmetry of $x^2 + bx - a$, giving your answer in terms of a and/or b.

(c) Show that the remainder when $x^2 + bx - a$ is divided by $x - \frac{a}{b}$ is always positive.

(d) The remainder when $x^2 + bx - a$ is divided by $x - a$ is -9. Find the possible values that b can take.

[14 marks]

2. (a) Show that for all values of p, $(x-2)$ is a factor of

$$f(x)=x^3+(p-2)x^2+(5-2p)x-10.$$

(b) By factorising $f(x)$, or otherwise, find the exact values of p for which the equation

$$x^3+(p-2)x^2+(5-2p)x-10=0$$

has exactly two real roots.

(c) For the smaller of the two values of p found above, sketch the graph of $y=f(x)$.

[10 marks]

3. (a) On the graph of $y=\dfrac{x^2+4x+5}{x+2}$ prove that there is no value of x for which $y=0$.

(b) Find the equation of the vertical asymptote of the graph.

(c) Rearrange the equation to find x in terms of y.

(d) Hence show that y cannot take values between -2 and 2.

(e) Sketch the graph of $y=\dfrac{x^2+4x+5}{x+2}$.

[18 marks]

4. Let $f(x)=x^4+x^3+x^2+x+1$.

(a) Evaluate $f(1)$.

(b) Show that $(x-1)f(x)\equiv x^5-1$.

(c) Sketch $y=x^5-1$.

(d) Hence show that $f(x)$ has no real roots.

[10 marks]

4 Algebraic structures

In this chapter you will learn:

- some standard algebraic strategies for solving equations
- how to sketch graphs, and some limitations of graphical calculators
- how to use a graphical calculator to solve equations
- how to solve systems of up to three linear simultaneous equations
- algebraic and graphical strategies for solving inequalities
- how to prove and work with identities.

Introductory problem

What information can be extracted from the equation $ax + b = 3x + 2$?

What information can be extracted from the identity $ax + b = 3x + 2$, for all x?

Equations are the building blocks of mathematics. There are many different types; some have no solutions, some have many solutions. Some have solutions which cannot be expressed in terms of any function you have met.

Graphs are an alternative way of expressing a relationship between two **variables**. If you understand the connection between graphs and equations, and can switch between the two representations, you will have a wider variety of tools to solve mathematical problems. The International Baccalaureate® places great emphasis on using graphical calculators to analyse graphs.

Inequalities share many of the same properties as equations. However, a few important differences mean you need different techniques to solve some inequalities.

In much of mathematics we do not distinguish between equations and **identities**, although they are fundamentally different. In this chapter we shall explore some of these differences and look at the different techniques we can apply.

4A Solving equations by factorising

We start by looking at some common algebraic methods for solving equations. You have already used factorising to sketch polynomials, and in this chapter we will apply the same principles in a wider context.

This is true for real numbers, but if you know that the numbers you are looking for are integers, then knowing the product of two numbers is far more useful.

If two numbers multiply together to give 5, what can you say about those two numbers?

They could be 1 and 5, 10 and $\frac{1}{2}$, π and $\frac{5}{\pi}$; there are actually an infinite number of possibilities. So if you know that two numbers multiply together to give 5 it does not help to find what those numbers are. However, if two numbers multiply together to give zero, this is only possible if one of them is zero.

If we can factorise an expression which is equal to zero then this is a very powerful tool for solving equations.

Worked example 4.1

Solve the equation $e^x(\ln(x)-1)(2x-1)=0$.

Set the three factors equal to 0 then solve each separately	Either $e^x=0$ (1) or $\ln(x)-1=0$ (2) or $2x-1=0$ (3)
From (1)	$e^x=0$ has no solution
From (2)	$\ln(x)=1$ $\Leftrightarrow x=e$
From (3)	$2x=1$ $\Leftrightarrow x=\frac{1}{2}$
List the solutions	$x=e$ or $\frac{1}{2}$

EXAM HINT

If you divide by a function instead of looking at factors then it is possible that you might lose solutions. For example, consider the equation $x^3 = x$. If you divide both sides by x you get $x^2 = 1$ so $x = \pm 1$. However, if you put $x = 0$ into the original equation you will see that this is also a solution. The correct method for solving this equation is by factorisation.

$$x^3 = x$$
$$\Rightarrow x^3 - x = 0$$
$$\Rightarrow x(x^2-1) = 0$$
$$\Rightarrow x(x-1)(x+1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1$$

Exercise 4A

1. Solve the following equations:

   (a) (i) $3(x-3)^3 = 0$ (ii) $-4(x+1)^5 = 0$

   (b) (i) $7(2x-1)(5x+3)^2 = 0$ (ii) $5(3-x)^2(2x+6)^2 = 0$

   (c) (i) $(\log_3 x - 3)(3^x - 3) = 0$ (ii) $\left(\sqrt{x}-4\right)\left(9\sqrt{x}-1\right) = 0$

   (d) (i) $x(x^2-3) = 7(x^2-3)$ (ii) $5x(x^2-5x+4) = 6(x^2-5x+4)$

   (e) (i) $6^x - 4\times 3^x = 0$ (ii) $2\times 5^x - 7\times 10^x = 0$

2. Solve the following equations:

   (a) (i) $x^3 + 12 = 2x^2 + 11x$

   (ii) $x^3 - x^2 - 17x = 15$

   (b) (i) $x^3 - 5x^2 + 7x - 2 = 0$

   (ii) $x^3 - 6x^2 + 7x - 2 = 0$

You saw how to use the factor theorem to factorise polynomials in Section 3B.

3. (a) Find the roots of these equations:

   (i) $x^3 - 6x^2 + 11x = 6$ (ii) $x^3 - 2x^2 + 6 = 5x$

   (b) Find the zeros of these polynomials:

   (i) $x^3 + x^2 - x - 1$ (ii) $x^3 - 3x^2 - 10x + 24$

4. Solve $(3x-1)^{x^2-4} = 1$. *[4 marks]*

5. Solve the equation $x|x| = 4x$.

4B Solving equations by substitution

There are certain types of equation which you should know how to solve; in this section we shall focus particularly on quadratic equations as we have a formula to solve them.

We shall also see how a substitution can turn some complicated-looking equations into quadratic equations.

Worked example 4.2

Solve the equation $x^4 - 4 = 3x^2$.

A substitution $y = x^2$ turns this into a quadratic equation since $x^4 = y^2$

If $y = x^2$ the equation becomes
$y^2 - 4 = 3y$

continued . . .

This is a standard quadratic equation	$y^2 - 3y - 4 = 0$ $\Rightarrow (y+1)(y-4) = 0$ $\Rightarrow y = -1$ or $y = 4$
Use the substitution to find x	Therefore $x^2 = 4$ or $x^2 = -1$ (reject) $\Rightarrow x^2 = 4$ $x = 2$ or $x = -2$

Other substitutions may not be so obvious.

In particular, it is quite common to be given an exponential equation which needs a substitution. Look out for an a^x and an a^{2x} or an $(a^2)^x$ term, both of which can be rewritten as $(a^x)^2$.

Worked example 4.3

Solve the equation $2(4^x + 2) = 9 \times 2^x$.

A substitution $y = 2^x$ turns this into a quadratic equation since $4^x = 2^{2x} = y^2$	If $y = 2^x$ the equation becomes $2(y^2 + 2) = 9y$
This is a standard quadratic equation	$2y^2 - 9y + 4 = 0$ $\Rightarrow (2y-1)(y-4) = 0$ $\Rightarrow y = \frac{1}{2}$ or $y = 4$
Use the substitution to find x	Therefore $2^x = \frac{1}{2}$ or $2^x = 4$ $\Rightarrow x = -1$ or $x = 2$

Exercise 4B

1. Solve the following equations, giving your answers in an exact form.

(a) (i) $a^4 - 10a^2 + 21 = 0$ (ii) $x^4 - 7x^2 + 12 = 0$

(b) (i) $2x^6 + 7x^3 = 15$ (ii) $a^6 + 7a^3 = 8$

(c) (i) $x^2 - 4 = \frac{2}{x^2}$ (ii) $x^2 + \frac{36}{x^2} = 12$

(d) (i) $x - 6\sqrt{x} + 8 = 0$ (ii) $x - 10\sqrt{x} + 24 = 0$

(e) (i) $e^{2x} + 16e^{x} = 80$ (ii) $e^{2x} - 9e^{x} + 20 = 0$

(f) (i) $25^{x} - 15 \times 5^{x} + 50 = 0$ (ii) $4^{x} - 7 \times 2^{x} + 12 = 0$

(g) (i) $\log_4 x = (\log_4 x^2)^2$ (ii) $(\log_3 x)^2 - 3\log_3 x + 2 = 0$

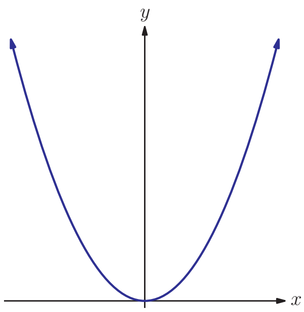
2. Solve the equation $9(1 + 9^{x-1}) = 10 \times 3^{x}$. *[5 marks]*

3. If $a^{x} = 6 - \frac{5}{a^{x}}$, where $a > 0$, solve for x, leaving your answer in terms of a. *[5 marks]*

4. Solve the equation $\log_2 x = 6 - 5\log_x 2$. *[6 marks]*

4C Features of graphs

You will meet many different types of equations in this course, and will learn various techniques for solving them. In Chapters 2 and 3 you have already met polynomial, exponential and logarithmic equations. However, you also have to be able to find approximate solutions to equations which are not from among the standard types. This can be done using graphs.



Graphs are simply another way of representing a relationship between two variables. For example, we can write $y = x^2$ or draw the graph alongside.

In this section we shall examine important features you should be looking for when sketching graphs using your graphical calculator, and also discuss some limitations of this method.

The main features you should indicate on your sketch are:

- y-intercept; this is where $x = 0$
- x-intercepts (zeros); these are where $y = 0$
- maximum and minimum points.

The x- and y-intercepts were discussed in more detail in chapter 3.

See Calculator skills sheets 2 and 4 on the CD-ROM about sketching graphs and finding the main features.

If the graph you are sketching is completely unfamiliar, it can be difficult to choose a good viewing window. If the window is too small you may be missing important parts of the graph, and if it is too large you may not be able to distinguish between features that are close together. Having an idea about the overall shape of the graph is very useful, so it is important to learn about different types of graphs, even if you have a graphical calculator available.

Worked example 4.4

Sketch the graph of $y = \dfrac{x^3 - 16x}{x^2 + 1}$.

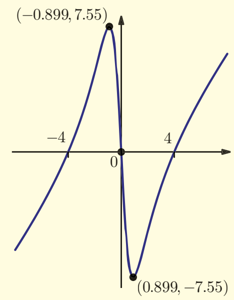
Find the important features

We know that the zeros will occur when the numerator is zero, and we can find those points by factorising

$$x^3 - 16x = 0$$
$$\Leftrightarrow x(x^2 - 16) = 0$$
$$\Leftrightarrow x(x-4)(x+4) = 0$$
$$\Leftrightarrow x = 0, 4, -4$$

The viewing window should include 4 and –4

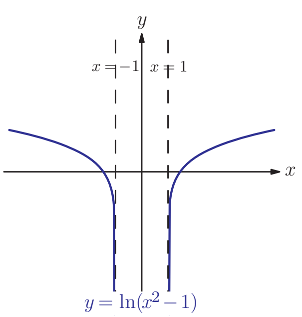
There appear to be one maximum and one minimum point; we should find their coordinates

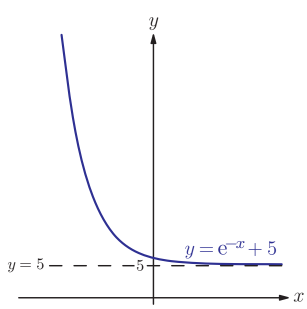


Some graphs might have asymptotes.

KEY POINT 4.1

> An **asymptote** is a straight line which the graph approaches as either x or y gets very large.





Asymptotes are usually shown on graphs by dotted lines.

Vertical asymptotes occur where a function ceases to be defined.

They are vertical lines of the form $x = a$.

For example, $y = \dfrac{1}{x-3}$ has an asymptote $x = 3$ (because we cannot divide by zero), and $y = \ln(x^2 - 1)$ has asymptotes $x = -1$ and $x = 1$ (because we cannot take a logarithm of zero).

Horizontal asymptotes indicate long term behaviour of the graph and are lines that the graph approaches for large values of x (positive or negative). They are horizontal lines of the form $y = a$. For example, $y = e^{-x} + 5$ has an asymptote $y = 5$.

Since the asymptotes are not a part of the graph, they will not show on your calculator sketch. Using the cursor allows you to investigate what happens for large values of x or y, so you can find the approximate positions of the asymptotes. To find their exact equations you need to apply some knowledge of the function in question; for example, knowing that you cannot divide by zero, or that e^{-x} approaches zero for large x. You will only be asked to identify the exact position of asymptotes for familiar functions.

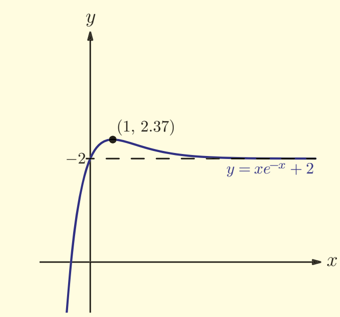
Worked example 4.5

Sketch the graph of $y = xe^{-x} + 2$.

Start with a standard window with both x and y between −10 and 10

It looks like there is something interesting happening for x values around 1, so zoom in a bit to confirm that there is a maximum point

There also seems to be an asymptote; we know that e^{-x} approaches 0, so it is likely that the asymptote is $y = 2$



Note that although the line $y = 2$ is an asymptote, the graph actually crosses it when $x = 0$. This is fine, as the asymptote is only relevant for large values of x.

See Prior learning Section I on the CD-ROM for an introduction to the modulus function $|x|$. Make sure you know where to find it on your calculator.

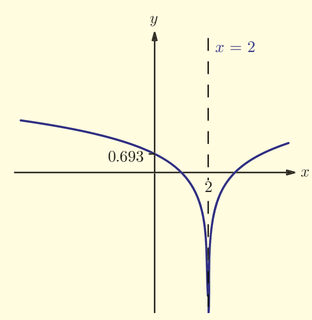
Vertical asymptotes can sometimes be unclear. In the following example the graph approaches the vertical asymptote from two sides, so the calculator may attempt to 'connect the points' and form a 'V' shape. You must identify for yourself that this is an asymptote and draw it correctly.

Worked example 4.6

Sketch the graph of $y = \ln|x - 2|$.

It looks like the two branches of the graph join at a point. However, we know that ln 0 is not defined, so there should be an asymptote at $x = 2$

Asymptote: $x - 2 = 0 \Leftrightarrow x = 2$



Exercise 4C

1. Sketch the following graphs, indicating all the important features: axis intercepts, vertical and horizontal asymptotes, maximum and minimum points.

(a) (i) $y = x^4 - x^3 + 1$ (ii) $y = x^4 - x^2$

(b) (i) $y = (x-1)e^x$ (ii) $y = (e^x - 1)^2$

(c) (i) $y = \dfrac{\frac{1}{2}e^x - 1}{x-1}$ (ii) $y = \ln\left(\dfrac{x+2}{x-1}\right)^2$

(d) (i) $\left|\dfrac{x^2-1}{x+2}\right|$ (ii) $y = \dfrac{|x^2-4|}{x+1}$

For questions 2 to 4, mark the coordinates of all zeros, maximum and minimum points.

2. Sketch the graph of $y = x \ln x$. *[4 marks]*

3. Sketch the graph of $y = \dfrac{e^x}{\ln x}$. *[6 marks]*

4. Sketch the graph of $y = \dfrac{x^2(x^2-9)}{e^x}$ for $-5 \le x < 10$. *[6 marks]*

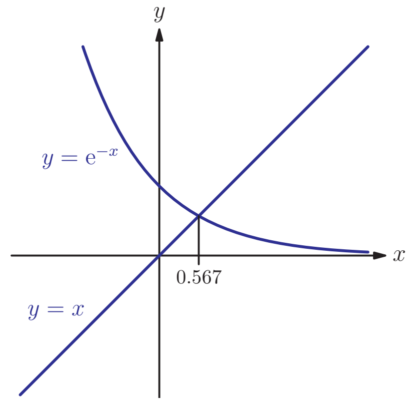
4D Using a graphical calculator to solve equations

Some equations cannot be solved analytically. This means that you cannot rearrange them to get $x = value$ using the standard set of functions mathematicians allow. However, there are still values of x which satisfy the equation.

An example of such an equation is $x = e^{-x}$.

One good way to find these solutions is by plotting both sides of the equation using a graphical calculator. Calculator skills sheet 5 explains how to find the intersection of two graphs.

The solution to the equation $x = e^{-x}$ can actually be written in terms of the 'Lambert W Function'. In fact, it is W(1). However, does knowing this actually give us any more information about the solution to the equation?



The x-value of the intersection gives the solution to the equation, in this case 0.567 to 3 significant figures.

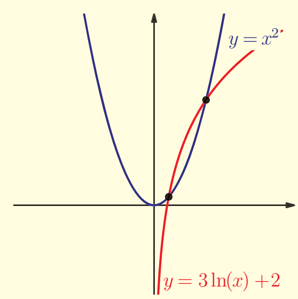
EXAM HINT

When using your calculator to solve an equation, you must draw a sketch of the graph and round the solution to an appropriate degree of accuracy (usually 3 significant figures).

Worked example 4.7

Solve the equation $x^2 = 3\ln x + 2$.

There is no obvious substitution or factorisation, so plot on the calculator and sketch



Use the calculator to find the intersection points

$x = 2.03$ or 0.573 (3SF) from GDC

Solving equations graphically has some problems: You may not know how many solutions you are looking for, or how to set the viewing window so you can see them all. If two solutions are very close together you may miss one of them. This is where you need your knowledge of the shapes of different graphs to make sure that the calculator is showing all the important features.

Many graphical calculators have functions for solving some special types of equations. In particular, you may be able to solve polynomial equations (those involving only positive integer powers of x) without having to use a graph.

EXAM HINT

If the question does not ask for an exact answer this is an indication that a graphical solution might be appropriate.
If you cannot see another quick way to do it, put it into your calculator.

You may also be able to use an equation solver tool, although this has similar issues to using graphs. See Calculator skills sheets 5 and 6 on the CD-ROM for more details.

Worked example 4.8

Solve the equation $x^3 = 5x^2 - 2$.

We don't know any algebraic methods for solving cubic equations, so use the polynomial equation solver on the calculator

The equation needs to be rearranged

$x^3 - 5x^2 + 2 = 0$

$\Rightarrow x = -0.598, 0.680, 4.92$ (3SF) (using GDC)

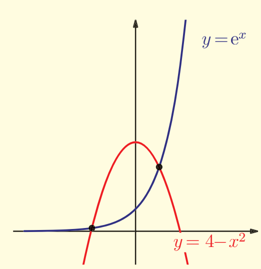
EXAM HINT

You can use the graphical method in the non-calculator paper, too. In such questions you are often just asked to find the number of solutions, rather than actually solve the equation.

Worked example 4.9

Find the number of solutions of the equation $e^x = 4 - x^2$.

We know how to sketch both graphs



The solutions correspond to the intersections of the two curves

There are 2 solutions.

Exercise 4D

1. Solve the following equations giving your answers to 3 significant figures.

   (a) (i) $x^3 = 3x - 1$ (ii) $x^3 + 4x^2 = 2x - 1$

   (b) (i) $e^x = x + 1$ (ii) $e^{2x} = x^2 - 3$

   (c) (i) $e^x = \ln x$ (ii) $e^x \ln x = x^3 - x$

2. Solve the equation $x \ln x = 3 - x^2$. *[4 marks]*

3. The equation $\ln x = kx$ where $k > 0$ has one solution.

   How many solutions do the following equations have?

   (a) $\ln x^2 = kx$ (b) $\ln\left(\frac{1}{x}\right) = kx$ (c) $\ln\sqrt{x} = kx$ *[6 marks]*

4E Solving simultaneous equations by substitution

In the previous section we saw how to solve equations by finding intersections of graphs. For example, to solve the equation $e^x = 4 - x^2$ we sketched the graphs of $y = e^x$ and $y = 4 - x^2$ and found their intersections. We were only interested in the x-coordinates of the intersection points. However, the y-values for the two equations are also equal at those points. This means that we have in fact solved simultaneous equations

$$\begin{cases} y = e^x \\ y = 4 - x^2 \end{cases}$$

See Prior learning section Q for a reminder about linear simultaneous equations.

A powerful method for solving simultaneous equations is **substitution**, where you replace all the occurrences of one of the variables in one equation by an expression from the other equation.

Worked example 4.10

Find the coordinates of the points of intersection between the line $2x - y = 1$ and the parabola $y = x^2 - 3x + 5$.

At the intersection points, the y-coordinates for the two curves are equal, so we can replace y in the first equation by the expression for y from the second equation

$2x - y = 1 \Rightarrow y = 2x - 1$

$x^2 - 3x + 5 = 2x - 1$

This rearranges to a quadratic equation

Try to factorise

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2 \text{ or } 3$

We also need to find the y-coordinates, by substituting back into one of the equations for y (they will both give the same answer). Pick the first equation, as it is easier

$y = 2x - 1$

$x = 2: \; y = 2 \times 2 - 1 = 3$

$x = 3: \; y = 2 \times 3 - 1 = 5$

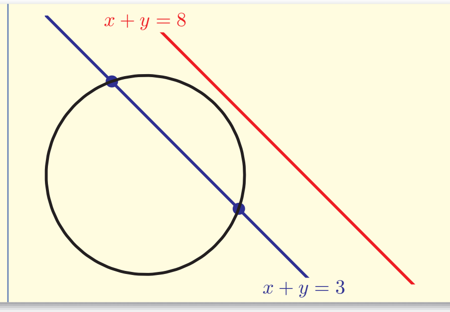
The coordinates are $(2, 3)$ and $(3, 5)$

Sometimes we only want to know how many intersection points there are, rather than to find their actual coordinates. The discriminant can be used to determine the number of intersections if the equation is quadratic.

Worked example 4.11

Find the set of values of k for which the line with equation $x + y = k$ intersects the circle with equation $x^2 - 4x + y^2 + 6y = 12$ at two distinct points.

The question suggests that if we try to find the intersections between the line and the circle, the number of solutions will depend on the value of k. This makes sense, as changing k moves the line up and down, so it will sometimes intersect the circle and sometimes won't



continued . . .

Try finding the intersections in terms of k and see if that gives any ideas

At the intersection points, the y-coordinates for the two curves are equal, so we can replace y in the second equation by the expression for y from the first equation

Line equation: $y = k - x$

Substitute into the circle equation:

$$x^2 - 4x + (k - x)^2 + 6(k - x) = 12$$

$$\Rightarrow x^2 - 4x + k^2 - 2kx + x^2 + 6k - 6x = 12$$

$$\Rightarrow \quad 2x^2 - (10 + 2k)x + k^2 + 6k = 12$$

This is a quadratic equation; write it with one side equal to zero

$$\Rightarrow 2x^2 - (10 + 2k)x + (k^2 + 6k - 12) = 0$$

We know that the discriminant tells us the number of solutions of a quadratic equation

Two solutions $\therefore \Delta > 0$

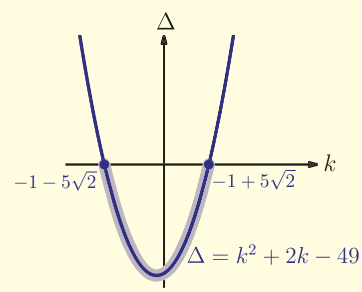
$$\Delta = (10 + 2k)^2 - 8(k^2 + 6k - 12) > 0$$

$$\Rightarrow 100 + 40k + 4k^2 - 8k^2 - 48k + 96 > 0$$

$$\Rightarrow \quad -4k^2 - 8k + 196 > 0$$

$$\Rightarrow \quad k^2 + 2k - 49 < 0$$

This is a quadratic inequality. To solve it, find where LHS = 0 and sketch the graph



Roots: $k^2 + 2k - 49 = 0$

$$k = \frac{-2 \pm \sqrt{4 + 4 \times 49}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1 + 49}}{2}$$

$$= -1 \pm \sqrt{50} = -1 \pm 5\sqrt{2}$$

The graph shows that the required interval is between the roots

$$\therefore -1 - 5\sqrt{2} < k < -1 + 5\sqrt{2}$$

EXAM HINT

You are not expected to know why the second equation represents a circle, or what the centre and the radius of the circle are. Exam questions sometimes involve unfamiliar concepts, but you will be given all the information required to answer the question.

The last example illustrates how a geometrical problem can be solved using purely algebraic methods.

There is a whole branch of mathematics studying such methods, called analytic geometry. It was developed in the 17th Century by the French philosopher and mathematician René Descartes. Establishing a link between geometry and algebra was a major step in the development of modern mathematics. You will learn more about it in chapter 14.

The circle and the parabola belong to a family of quadratic curves called conic sections, which also includes the ellipse and the hyperbola. There are many fascinating and beautiful results concerning conic sections, many of which can be investigated using the properties of the quadratic function.

Exercise 4E

1. Find the coordinates of intersection between the given parabola and the given straight line.

(a) (i) $y = x^2 + 2x - 3$ and $y = x - 1$

(ii) $y = x^2 - 4x + 3$ and $y = 2x - 6$

(b) (i) $y = -x^2 + 3x + 9$ and $2x - y = 3$

(ii) $y = x^2 - 2x + 8$ and $x - y = 6$

2. Solve these simultaneous equations.

(a) (i) $x - 2y = 1$, $3xy - y^2 = 8$ (ii) $x + 2y = 3$, $y^2 + 2xy + 9 = 0$

(b) (i) $xy = 3$, $x + y = 4$ (ii) $x + y + 8 = 0$, $xy = 15$

(c) (i) $x + y = 5$, $y = x^2 - 2x + 3$ (ii) $x - y = 4$, $y = x^2 + x - 5$

3. Show that the line with equation $x + y = 8$ is tangent to the circle with equation $x^2 - 6x + y^2 - 2y + 2 = 0$. *[5 marks]*

4. Find the exact values of m for which the line $y = mx + 3$ is tangent to the curve with equation $y = 3x^2 - x + 5$. *[5 marks]*

5. Let C be the ellipse with equation $4x^2 + 9y^2 = 36$.
Find the exact values of k for which the line $2x + 3y = k$ is tangent to C. *[6 marks]*

6. Show that the line $y = kx + 5$ intersects the parabola $y = x^2 + 2$ for all values of k. *[6 marks]*

7. Solve the simultaneous equations.

$2^x + 2^y = 10$

$x + y = 4$ *[6 marks]*

8. Solve the simultaneous equations.

$\log_3 x + \log_3 y = 3$

$y = x^5$ *[6 marks]*

4F Systems of linear equations

You already know several methods for solving two linear simultaneous equations with two unknowns. In this section we will extend the method of elimination to deal with three equations and three unknowns. These types of equations will be used later in the course to find the intersection between planes.

Section Q of Prior learning deals with simultaneous equations.

We will study equations of lines and planes in chapter 14.

Consider the system of equations

$$\begin{cases} 2x + y + z = 7 & (1) \\ x + 3y + 2z = 13 & (2) \\ 4x - 2y + 3z = 9 & (3) \end{cases}$$

Following the strategy we used with two simultaneous equations, we can try to eliminate x. We can choose any two equations; usually we use the second and third. For example, we can multiply the second equation by 2 and subtract it from the first equation:

$$\begin{aligned} 2x + y + z &= 7 \quad (1) \\ 2\times(2): 2x + 6y + 4z &= 26 \\ \Rightarrow \quad -5y - 3z &= -19 \quad (4) \end{aligned}$$

To eliminate x from the third equation, we can multiply (1) by 2 and subtract (3):

$$\begin{aligned} 2\times(1): 4x + 2y + 2z &= 14 \\ (3): 4x - 2y + 3z &= 9 \\ \Rightarrow \quad 4y - z &= 5 \quad (5) \end{aligned}$$

We can replace equation (2) by equation (4) and equation (3) by equation (5), so now we have a new system of three equations (1), (4) and (5):

$$\begin{cases} 2x + y + z = 7 & (1) \\ -5y - 3z = -19 & (4) \\ 4y - z = 5 & (5) \end{cases}$$

Equations (4) and (5) contain only two unknowns, so we know how to proceed. For example, we can eliminate y if we replace equation (5) by $4\times(4)+5\times(5)$:

$$4\times(4): -20y - 12z = -76$$
$$5\times(5): \quad 20y \ -5z = 25$$
$$\Rightarrow -17z = -51 \ (6)$$

The last equation enables us to find z. It is useful to keep track of the equations we will be using to find x and y, so write the system of three equations we now have:

$$\begin{cases} 2x + y + z = 7 & (1) \\ -5y - 3z = -19 & (4) \\ -17z = -51 & (6) \end{cases}$$

Writing the equations like this makes the rest of the strategy clear: We will use equation (6) to find z, then equation (4) to find y and finally equation (1) to find x.

$$(6) \Rightarrow z = 3$$
$$(4) \Rightarrow -5y = -19 + 3\times 3 \Rightarrow y = \frac{-10}{-5} = 2$$
$$(1) \Rightarrow \quad 2x = 7 - 2 - 3 \quad \Rightarrow x = \frac{2}{2} = 1$$

Thus the solution of our system is $x = 1,\ y = 2,\ z = 3$.

The method carried out above is known as **Gaussian elimination**.

KEY POINT 4.2

To solve a system of linear equations using Gaussian elimination:

First transform the system into a triangular form by eliminating x from the second and third equations and y from the third equation.

Then use back substitution to find z, y and x.

Your calculator can carry out the first part of the procedure. In some cases it can also complete the back substitution to find the solution of the system.

EXAM HINT

See calculator sheet 16 for details of Gaussian elimination

You may know that simultaneous equations do not always have a solution. For example, the system of equations $\begin{cases} x+y=3 \\ x+y=2 \end{cases}$ clearly has no solutions. This can also happen with systems of three equations, and it becomes apparent when the system has been transformed into the triangular form.

Worked example 4.12

Show that the system $\begin{cases} x+2y+3z=10 \\ 2x+3y+2z=\ \ 4 \\ 4x+7y+8z=\ \ 7 \end{cases}$ has no solutions.

We will be able to see whether there are solutions by looking at the triangular form. The calculator can produce this

Using GDC, the system is transformed to

$$\begin{cases} x+\frac{7}{4}y+2z=\frac{7}{4} \\ y+4z=-1 \\ 0=1 \end{cases}$$

The last equation is impossible

$0=1$ is impossible, so there are no solutions.

Systems which have solutions are called **consistent**; otherwise they are **inconsistent**. If a system is consistent, it may have more than one solution. This will always happen when there are more unknowns than equations. For example, the equation $x+y=5$ has infinitely many solutions: We can pick x to be any number and then $y=5-x$. The next example shows what happens with three unknowns.

Worked example 4.13

Find all solutions of the system $\begin{cases} 3x - y + 2z = 5 \\ 2x + y + \ \ z = 1 \end{cases}$

Start by eliminating x

$2\times(1): 6x - 2y + 4z = 10$

$3\times(2): 6x + 3y + 3z = 3$

$\Rightarrow -5y \ + \ z = 7 \quad (3)$

We cannot do any more elimination, so start on the back substitution. But we can't find y or z from (3), we can only express z in terms of y

$\Rightarrow z = 7 + 5y$

$(1) \Rightarrow 3x = 5 + y + 2(7 + 5y)$

$= 19 + 11y$

$\therefore x = \dfrac{19 + 11y}{3}$

We have expressed x and z in terms of y. But y can be any number

We can write the solution as:

$x = \dfrac{19 + 11t}{3},\ y = t,\ z = 7 + 5t$ where $t \in \mathbb{R}$

You will see in chapter 14 that this general solution represents an equation of a straight line, and the system describes two planes intersecting along this line.

In the last example the system has infinitely many solutions. What we have found is called a **general solution**; it is a way of representing all possible solutions.

KEY POINT 4.3

When a system has infinitely many solutions, the **general solution** involves writing x, y and z in terms of a real parameter t.

EXAM HINT

Some calculators can find the general solution so you can use this to check your answer. With others you have to perform back substitution yourself.

The situation with infinitely many solutions can also occur with three equations and three unknowns. The next example also shows you how to set out your working when performing Gaussian elimination.

Worked example 4.14

(a) Find the value of k for which the system of equations $\begin{cases} 2x+y+2z=4 \\ 6x-y-6z=8 \\ 4x+6y+16z=k \end{cases}$ is consistent.

(b) For the value of k found above, find the general solution of the system.

Do Gaussian elimination. Start by labelling all equations

(a)

$$\begin{cases} 2x+y+2z=4 & (1) \\ 6x-y-6z=8 & (2) \\ 4x+6y+16z=k & (3) \end{cases}$$

Eliminate x from second and third equations

$$\begin{cases} 2x+y+2z=4 & (1) \\ 4y+12z=4 & (4) \quad 3\times(1)-(2) \\ -4y-12z=8-k & (5) \quad 2\times(1)-(3) \end{cases}$$

Eliminate y from the third equation

$$\begin{cases} 2x+y+2z=4 & (1) \\ 4y+12z=4 & (4) \\ 0=12-k & (6) \quad (4)+(5) \end{cases}$$

For the system to be consistent, the last equation must say $0 = 0$

The system is consistent when $k=12$.

Start back substitution from the middle equation. We can express y in terms of z

(b)

Let $z=t$

$$(4) \Rightarrow y=\frac{4-12t}{4} = 1-3t$$

$$(1) \Rightarrow 2x = 4-(1-3t)-2t = 3+t$$

$$\Rightarrow x = \frac{3+t}{2}$$

We now have a general solution

The general solution is

$$x=\frac{3+t}{2}$$

$$y=1-3t$$

$$z=t,\ t\in\mathbb{R}$$

The method of Gaussian elimination can be extended to systems with more equations and more unknowns.

EXAM HINT

Remember that when there is a unique solution, you can also find it using the simultaneous equation solver. On some calculators the equation solver can also find the general solution. See calculator sheet xx

When performing Gaussian elimination without a calculator it is useful to look for shortcuts. In the above example, we could have divided equation (4) by 4 before continuing. You can also change the order of the equations, for example so that the first one has the smallest coefficient of x. If the first equation does not contain x at all, you need to arrange them so that the first equation is one that does. Note that your calculator will usually divide equations by a number so that the coefficients on the diagonal are all 1, but there is no need for you to do this.

We can summarise all the possibilities for a system of three linear equations with three unknowns:

KEY POINT 4.4

The number of solutions of a system of three linear equations in three unknowns can be seen from the triangular form.

If the last equation has the form $kz = c$ with $k \neq 0$, there is a unique solution for x, y and z.

If the last equation has the form $0 = c$ with $c \neq 0$, there are no solutions (the system is inconsistent).

If the last equation is $0 = 0$, there are infinitely many solutions. The general solution can be found by setting $z = t$ and expressing x and y in terms of t.

The next example illustrates all the possibilities.

Worked example 4.15

Consider the system of equations:

$$\begin{cases} 3x + 2y - 2z = 1 \\ 2x - 4y + z = 4 \\ x + y + kz = a \end{cases}$$

(a) Find the set of values of k for which the system has a unique solution.

(b) For the value of k for which the system does not have a unique solution, find the value of a for which the system has infinitely many solutions.

(c) State the set of values of k and a for which the system has no solutions.

continued . . .

Start by performing Gaussian elimination

(a)

$$\begin{cases} 3x+2y-2z=1 & (1) \\ 2x-4y+\ \ z=4 & (2) \\ x+\ \ y+kz=a & (3) \end{cases}$$

$$\begin{matrix} 2\times(1)-3\times(2) \\ \\ (1)-3\times(3) \end{matrix} \begin{cases} 3x+2y\quad -2z=1 & (1) \\ 16y\quad -7z=-10 & (4) \\ -\ y-(2+3k)z=1-3a & (5) \end{cases}$$

$$\begin{matrix} \\ \\ (4)+16\times(5) \end{matrix} \begin{cases} 3x+\ 2y\quad -2z=1 & (1) \\ 16y\quad -7z=-10 & (4) \\ -(39+48k)z=6-48a & (6) \end{cases}$$

For a unique solution, the coefficient of z in the last equation must not be zero

Unique solution when
$39+48k\neq 0$

$$\Leftrightarrow k\neq -\frac{39}{48}$$

When the LHS side of the last equation is zero, the RHS must be zero as well

(b) When $k=-\frac{39}{48}$, the system is consistent when $6-48a=0$

$$\Leftrightarrow a=\frac{1}{8}$$

No solutions when in the last equation LHS is zero but RHS is not

(c) There are no solutions when $k=-\frac{39}{48}$ and $a\neq\frac{1}{8}$.

In chapter 14 you will see that the different cases for the number of solutions of a system of three equations correspond to different possibilities for the relative position of three planes in space.

Exercise 4F

1. Using Gaussian elimination, or otherwise, solve the following systems of equations:

(a) (i) $\begin{cases} 2x-3y+\ \ z=10 \\ 5x+2y+2z=15 \\ x+4y-2z=-3 \end{cases}$

(ii) $\begin{cases} x - y + 3z = -1 \\ 2x + y + z = -1 \\ x - 2y + 3z = -3 \end{cases}$

(b) (i) $\begin{cases} 2x + y + z = 3 \\ x - y + 3z = 6 \\ 4x + 3y + z = 3 \end{cases}$

(ii) $\begin{cases} x - 3y + z = -5 \\ 3x - y - z = -7 \\ x + 3y + 5z = 1 \end{cases}$

(c) (i) $\begin{cases} 2y + 3z = 4 \\ x + 3y + 2z = 4 \\ -2x + y + 3z = -1 \end{cases}$

(ii) $\begin{cases} y - 3z = -9 \\ 2x + y + z = 1 \\ x - 3y + z = 2 \end{cases}$

2. Use your calculator to transform the following systems into a triangular form, and hence say whether they have a unique solution, no solutions, or infinitely many solutions.

(a) $\begin{cases} 3x - y + 4z = 7 \\ x - 2y + z = 3 \\ x - y + 4z = -5 \end{cases}$

(b) $\begin{cases} 2x - y - z = 5 \\ 3x + 2y + 3z = 1 \\ 5x + y + 2z = 4 \end{cases}$

(c) $\begin{cases} -2x + y + 3z = 5 \\ 2x + y + z = 2 \\ 2x + 3y + 5z = 9 \end{cases}$

(d) $\begin{cases} x - 2y + z = 4 \\ -2x + 4y - 2z = 1 \\ 3x - 6y + 3z = 12 \end{cases}$

(e) $\begin{cases} x - 2y + z = 0 \\ 3x + 2y + 2z = 0 \\ x - 5y - z = 0 \end{cases}$

3. Find the general solution of the following systems of equations:

(a) (i) $\begin{cases} 2x + y + z = 5 \\ x - y + 2z = 1 \\ x + 2y - z = 4 \end{cases}$

(ii) $\begin{cases} 2x - y - 3z = 3 \\ x + y - 3z = 0 \\ x + 2y - 4z = -1 \end{cases}$

(b) $\begin{cases} x - 2y + z = 1 \\ 3x - 6y + z = 3 \\ -x + 2y + 2z = -1 \end{cases}$

(c) $\begin{cases} x - 2y + z = 5 \\ -3x + 6y - 3z = -15 \\ 2x - 4y + 2z = 10 \end{cases}$

4. Find the value of c for which the following system of equations is consistent:

$\begin{cases} 2x + y - 2z = 0 \\ x - 2y - z = 2 \\ 3x + 4y - 3z = c \end{cases}$ *[5 marks]*

5. For which value of k does the system of equations

$\begin{cases} x - 2y + 2z = 0 \\ 2x + ky - z = 3 \\ x - y + 3z = -5 \end{cases}$

not have a unique solution? *[6 marks]*

6. Find the solution of the following system of equations:

$\begin{cases} 2x + y - z = 2 \\ x - 2y + 2z = 1 \\ 2x + y - 4z = a \end{cases}$ *[7 marks]*

7. (a) Find the values of a for which the following system of equations in consistent:

$$\begin{cases} x - 2y + z = 1 \\ 2x + y - z = a \\ 4x + 7y - 5z = a^2 \end{cases}$$

(b) For the larger of the two values found above, find the general solution of the system. *[10 marks]*

8. (a) Find the value of k for which the system of equations

$$\begin{cases} x - 2y + z = 7 \\ 2x + y - 3z = b \\ x + y + kz = 4 \end{cases}$$

does not have a unique solution.

(b) For this value of k, find the value of b for which the system is consistent.

(c) In the case when the system has infinitely many solutions, find the general solution. *[11 marks]*

4G Solving inequalities

You already know that:

- you can add or subtract the same number from both sides of an **inequality** and multiply or divide both sides by a positive number
- if you multiply or divide both sides by a negative number you must reverse the sign.

How about using the rule 'do the same thing to both sides' with other functions we will meet in this course?

You know from Prior learning that the solution to $x^2 > 4$ is *not* $x > 2$ or even $x > \pm 2$; it is in fact $x < -2$ or $x > 2$. This is easiest to see when you sketch the graph and look for regions which satisfy the inequality.

Inequalities with other functions can be even more complicated.

For example, trying some numbers reveals that the inequality $\left(\frac{1}{2}\right)^x < 4$ is satisfied for $x > -2$, while the inequality $x^3 - 4x > 2$

is satisfied for (among other numbers) $x = -1, 3, 4$ but not for $x = -2, 0, 1, 2$.

Given the difficulties with rearranging inequalities, it is often best to apply only simple operations and otherwise solve them graphically. Inequalities can appear on the non-calculator paper; there are many graphs that you must know how to sketch without a calculator.

EXAM HINT

Whenever solving inequalities *always* sketch the graph!

Worked example 4.16

Solve the following inequality $(x-2)(x^2-5x+5) > x-2$.

We are going to try to get everything onto one side and then factorise it so that we can draw the graph.

Notice that there is a factor of $x - 2$ on both sides. We can use this to make the algebra neater

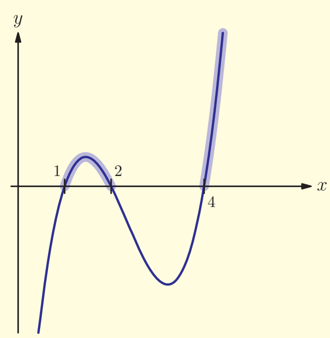
$$\Leftrightarrow \quad (x-2)(x^2-5x+5) - 1(x-2) > 0$$

$$\Leftrightarrow \quad (x-2)(x^2-5x+5-1) > 0$$

$$\Leftrightarrow \quad (x-2)(x^2-5x+4) > 0$$

$$(x-2)(x-1)(x-4) > 0$$

Sketch the cubic graph, highlighting where it is above the axis



$$\therefore \quad 1 < x < 2 \text{ or } x > 4$$

EXAM HINT

If you are describing distinct regions you must combine the inequalities using 'or'. You can only write a single inequality, such as $-6 \le x \le 1$, if it describes one region.

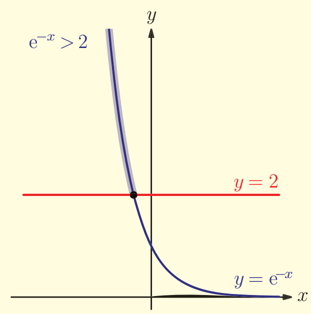
Note that the answer from the last example can also be written using interval notation as $x \in \,]1, 2[\cup]4, \infty[$. You do not have to use this notation, but you must be able to understand it.

See Prior learning I for an explanation of interval notation.

Worked example 4.17

Solve the inequality $e^{-x} > 2$.

Sketch the graph of $y = e^{-x}$ and look to see where it is above the line $y = 2$



Find the intersection point

Intersection: $e^{-x} = 2$

$\Leftrightarrow \quad -x = \ln 2$

$\Leftrightarrow \quad x = -\ln 2$

The highlighted part is to the left of the intersection point

So $x < -\ln 2$

EXAM HINT

If you are going to use the calculator there is no need to rearrange the inequality in any way – just plot the two sides.

Exercise 4G

1. Solve the following inequalities:

(a) (i) $2.4^x < 14.8$ (ii) $3.1^x > 6$

(b) (i) $0.8^x \geq 0.4$ (ii) $0.4^x \leq 0.001$

(c) (i) $3 \times \left(\frac{3}{4}\right)^{2x} > 1$ (ii) $5 \times \left(\frac{4}{3}\right)^{3x+4} < 30$

(d) (i) $3x^5 \geq 14$ (ii) $4x^3 + 1 < 6$

(e) (i) $\ln(x+4) > 0.8$ (ii) $\ln(x-3) \leq 0.4$

(f) (i) $x^3 - 4x > 0$ (ii) $x^3 - 6x < 0$

2. By sketching both sides on the same diagram, solve the following inequalities:

(a) $x > 5\ln x$ (b) $\frac{x}{\ln x} > 5$

Why are the answers not the same? *[5 marks]*

4H Working with identities

An identity is an equation which is true for all values of x, for example $x \times x \equiv x^2$.

The identity sign $\equiv$ is used to emphasise the identity, although often an equals sign is used instead, with the words 'for all x'.

You have already seen in Section 3A that one use of identities is the ability to compare coefficients. Another important use of identities is to manipulate expressions.

Operations such as opening out brackets are actually applications of identities. For example when manipulating the expression $(x-2)(x+1)$ to give $x^2 - x - 2$ we are actually using the identity $(x+a)(x+b) \equiv x^2 + (a+b)x + ab$ for all a, b and x.

To prove an identity we must start from one side and state which identities or rules are being used as we transform it line by line to reach the other side of the identity.

EXAM HINT

You may be unsure about which rules and identities you are allowed to use. Anything listed as a Key point in this book is acceptable, as are basic algebraic manipulations such as multiplying out brackets and simplifying fractions.

Worked example 4.18

Prove that $e^{(2\ln x + \ln 3)} = 3x^2$.

Try to simplify the exponent using laws of logarithms	$\text{LHS} = e^{(2\ln x + \ln 3)}$
First, take the multiple inside the log: $p\ln a = \ln(a^p)$	$= e^{\ln x^2 + \ln 3}$
Then $\ln a + \ln b = \ln(ab)$	$= e^{\ln(3x^2)}$
Finally, apply the cancellation principle	$= 3x^2$ $= \text{RHS}$

There are other ways to prove identities.

- We are allowed to work from both ends and meet in the middle.

Proving identities is very important in trigonometry – see chapter 10.

- We can subtract one side from the other and show that this is always 0.
- We can divide one side by the other and show that this is always 1.

It can be very tempting to treat the identity to be proven like an equation and do things to both sides until we get something which we know is correct. However, this is logically flawed and you should avoid it.

Worked example 4.19

Prove that for all x,

$$\frac{2}{x}-\frac{1+x}{x^2-x}=\frac{3}{x}-\frac{2}{x-1}$$

We must start from one side and try to get to the other. The left-hand side looks a little more complicated, so start there. See if we can simplify

$$LHS=\frac{2}{x}-\frac{1+x}{x^2-x}$$
$$=\frac{2}{x}-\frac{1+x}{x(x-1)}$$
$$=\frac{2(x-1)-(1+x)}{x(x-1)}$$
$$=\frac{x-3}{x(x-1)}$$

There is no obvious way to continue, so now start from the right-hand side and see if we can get to the same place

$$RHS=\frac{3}{x}-\frac{2}{x-1}$$
$$=\frac{3(x-1)-2x}{x(x-1)}$$
$$=\frac{x-3}{x(x-1)}$$

We have shown that the two sides are equal

LHS=RHS, so the identity is proved.

Exercise 4H

1. Prove the following identities.

(a) (i) $(x-y)^2+4xy=(x+y)^2$

(ii) $x^3+y^3=(x+y)(x^2-xy+y^2)$

(b) (i) $\log(xyz)=\log x+\log y+\log z$ (for $x, y, z>0$)

(ii) $\log_a b = \dfrac{1}{\log_b a}$ (for $a, b > 0$, $a, b \neq 1$)

(c) (i) $\dfrac{a^2 - b^2}{a - b} = a + b$ (ii) $\dfrac{x - y}{y - x} = -1$

(d) (i) $\sqrt[a]{x^b} = \left(\sqrt[a]{x}\right)^b$ (ii) $2^a + 2^a = 2^{a+1}$

2. If the cubic expression $x^3 + px + q$ can be written in the form $(x - a)^2 (x - b)$, find an equation linking p and q. *[6 marks]*

Summary

- Important methods for solving equations include factorising and substitution.
- When sketching graphs on your calculator, you may need to use your knowledge of the shapes of common graphs to make sure you do not miss any important features, such as zeros, vertical or horizontal asymptotes.
- You can use a graphical calculator to solve equations involving unfamiliar functions, by finding intersections of two curves.
- Systems of three equations in three unknowns can be solved using **Gaussian elimination**. The system may have a unique solution, infinitely many solutions, or no solutions.
- When a system of equations has infinitely many solutions, a **general solution** has x, y, z expressed in terms of a real parameter t.
- Sketching graphs is the best way to solve complicated **inequalities.**
- **Identities** are true for all values of x. We can prove identities by transforming one side into the other, using known identities.

Introductory problem revisited

What information can be extracted from the equation $ax + b = 3x + 2$?
What information can be extracted from the identity $ax + b = 3x + 2$ for all x?

$ax + b = 3x + 2$ is an equation, so it will only be true for some values of x.

Rearranging gives the answer $x = \dfrac{2 - b}{a - 3}$.

$ax + b \equiv 3x + 2$ is an identity, so it is true for all x. Comparing coefficients gives $a = 3$ and $b = 2$.
The equation allows us to examine x in terms of a and b.
The identity allows us to evaluate a and b.

Mixed examination practice 4

Short questions

1. (a) Sketch the graphs $y = 2^x$ and $y = 1 - x^2$ on the same axes.

(b) Hence write down the number of solutions of the equation $2^x = 1 - x^2$. *[6 marks]*

2. Find the largest possible value of $y = x^2e^{-x}$ for $x \in [0, 5]$. *[4 marks]*

3. Solve the inequality $x^3 - 4x > e^{-x}$. *[5 marks]*

4. Solve the following system of equations:

$$\begin{cases} 3x - y + 2z = 2 \\ x - 2y + z = 3 \\ x - y + 3z = -5 \end{cases}$$

[6 marks]

5. Find the exact solution of the equation $e^x \ln x = 3e^x$. *[5 marks]*

6. (a) By using an appropriate substitution find the exact solutions to the equation $x^4 + 36 = 13x^2$.

(b) Hence solve the inequality $x^4 + 36 \leq 13x^2$. *[6 marks]*

7. (a) Sketch the graph of $\frac{1}{e^x - 2}$.

(b) State the exact equation of the vertical asymptote. *[6 marks]*

8. (a) Find the value of p for which the following system of equations is consistent:

$$\begin{cases} 2x + 2y - z = 1 \\ x + y - z = -4 \\ 4x + 4y - 3z = p \end{cases}$$

(b) For this value of p, find the general solution of the system. *[9 marks]*

9. Find the values of k for which the line $y = 2x - k$ is tangent to the circle with equation $x^2 + y^2 = 5$.

10. Solve the inequality $\frac{x}{\ln x} < 4$. *[5 marks]*

11. Solve the equation $x \ln x + 4 \ln x = 0$. *[5 marks]*

Long questions

1. (a) (i) Find the value of k for which the equation $x^3 - 9x^2 + kx - 28 = 0$ can be written in the form $(x-a)^3 = b$.

(ii) For this value of k, find the exact solution of the equation $x^3 - 9x^2 + kx - 28 = 0$.

(b) (i) Sketch the graph of $\frac{1}{(x-3)^3 - 1}$.

(ii) Write down the equations of the asymptotes.

(c) Find the coordinates of the minimum point on the graph of $y = (x-3)^3 \ln x$. *[10 marks]*

2. (a) If $f(x) = x^4 + x^2 - 6$ solve the equation $f(x) = 0$.

(b) (i) Show that the equation $x^4 + 7x^2 = 4x^3 + 6x + 4$ can be written in the form $f(x+k) = 0$ where k is a constant to be determined.

(ii) Hence find the exact solutions to the equation $x^4 + 7x^2 = 4x^3 + 6x + 4$.

(c) Solve the inequality $x^4 + 7x^2 > 4x^3 + 6x + 4$. *[12 marks]*

3. (a) Find the value of k for which the system of equations

$$\begin{cases} x + y + z = 3 \\ x + ky + z = 4 \\ x - y + z = b \end{cases}$$

does not have a unique solution.

(b) For this value of k, find the value of b for which the system is consistent.

(c) In the case when the system has infinitely many solutions, find the general solution. *[11 marks]*

5 The theory of functions

In this chapter you will learn:

- about the concepts of relations and functions, and how to distinguish between them
- the notation used to represent functions
- what happens when one function acts after another function
- how to reverse the effect of a function
- how to apply the ideas from this chapter to analyse functions formed by dividing one polynomial by another.

Introductory problem

Think of any number. Add 3. Double your answer. Take away 6. Divide by the number you first thought of. Is your answer always prime? Why?

Doubling; adding five; finding the largest prime factor; these are all instructions which can be applied to different numbers in order to get a result. This idea is used a lot in mathematics, and its formal study leads to the concept of functions.

5A Relations, functions and graphs

This section introduces the concepts of a function and a relation and how to distinguish between them.

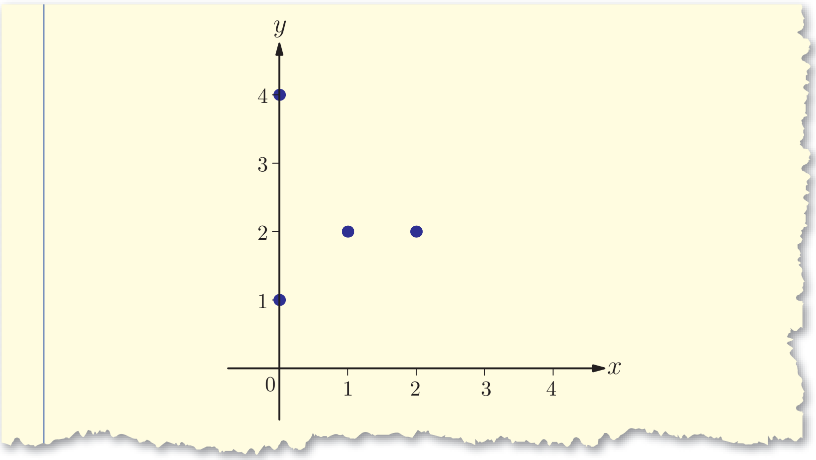
A **relation** can be written as a list of ordered pairs. An example of a relation might link a person to their age:

(John, 7)
(Patricia, 23)
(Joanne, 14)

Most of the relations we are going to use will be pairs of real numbers and we can illustrate these on a graph by plotting them as points. Conventionally we plot the first number on the x-axis and the second number on the y-axis.

Worked example 5.1

Plot the relation (0, 1) (0, 4) (1, 2) (2, 2) on a graph.



A relation can be more than just a few scattered points. We can have an infinite number of points, but these would be impossible to list. Instead we provide a rule that links the x-coordinate and the y-coordinate. Examples of such rules are $y = 3x$ or $y = \text{a factor of } x$.

For the relation $y = \text{a factor of } x$ you will realise that for some numbers there is more than one answer and so we cannot be sure what y is. This is a problem for mathematicians who like to work without ambiguity. A solution is to restrict their attention only to relations where for each x-value there is only one y-value. This type of relation is called a **function**. The easiest way to identify if a relation is a function is to look at a graph of the function, using a technique called the **vertical line test**.

KEY POINT 5.1

Vertical line test

If a relation is a function, any vertical line will cross its graph no more than once.

When we study inverse functions you will see that functions are further divided into two categories: one-to-one functions and many-to-one functions.

Inverse functions are covered in Section E of this chapter.

For a **one-to-one function**, as well as every x-value corresponding to only one y-value, every y-value corresponds to only one x-value.

Therefore, as well as the vertical line test a one-to-one function passes the **horizontal line test**.

KEY POINT 5.2

Horizontal line test

If a function is one-to-one then any horizontal line will cross the graph no more than once.

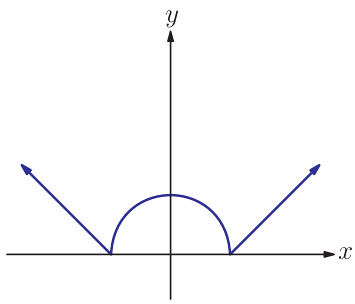
If a function does *not* pass the horizontal line test then there are some y-values which come from more than one x-value. This is a **many-to-one function**.

Worked example 5.2

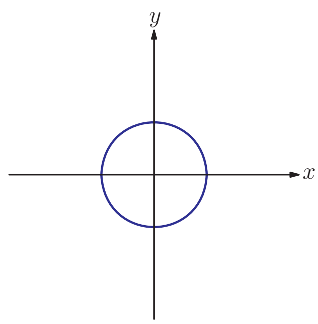
Which of the graphs below represent functions?

For those that are functions, classify them as one-to-one or many-to-one.

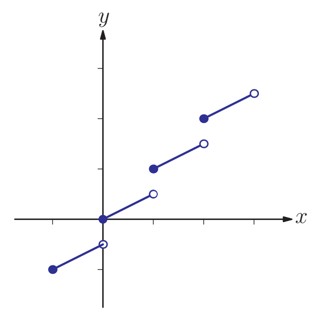
(a)



(b)



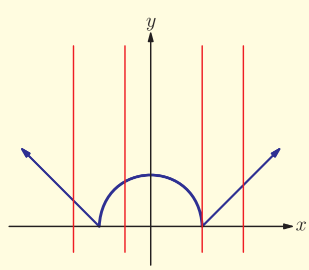
(c)



Note: The white circles mean that the end point is not included, and filled circles mean that it is.

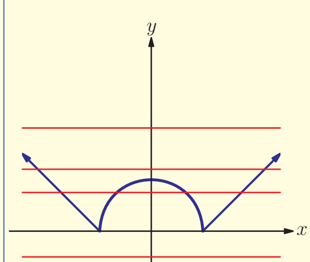
Apply the vertical line test to see whether the relation is a function

(a)



Any vertical line meets the graph at most once, therefore it is a function.

Apply the horizontal line test to see whether the function is one-to-one

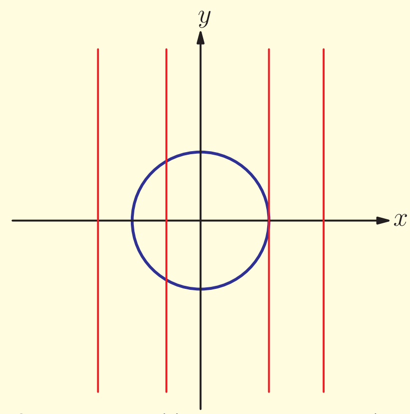


continued . . .

Some horizontal lines meet the graph at more than one point therefore it is a many-to-one function.

Apply the vertical line test to see whether the relation is a function

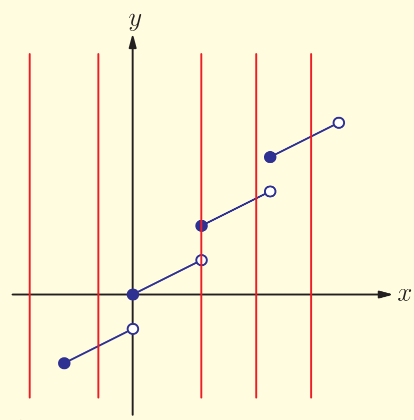
(b)



Some vertical lines meet more than once therefore it is not a function.

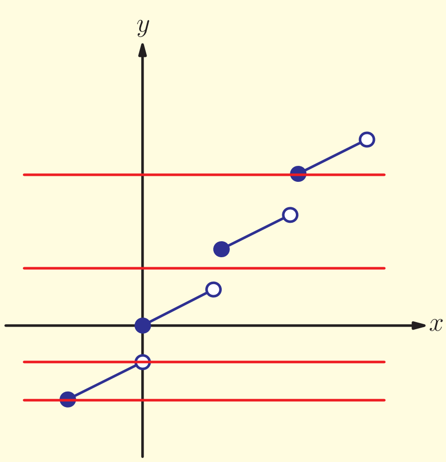
Apply the vertical line test to see whether the relation is a function. A vertical line through an open circle does not count as an intersection

(c)



Any vertical line meets the graph at most once, therefore it is a function.

Apply the horizontal line test to see whether the function is one-to-one



Any horizontal line meets the graph at most once, therefore it is a one-to-one function.

Often a relation is given by an equation linking x and y and we may not know how to draw its graph. In that case we need to look at the equation and ask whether there are any values of x for which we can find more than one value of y.

Worked example 5.3

Decide whether the following relations are functions:

(a) $x + y^2 = 5$ (b) $x + y^3 = 9$

Think of an example where there is more than one y for each x

(a) For $x = 1$, y could be 2 or −2
So this is not a function.

Can we prove that there is always just one value of y? Try to find y for a given x

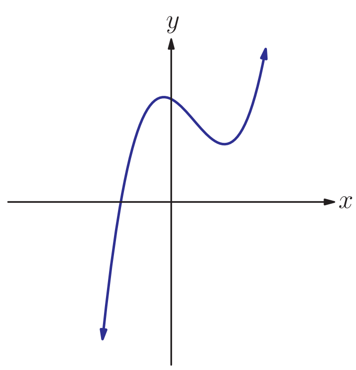
(b)
$$x + y^3 = 9$$
$$\Leftrightarrow \quad y^3 = 9 - x$$
$$\Leftrightarrow \quad y = \sqrt[3]{9 - x}$$

There is only one y for each x.
So this is a function.

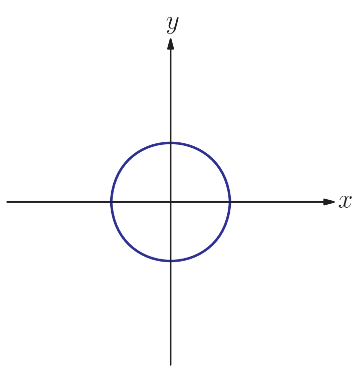
Exercise 5A

1. Classify these graphs as relations or functions. If you decide a graph is a function, state if it is one-to-one or many-to-one.

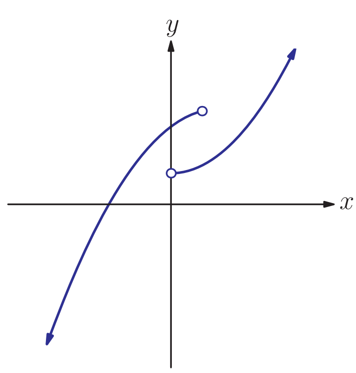
(a)



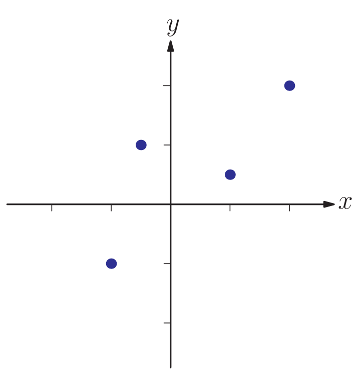
(b)



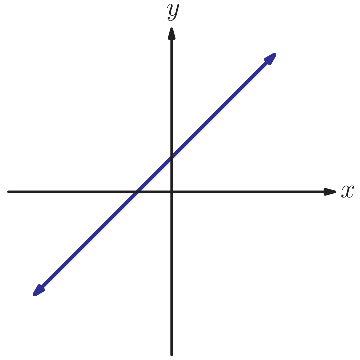
(c)



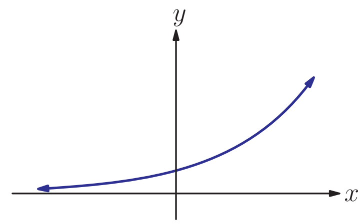
(d)



(e)



(f)



2. Decide if the following relations are functions:

(a) (i) $y = \sqrt{x}$ (ii) $y = 3x + 7$

(b) (i) $y = 5x^2 + 3x$ (ii) $y = 9 - x^2$

(c) (i) $y^2 = x$ (ii) $x^2 + y^2 = 1$

5B Function notation

In this section we will see how to describe functions using mathematical expressions. One way of thinking about functions is that they are rules for turning one number into another. If we have the rule 'add 3 to the input' and we want to call this rule 'f', in function notation this is written as:

$$f : x \rightarrow x + 3$$

We say that the function f transforms x into $x + 3$.

An alternative way of writing this is

$$f(x) = x + 3.$$

Notice that although conventionally the letters f and x are often used, they are not particularly important and we can replace f with any name for the function and x with any input we liked. But whatever we call the input, f will do the same thing to it. We sometimes also call the input the **argument** of the function.

The output is called the **image** of the input.

Is $f(x)$ just a tool to label a rule or does it open up new techniques and new knowledge?

It may surprise you to learn that the latter is actually the case. Particularly in many applications of calculus, we do not need to know exactly what the rule is, but simply that it depends upon x, or time, or height.

Worked example 5.4

We define a function $g(x)=x^2+x$.

Find and simplify: (a) $g(2)$ (b) $g(y)$ (c) $g(x+1)$ (d) $g(3x)$ (e) $4g(x-1)-3$

Replace x with 2

(a) $g(2)=2^2+2=6$

Replace x with y

(b) $g(y)=y^2+y$

Replace x with $x+1$ Don't forget brackets!

(c) $g(x+1)=(x+1)^2+(x+1)=x^2+2x+1+x+1=x^2+3x+2$

Replace x with $3x$ Don't forget brackets!

(d) $g(3x)=(3x)^2+(3x)=9x^2+3x$

Replace x with $x-1$ Don't forget brackets!

(e) $4g(x-1)-3=4\left((x-1)^2+(x-1)\right)-3$
$=4(x^2-2x+1+x-1)-3=4(x^2-x)-3=4x^2-4x-3$

Exercise 5B

1. If $h(x)=3x^2-x$, find and simplify:
   - (a) (i) $h(3)$ (ii) $h(7)$
   - (b) (i) $h(-2)$ (ii) $h(-1)$
   - (c) (i) $h(z)$ (ii) $h(a)$
   - (d) (i) $h(x+1)$ (ii) $h(x-2)$
   - (e) (i) $\frac{1}{2}(h(x)-h(-x))$ (ii) $3h(x)+4h(2x)$
   - (f) (i) $h\left(\frac{1}{x}\right)$ (ii) $h(\sqrt{x})$

Calculations with logarithms were covered in chapter 2.

2. If $g: x \mapsto 1+\log_{10} x$, find and simplify:
   - (a) (i) $g(100)$ (ii) $g(1\,000\,000)$
   - (b) (i) $g(0.1)$ (ii) $g(1)$
   - (c) (i) $g(y)$ (ii) $g(z)$
   - (d) (i) $g(10x)$ (ii) $g(100y)$
   - (e) (i) $3g(x)+g(x^2)$ (ii) $\frac{1}{2}\left(g(x)+g\left(\frac{1}{x}\right)\right)$

3. If $u(x)=3x+1$ and $v(x)=-\sqrt{x}$, find:

(a) (i) $u(2)+v(9)$ (ii) $u(1)v(4)$

(b) (i) $u(x)+v(y)$ (ii) $u(2x+1)-v(4x)$

(c) (i) $2u(4x)+3v(4x)$ (ii) $v(x^2+1)+xu(2x)$

5C Domain and range

A function tells you what to do with the input, but, to be completely defined, you also need to know what type of input is permitted to go into the function.

KEY POINT 5.3

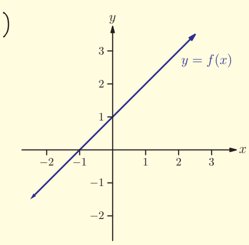
> The set of allowed input values is called the **domain** of the function. Conventionally we write it after the rule using set notation, interval notation or inequalities.

Worked example 5.5

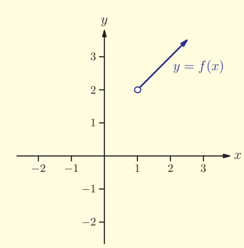
Sketch the graph of $f(x)=x+1$ over the domain (a) $x\in\mathbb{R}, x>1$ (b) $x\in\mathbb{Z}$

Sketch the graph over the domain $x \in \mathbb{R}$

(a)

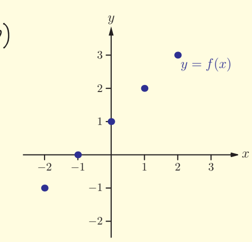


Discard the part of the graph which is outside of the required domain. Since the endpoint is not included we must label it with an open circle



Use the same original graph as in (a), but this time it only exists at whole numbers, so we label them with closed circles

(b)



EXAM HINT

When you are instructed to sketch the graph of $f(x)$, this just means the graph of $y = f(x)$.

It is quite tempting to say that dividing by zero results in infinity. However, doing this leads to some unfortunate consequences, such as all numbers being 'equal'!

If no domain is explicitly mentioned you can assume that the domain is all real numbers.

You may wonder why we would ever need any other domain. There are normally two reasons. We may be modelling a physical situation where the variables can only take particular values; for example if our variable was age of humans we would not expect it to be negative or much beyond 120.

Another possibility is that the mathematical operation we are using cannot handle certain types of numbers. For example if were looking for the largest prime factor of a number, we would normally only be looking at positive integers.

When working with real numbers, the three most important reasons to restrict the domain are

- you cannot divide by zero
- you cannot square root a negative number
- you cannot take the logarithm of a negative number or zero.

Worked example 5.6

What is the largest possible domain of $h : x \to \frac{1}{x-2} + \sqrt{x+3}$?

Look for division by zero	There will be division by zero when $x - 2 = 0$
Look for square rooting of a negative number	There will be square rooting of a negative number when $x + 3 < 0$
Decide what can therefore be allowed into the function	$x \geq -3$ and $x \neq 2$

Remember we can also use interval notation to write domains, so in the last example we could write $x \in [-3, 2[\cup]2, \infty[$.

Once we have set the domain, it is interesting to see what values can come out of the function.

KEY POINT 5.4

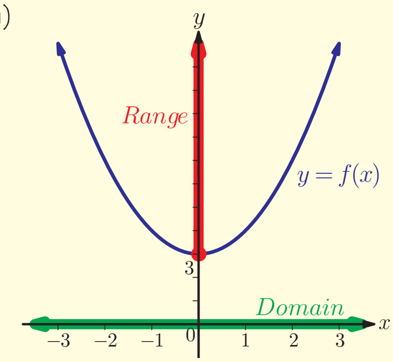
The set of all possible outputs of a function is called the **range**. The easiest way of finding this is to sketch the graph (possibly using your GDC). Be aware that the range will depend upon the domain.

Worked example 5.7

Find the range of $f(x) = x^2 + 3$ if the domain is: (a) $x \in \mathbb{R}$ (b) $x > 2$

Sketch the graph $y = f(x)$

(a)

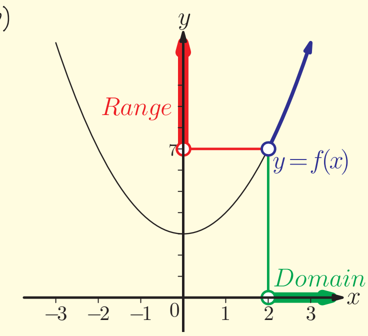


Use the graph to state what y values can occur

$y \geq 3$

Sketch the graph $y = f(x)$

(b)



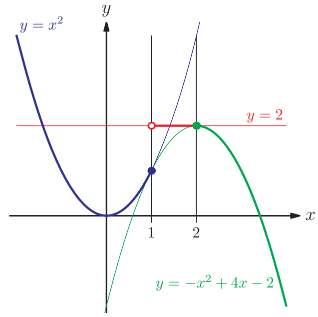
Use the graph to state what y values can occur given domain $x > 2$

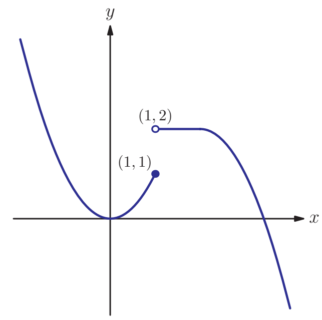
$y > 7$

A function does not have the same rule over all of its domain. For example, we could define a function $f(x)$ over all of $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2, & 1 < x < 2 \\ -x^2 + 4x - 2, & x \geq 2 \end{cases}$$

A function defined in this way is called a **piecewise** function. To sketch the graph of $f(x)$ we need to sketch all three graphs and then select the relevant graph for each part of the domain. Remember to use a closed circle when the end point is included and an open circle when it is not.





As we move from one part of the domain to another, the pieces of the graph may join up (as at $x = 1$ above) or there may be jump in the y-coordinate (as at $x = 1$). In the former case we say that the function is **continuous** at that point. So the function $f(x)$ above is continuous everywhere except at $x = 1$.

The easiest way to see whether a function is continuous is to sketch its graph. If this is not possible, we can look at the values of the function near the end points of different parts of the domain. For example, $f(1) = 1^2 = 1$, but when x is just larger than 1, $f(x) = 2$. Therefore, $f(x)$ is not continuous at $x = 1$. On the other hand, $f(2) = -2^2 + 4 \times 2 - 2 = 2$, and when x is just smaller than 2, $f(x) = 2$ also. So $f(x)$ is continuous at $x = 2$.

Worked example 5.8

A function is defined by

$$f(x) = \begin{cases} 2x + 1, & x \le 2 \\ k - x, & x > 2 \end{cases}$$

(a) If $k = 8$, find the range of $f(x)$.

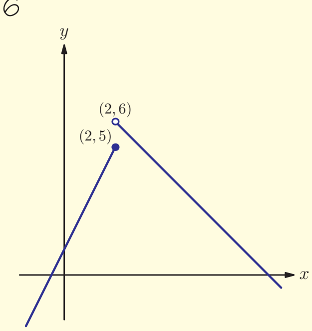
(b) Find the value of k for which $f(x)$ is continuous.

To see the range, sketch the graph. Mark the values at the end points of the different parts of the domain

(a) When $x = 2$:

$2x + 1 = 5$

$8 - x = 6$



continued . . .

There are no y values between 5 and 6. 5 is included in the range, but 6 is not

The range is $]-\infty,5]\cup]6,\infty[$

For $f(x)$ to be continuous, the two expressions have to have equal values at $x=2$

(b) If $f(x)$ is continuous:

$2x+1=k-x$ when $x=2$

$\Rightarrow \quad 5=k-2$

$\Rightarrow \quad k=7$

Exercise 5C

Exponential functions were covered in chapter 2.

1. State the largest real domain of the following functions, and their ranges, over those domains:
   (a) $f(x)=2^x$
   (b) $f(x)=a^x,\ a>0$
   (c) $f(x)=\log_{10}x$
   (d) $f(x)=\log_b x,\ b>0$

EXAM HINT

If a question asks for 'the domain' of a function, you should answer with the largest possible real domain.

2. Find the domain of the following functions.
   (a) (i) $f(x)=\dfrac{1}{x+2}$ (ii) $f(x)=\dfrac{5}{x-7}$
   (b) (i) $f(x)=\dfrac{3}{(x-2)(x+4)}$ (ii) $g(x)=\dfrac{x}{x^2-9}$
   (c) (i) $r(y)=\sqrt{y^3-1}$ (ii) $h(x)=\sqrt{x+3}$
   (d) (i) $f(a)=\dfrac{1}{\sqrt{a-1}}$ (ii) $f(x)=\dfrac{5x}{\sqrt{2-5x}}$
   (e) (i) $a(x)=\dfrac{1}{x}+\dfrac{2}{x+1}$ (ii) $f(x)=\sqrt{x+1}+\dfrac{1}{x+2}$
   (f) (i) $f(x)=\sqrt{x^2-5}$ (ii) $f(x)=4\sqrt{x^2+2x-3}$
   (g) (i) $f(x)=\sqrt{x}+\dfrac{1}{x+7}-x^3+5$
   (ii) $f(x)=e^x+\sqrt{2x+3}-\dfrac{1}{x^2+4}-2$

Some of these questions require solution of quadratic inequalities, covered in Prior learning Section Z on the CD-ROM.

3. Find the range of the following functions.
   (a) (i) $f(x)=7-x^2,\ x\in\mathbb{R}$ (ii) $f(x)=x^2+3,\ x\in\mathbb{R}$
   (b) (i) $g(x)=x^2+3,\ x\geq 3$ (ii) $h(x)=x+1,\ x>3,\ x\in\mathbb{Z}$
   (c) (i) $f(x)=|x-1|$ (ii) $f(x)=|2x+3|$
   (d) (i) $d(x)=\dfrac{1}{x},\ x\geq -1,\ x\neq 0$ (ii) $q(x)=3\sqrt{x},\ x>0$

What is the largest possible range and domain of the function $f(x)=(-2)^x$? This function illustrates why it is important to be careful in deciding how to define a continuous function; an important concept in higher mathematics.

4. For each of the following piecewise functions sketch the graph, find the range and state whether the function is continuous.

(a) (i) $f(x)=\begin{cases}2+x, & x\le 0\\ 3+x, & x>0\end{cases}$ (ii) $f(x)=\begin{cases}2-x, & x<0\\ 1-2x, & x\ge 0\end{cases}$

(b) (i) $f(x)=\begin{cases}x+3, & x<0\\ 2e^{-x}, & x\ge 0\end{cases}$ (ii) $f(x)=\begin{cases}3e^{x}, & x\le 0\\ x+3, & x>0\end{cases}$

(c) (i) $f(x)=\begin{cases}1, & x\le -1\\ x^2, & -1<x\le 2\\ 4, & x>2\end{cases}$ (ii) $f(x)=\begin{cases}0, & x<-2\\ 4-x^2, & -2<x\le 2\\ 2, & x>2\end{cases}$

5. The function f is given by $f(x)=\sqrt{\ln(x-4)}$. Find the domain of the function. *[4 marks]*

See Prior learning Section G on the CD-ROM for the definition of modulus function, $|x|$.

6. Find the largest possible domain of the function

$$f(x)=\frac{4^{\sqrt{x-1}}}{x+2}-\frac{1}{x^2-5x+6}+x^2+1.$$ *[5 marks]*

7. The function f is defined for all $x\in\mathbb{R}$ by:

$$f(x)=\begin{cases}3x^2-1, & x\le 2\\ a-x^2, & x>2\end{cases}$$

Find the value of a for which f is continuous. *[3 marks]*

8. Find the largest possible domain of the function $g(x)=\ln(x^2+3x+2)$. *[5 marks]*

9. Find the largest set of values of x such that the function f given by $f(x)=\sqrt{\dfrac{8x-4}{x-12}}$ takes real values. *[5 marks]*

10. (a) State the domain of the function $f(x)=\sqrt{x-a}+\ln(b-x)$ if

(i) $a<b$ (ii) $a>b$

(b) Evaluate $f(a)$. *[6 marks]*

5D Composite functions

We can apply one function to a number and then apply another function to that result.

The resulting rule is called a **composite function**.

KEY POINT 5.5

If we first apply the function g to x and then the function f to the result, we write:

$f(g(x))$ or $fg(x)$ or $f \circ g(x)$

It can be useful to refer to $g(x)$ as the **inner function** and $f(x)$ as the **outer function**.

For the composite function $fg(x)$ to exist, the range of $g(x)$ must lie entirely within the domain of $f(x)$, otherwise we would be trying to put values into $f(x)$ which cannot be calculated.

EXAM HINT

Any of these notations are all equally correct, so you can use whichever one suits you. In the exam you must be able to interpret any of them. Remember the correct order: The function nearest to x acts first!

Worked example 5.9

If $f(x) = x^2$ and $g(x) = x - 3$, find (a) $f \circ g(1)$ (b) $fg(x)$ (c) $gf(x)$

We need to evaluate $g(1)$ and then apply f to the result

(a) $g(1) = 1 - 3 = -2$
$f(-2) = (-2)^2$
$= 4$
$\therefore f(g(1)) = 4$

Replace x in $f(x)$ with the expression for $g(x)$

(b) $f(g(x)) = f(x-3) = (x-3)^2 = x^2 - 6x + 9$

Replace x in $g(x)$ with the expression for $f(x)$

(c) $g(f(x)) = g(x^2) = x^2 - 3$

EXAM HINT

Notice that to answer part (a) we did not need to work out the general expression for $f \circ g(x)$.

Notice that $f(g(x))$ and $g(f(x))$ are not the same function.

It is more difficult to recover one of the original functions from a composite function. The best way to do this is to use a substitution.

Worked example 5.10

If $f(x+1) = 4x^2 + x$, find $f(x)$.

Substitute $y =$ the inner function	$y = x + 1$
Rearrange to get x as the subject	$x = y - 1$
Replace all instances of x	$f(y) = 4(y-1)^2 + (y-1)$ $= 4y^2 - 8y + 4 + y - 1$ $= 4y^2 - 7y + 3$
We were asked to write the answer in terms of x	$f(x) = 4x^2 - 7x + 3$

Exercise 5D

1. If $f(x) = x^2 + 1$ and $g(x) = 3x + 2$, find:

(a) (i) $g(f(0))$ (ii) $fg(1)$

(b) (i) $gg(x)$ (ii) $f \circ g(x)$

(c) (i) $gg(\sqrt{a} + 1)$ (ii) $f \circ f(y - 1)$

(d) (i) $ggf(y)$ (ii) $gfg(z)$

2. Find $f(x)$ given the following conditions:

(a) (i) $f(2a) = 4a^2$ (ii) $f\left(\frac{b}{3}\right) = \frac{b^3}{27}$

(b) (i) $f(x+1) = 3x - 2$ (ii) $f(x-2) = x^2 + x$

(c) (i) $f(1-y) = 5 - y$ (ii) $f(y^3) = y^2$

(d) (i) $f(e^k) = \ln k$ (ii) $f(3n+2) = \ln(n+1)$

3. If $f(x) = x^2 + 1$ and $g(x) = 3x + 2$, solve $fg(x) = gf(x)$. *[4 marks]*

4. If $f(x) = 3x + 1$ and $g(x) = \dfrac{x}{x^2 + 25}$, solve $gf(x) = 0$. *[5 marks]*

5. Functions g and h are defined by

$$g(x) = \sqrt{x} \text{ and } h(x) = \frac{2x - 3}{x + 1} \quad (x \neq 1).$$

(a) Find the range of h.

(b) Solve the equation $h(x) = 0$.

(c) Find the domain and range of $g \circ h$. *[6 marks]*

6. The function f is defined by $f : x \mapsto x^3$. Find an expression for $g(x)$ in terms of x in each of the following cases.

(a) $f \circ g\,(x) = 2x + 3$

(b) $g \circ f(x) = 2x + 3$ *[6 marks]*

7. Functions f and g are defined by $f(x) = \sqrt{x^2 - 2x}$ and $g(x) = 3x + 4$. The composite function $f \circ g$ is undefined for $x \in]a, b[$.

(a) Find the value of a and the value of b.

(b) Find the range of $f \circ g$. *[7 marks]*

8. If $f(x) = x - 1,\ x > 3$ and $g(x) = x^2,\ x \in \mathbb{R}$:

(a) Show that $g \circ f$ exists and $f \circ g$ does not.

(b) Find the largest possible domain for g so that $f \circ g$ is defined. *[6 marks]*

9. Let f and g be two functions. Given that $f \circ g(x) = \dfrac{x + 2}{3}$ and $g(x) = 2x + 5$, find $f(x - 1)$. *[6 marks]*

5E Inverse functions

Functions transform an input into an output, but sometimes we need to reverse this process to be able to say which input produced a particular output. When this is possible, it is done by finding the **inverse function**, usually labelled f^{-1}.

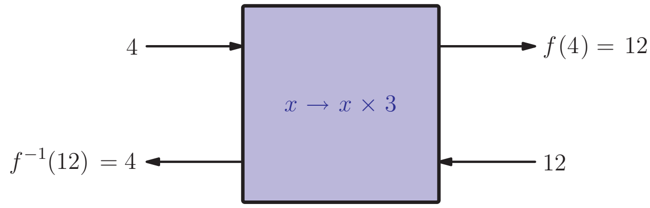
For example, if $f(x) = 3x$ then $f^{-1}(12)$ is a number which when put into f produces output 12. In other words, we are looking for a number x such that $f(x) = 12$, hence $f^{-1}(12) = 4$.

EXAM HINT

Make sure you do not get confused about this notation.

With numbers, the superscript '−1' denotes reciprocal, e.g. $x^{-1} = \frac{1}{x}$, $3^{-1} = \frac{1}{3}$

With functions, f^{-1} denotes the inverse function of f.



To find the inverse function you rearrange the formula to find the input (x) in terms of the output (y).

KEY POINT 5.6

To find the inverse function $f^{-1}(x)$ from an expression for $f(x)$

1. start off with $y = f(x)$
2. rearrange to get x (the input) in terms of y (the output) i.e. make x the subject
3. but this is $f^{-1}(y)$ and we can restate with x instead of y to show $f^{-1}(x)$.

Worked example 5.11

Find the inverse function of $f(x) = \frac{1+x}{3-x}$.

Set $y = f(x)$

$$y = \frac{1+x}{3-x}$$

Make x the subject

$$y(3-x) = 1+x$$
$$3y - yx = 1+x$$
$$3y - 1 = x + xy$$
$$3y - 1 = x(1+y)$$
$$x = \frac{3y-1}{1+y}$$
$$\therefore f^{-1}(y) = \frac{3y-1}{1+y}$$

Restate with x instead of y

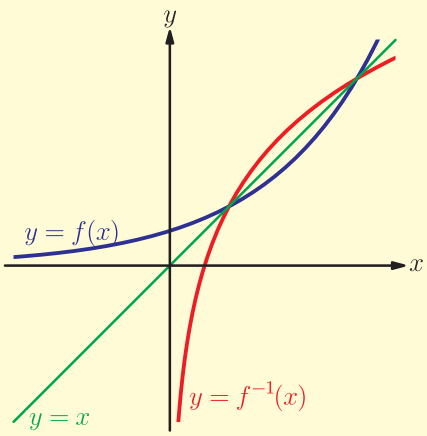
$$f^{-1}(x) = \frac{3x-1}{1+x}$$

Once we know how to find inverse functions, there are some important facts we need to know about them:

- When we are finding the inverse function we switch the inputs and the outputs, so on the graph we switch the x- and y-axis.

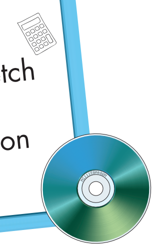
KEY POINT 5.7

The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$.



EXAM HINT

See Calculator sheet 7 on the CD-ROM to see how you can sketch the graph of the inverse function on your calculator.



- When you do and then undo a function you get back to where you started.

KEY POINT 5.8

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

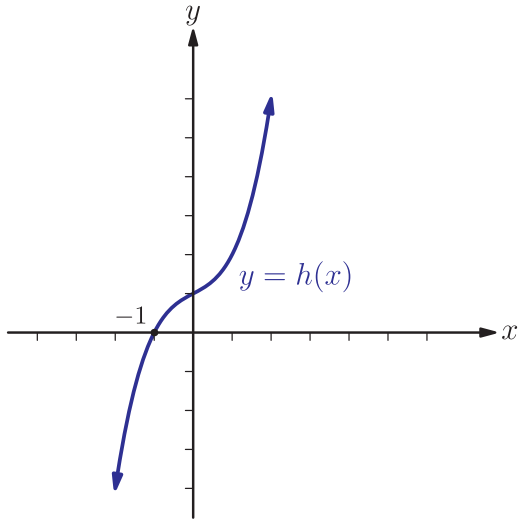
There are two pieces of vocabulary which you need to know:

A function which does not change the input is called an **identity function**, so the key point above says that the composition of a function and its inverse is the identity function.

For some functions, such as $f(x) = \dfrac{3}{x}$ or $g(x) = 5 - x$ the inverse function is the same as the original function so that $f \circ f(x)$ is the identity function. In this case the function is called a **self-inverse function**.

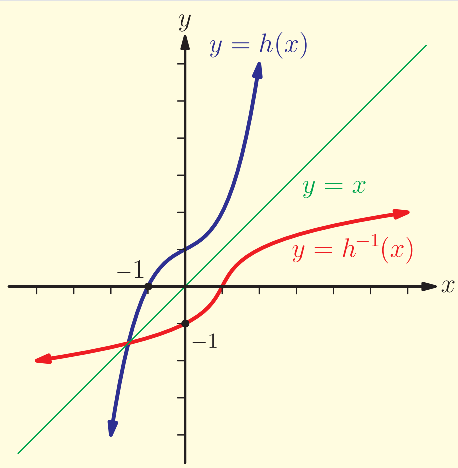
Worked example 5.12

The graph of $y = h(x)$ is shown below. Sketch the graphs of $y = h^{-1}(x)$ and $y = h \circ h^{-1}(x)$.



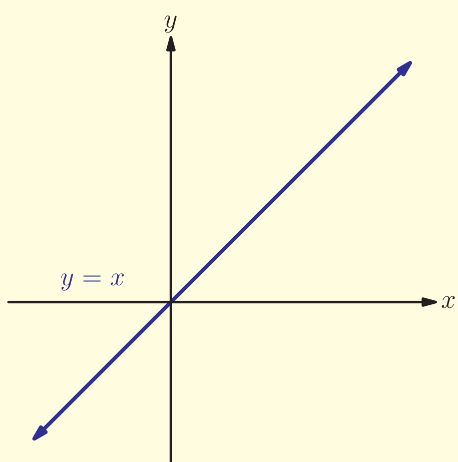
The graph of $y = h^{-1}(x)$ is a reflection in the line $y = x$ of $y = h(x)$

(a)



Simplify $y = h \circ h^{-1}(x)$ to $y = x$ and then sketch this over the domain of h

(b)



- The reflection in the line $y = x$ swaps the domain and the range of a function (because it swaps x- and y-coordinates).

KEY POINT 5.9

The domain of $f^{-1}(x)$ is the same as the range of $f(x)$.

The range of $f^{-1}(x)$ is the same as the domain of $f(x)$.

- All functions have inverse relations, but these inverses are not necessarily themselves functions. Since an inverse function is a reflection in the line $y = x$, for the result to pass the vertical line test the original function must pass the horizontal line test. But as we saw in Section 5A, this means it must be a one-to-one function.

KEY POINT 5.10

Only one-to-one functions have inverse functions.

This leads us to one of the most important uses of domains. By restricting the domain we can turn any function into a one-to-one function, which allows us to find its inverse function.

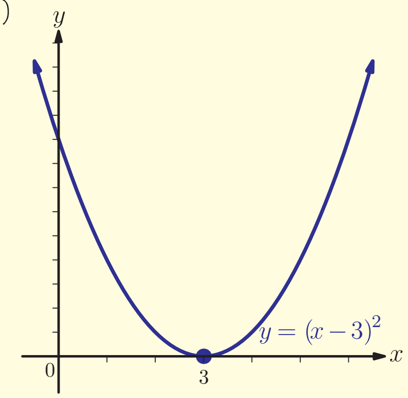
Worked example 5.13

(a) Find the largest value of k such that the function $f(x) = (x-3)^2,\ x \le k$ is one-to-one.

(b) Find $f^{-1}(x)$ for this value of k and state its range.

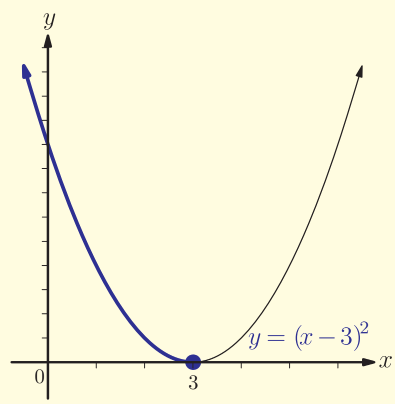
Sketch the graph of $y = (x - 3)^2,\ x \in \mathbb{R}$

(a)



continued . . .

Eliminate the points towards the right of the graph which cause the horizontal line test to fail



Decide which section remains

$x \leq 3$

Follow the standard procedure for finding inverse functions

(b) $y = (x-3)^2$

$\pm\sqrt{y} = x - 3$

$x = 3 \pm \sqrt{y}$

Use the fact that $x \leq 3$ to decide which root to take

Since $x \leq 3$

$x = 3 - \sqrt{y}$

Write $f^{-1}(x) = \ldots$

$f^{-1}(x) = 3 - \sqrt{x}$

The range of f^{-1} is the domain of f

The range of f^{-1} is $y \leq 3$

EXAM HINT

You can check that your expression for $f^{-1}(x)$ is correct by sketching f and f^{-1} on the same graph and looking for symmetry. Make sure you use the correct domain.

Exercise 5E

1. Find $f^{-1}(x)$ if:

(a) (i) $f(x) = 3x + 1$ (ii) $f(x) = 7x - 3$

(b) (i) $f(x) = \dfrac{2x}{3x-2},\ x \neq \dfrac{2}{3}$ (ii) $f(x) = \dfrac{x}{2x+1},\ x \neq \dfrac{-1}{2}$

(c) (i) $f(x) = \dfrac{x-a}{x-b},\ x \neq b$ (ii) $f(x) = \dfrac{ax-1}{bx-1},\ x \neq \dfrac{1}{b}$

(d) (i) $f(a) = 1 - a$ (ii) $f(y) = 3y + 2$

(e) (i) $f(x) = \sqrt{3x-2},\ x \geq \frac{2}{3}$ (ii) $f(x) = \sqrt{2-5x},\ x \leq \frac{2}{5}$

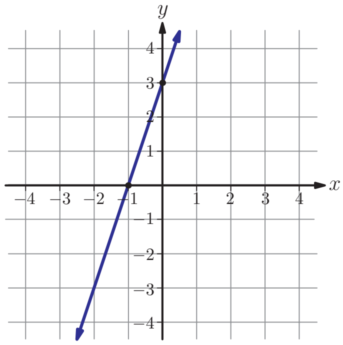
(f) (i) $f(x) = \ln(1-5x),\ x < 0.2$ (ii) $f(x) = \ln(2x+2),\ x > -1$

(g) (i) $f(x) = 7e^{\frac{x}{2}}$ (ii) $f(x) = 9e^{10x}$

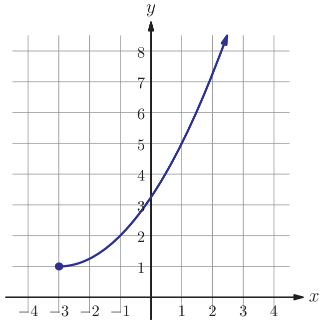
(h) (i) $f(x) = x^2 - 10x + 6,\ x < 5$ (ii) $f(x) = x^2 + 6x - 1,\ x > 0$

2. Sketch the inverse relations of the following functions.

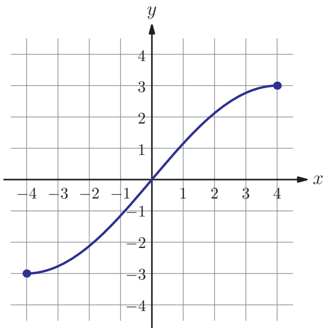
(a)



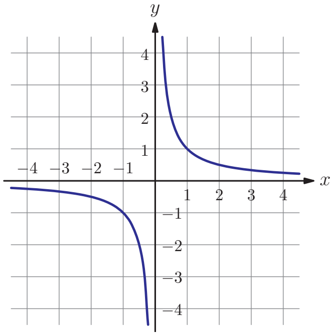
(b)



(c)



(d)



3. For each of the following functions, find and simplify $f \circ f(x)$ and hence state if the function is self-inverse.

(a) (i) $f : x \mapsto -x$ (ii) $f : x \mapsto x$

(b) (i) $f : x \mapsto 2x$ (ii) $f : x \mapsto \frac{x}{-4}$

(c) (i) $f : x \mapsto 10 + x$ (ii) $f : x \mapsto 1 + x$

(d) (i) $f : x \mapsto 7 - x$ (ii) $f : x \mapsto 1 - x$

(e) (i) $f : x \mapsto \frac{10}{x}$ (ii) $f : x \mapsto \frac{-3}{x}$

(f) (i) $f : x \mapsto \frac{3}{x^2}$ (ii) $f : x \mapsto \frac{1}{x^3}$

(g) (i) $f : x \mapsto \sqrt{x}$ (ii) $f : x \mapsto \sqrt[3]{x}$

(h) (i) $f: x \mapsto \dfrac{1}{1-x}$ (ii) $f: x \mapsto \dfrac{1}{1+x}$

(i) (i) $f: x \mapsto \dfrac{2x+3}{x-2}$ (ii) $f: x \mapsto \dfrac{1-3x}{3+x}$

4. Consider the function $f(x) = \sqrt{x^2}$. Over what domain is this an identity function?

5. Below is a table giving selected values of the one-to-one function $f(x)$:

x	-1	0	1	2	3	4
$f(x)$	-4	-1	3	0	7	2

(a) Evaluate $ff(2)$.

(b) Evaluate $f^{-1}(3)$. *[4 marks]*

6. The function f is defined by $f: x \mapsto \sqrt{3-2x}, \quad x \le \dfrac{3}{2}$.

Evaluate $f^{-1}(7)$. *[4 marks]*

7. Given that $f(x) = 3e^{2x}$, find the inverse function $f^{-1}(x)$. *[4 marks]*

8. Given functions $f: x \mapsto 2x+3$ and $g: x \mapsto x^3$, find the function $(f \circ g)^{-1}$. *[5 marks]*

9. The functions f and g are defined by: $f: x \mapsto e^{2x}$, $g: x \mapsto x+1$:

(a) Calculate $f^{-1}(3) \times g^{-1}(3)$.

(b) Show that $(f \circ g)^{-1}(3) = \ln\sqrt{3} - 1$. *[6 marks]*

10. Let $f(x) = \sqrt{x}$, and $g(x) = 2^x$. Solve the equation:

$f^{-1} \circ g(x) = 0.25$. *[5 marks]*

11. The function f is defined for $x \le 0$ by $f(x) = \dfrac{x^2-4}{x^2+9}$.

Find an expression for $f^{-1}(x)$. *[5 marks]*

12. For the following functions, find the value of k which gives the largest possible domain such that the inverse function exists. For this domain, find the inverse function.

(a) $y = x^2,\ x \le k$

(b) $y = (x+1)^2 + 2,\ x \ge k$

(c) $y = |x|,\ x \le k$ *[6 marks]*

13. Let $f(x) = \ln(x-1) + \ln 3$, for $x > 1$.

(a) Find $f^{-1}(x)$.

(b) Let $g(x) = e^x$. Find $(g \circ f)(x)$, giving your answer in the form $ax+b$, where $a, b \in \mathbb{Z}$. *[7 marks]*

14. A piecewise function is defined by:

$$f(x)=\begin{cases}2+(x-1)^2, & x<1\\ k-(x-1)^2, & x\ge 1\end{cases}$$

(a) Find the largest value of k for which f is a one-to-one function.

(b) When $k=0$:

(i) Find the range of f.

(ii) Find an expression for $f^{-1}(x)$. *[8 marks]*

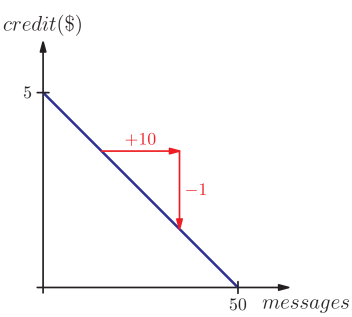
15. A function is called *self-inverse* if $f(x)=f^{-1}(x)$ for all x in the domain.

(a) Show that $f(x)=\dfrac{1}{x}$, $x\neq 0$ is a self-inverse function.

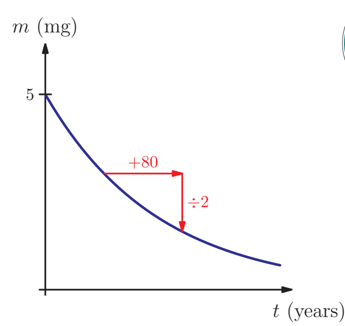
(b) Find the value of the constant k so that $g(x)=\dfrac{3x-5}{x+k}$, $x\neq k$ is a self-inverse function. *[8 marks]*

5F Rational functions

There are many situations where one quantity deceases as another increases. For example, the amount of your phone credit decreases as the number of text messages you send increases. As the number of messages increases by a fixed number, the credit decreases by a fixed amount; this is called **linear decay**. Another example is radioactive decay, where the amount of a radioactive substance halves in a fixed time period (called half-life). So as the time increases by a fixed number, the amount of substance decreases by a fixed factor; this is called **exponential decay**.

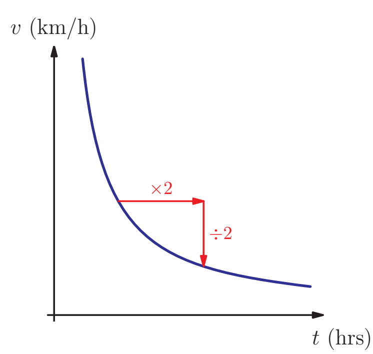


linear decay



exponential decay

Exponential functions are covered in chapter 2, and linear functions in Section R of Prior learning.

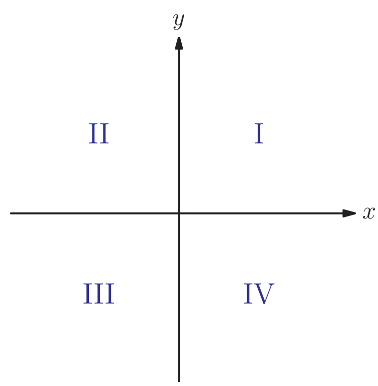


In this chapter we will look at a third type of decay, called **inverse proportion**, where as one quantity increases by a fixed factor, another decreases by the same factor. For example, if you double your speed, the amount of time to travel a given distance will halve. If the total distance travelled is 12 km, then the equation for time (in hours) in terms of speed (in km/h) is $t=12/v$. This is an example of a **reciprocal function**.

EXAM HINT

The reciprocal of a real number $x \neq 0$ is $\frac{1}{x}$. For example, the reciprocal of -2 is $-\frac{1}{2}$ and the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Graphs of reciprocal functions all have the same shape, called a **hyperbola**. This curve consists of two parts and has the axis as the asymptotes. This means that the function is not defined for $x = 0$, and that as x gets very large (positive or negative), y approaches zero. Therefore neither x nor y can equal zero. The two parts of the curve can be either in the first and third, or the second and fourth quadrants, depending on the sign of k.



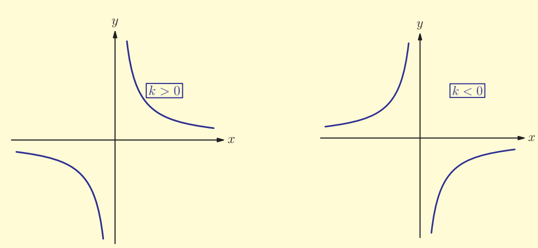
Quadrants

KEY POINT 5.11

A reciprocal function has the form $f(x) = \frac{k}{x}$.

The domain of f is $x \neq 0$ and the range is $y \neq 0$.

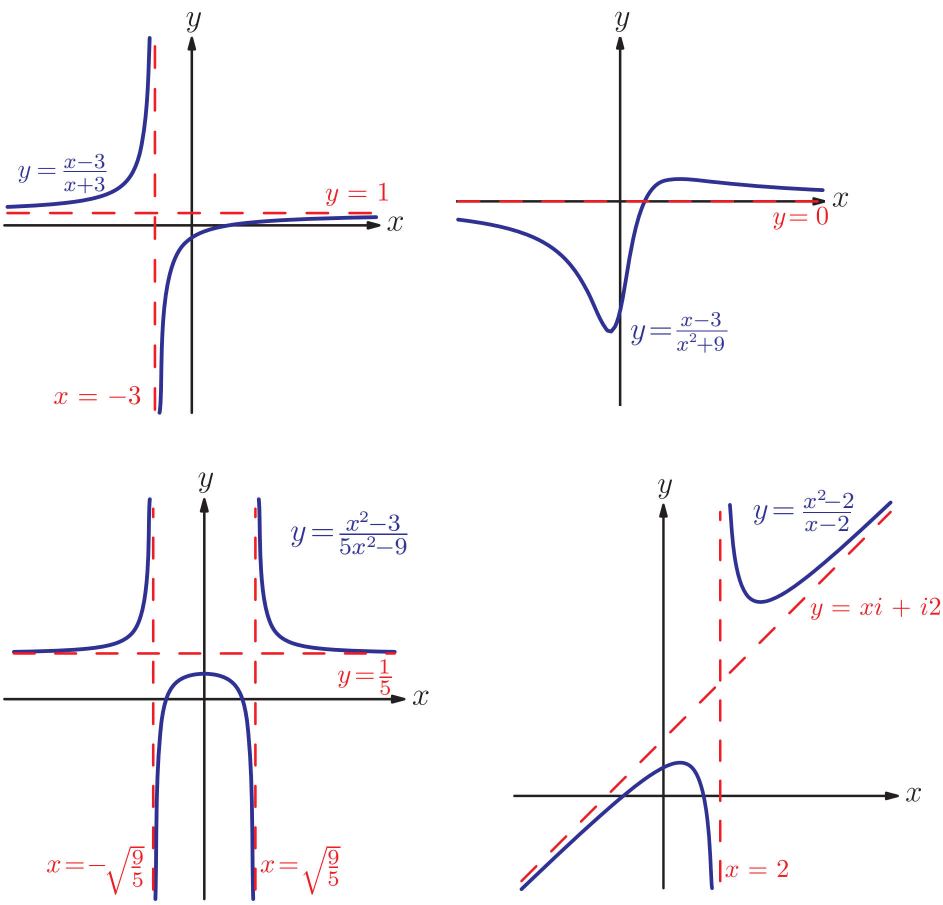
The graph of $f(x)$ is a hyperbola.



Related to reciprocal functions, **rational functions** are a ratio of two polynomials: $f(x) = \frac{p(x)}{q(x)}$.

Rational functions can take many different shapes, depending on the polynomials $p(x)$ and $q(x)$, their orders and how many zeros they have.

Here are some examples of the types of graphs we can get:



One very common feature of rational functions is asymptotes. Three of the four graphs above have **vertical asymptotes**, corresponding to the values of x where the rational function is not defined because of division by zero. The first three graphs have **horizontal asymptotes**, corresponding to the value of y the function approaches as x gets very large.

We have already encountered asymptotes in Sections 2B and 2F.

If the order of the numerator is greater than the order of the denominator then, as x gets larger, the graph gets closer and closer to a polynomial curve.

Understanding this asymptotic behaviour is important in applications which deal with very large numbers (such as in astronomy), but it is also used in some more unexpected situations, for example in looking for prime numbers.

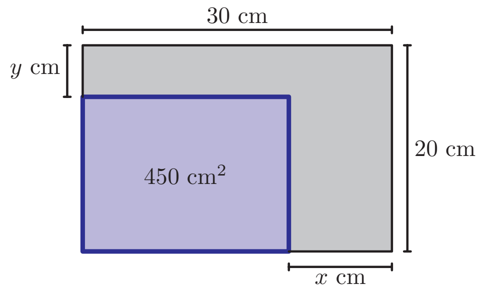
We will only consider rational functions of the form $f(x)=\frac{ax+b}{cx+d}$. The following example illustrates one application of such functions.

Worked example 5.14

A rectangular piece of card has dimensions 30 cm by 20 cm. Strips of width x cm and y cm are cut off the ends, so that the remaining card has area 450 cm^2.

(a) Find an expression for y in terms of x in the form $y=\frac{ax-b}{cx-d}$.

(b) Sketch the graph of y against x.



Write the equation for the remaining area in terms of x and y

(a) $(30-x)(20-y)=450$

We want to make y the subject, so divide by (30 – x) rather than expanding the brackets

$\Leftrightarrow 20-y=\frac{450}{30-x}$

$\Leftrightarrow \quad y=20-\frac{450}{30-x}$

$=\frac{20(30-x)-450}{30-x}$

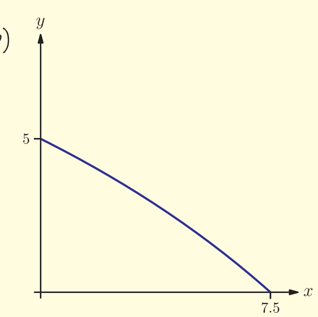
$=\frac{150-20x}{30-x}$

We can multiply top and bottom by –1

$\Leftrightarrow y=\frac{20x-150}{x-30}$

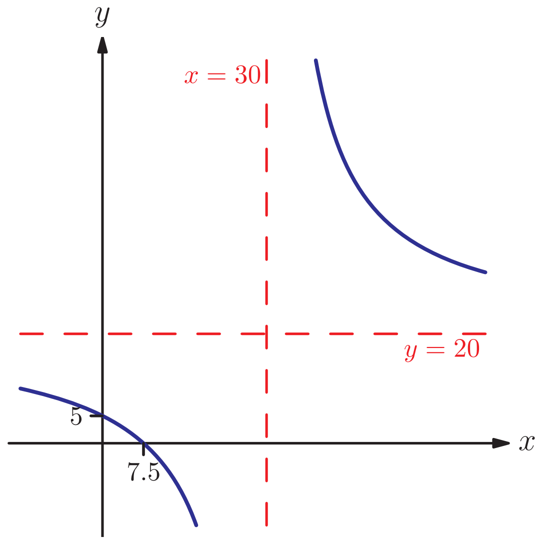
We can use GDC to sketch the graph
Only positive values of x and y are relevant

(b)



In the above example only positive values of x and y were relevant, but we can look at the function $f(x) = \frac{20x - 150}{x - 30}$ over the whole of $\mathbb{R}$. The function is not defined for $x = 30$ (as we would be dividing by zero), so there is a vertical asymptote there. The y-intercept is $(0,5)$. We can find the x-intercept by setting the top of the fraction equal to zero – it is $(7.5,0)$. We can use the calculator to sketch the graph. The curve looks like a hyperbola with the horizontal asymptote $y = 20$.

Is zero the same as nothing? What happens if you say that the result of dividing by zero is infinity? What would $\frac{0}{0}$ be?



The position of the horizontal asymptote can be found by looking at the first equation we found for y in the worked example, which can also be written as $y = 20 + \frac{450}{x - 30}$. As x gets very large (either positive or negative), $x - 30$ gets very large, so $\frac{450}{x - 30}$ gets very small. Therefore the value of y gets closer and closer to 20. Another way to find the asymptote is to think about what happens as x gets large in the equation $y = \frac{20x - 150}{x - 30}$. The terms containing x become much larger than 150 and 30, so the two constant terms can be ignored, leaving $y \approx \frac{20x}{x} = 20$.

The above example illustrates all the important properties of rational functions of the form $f(x) = \frac{ax + b}{cx + d}$.

EXAM HINT

In the examination you can just use the result that the horizontal asymptote is found by dividing the two coefficients of x.

EXAM HINT

Make sure you include all asymptotes and intercepts when sketching graphs of rational functions. The intercepts should help you decide which quadrants the graph is in.

KEY POINT 5.12

The graph of a **rational function** of the form $f(x)=\dfrac{ax+b}{cx+d}$ is a hyperbola with:

- vertical asymptote $x=-\dfrac{d}{c}$ (this is where $cx+d=0$)
- horizontal asymptote $y=\dfrac{a}{c}$
- x-intercept at $x=-\dfrac{b}{a}$ (this is where $ax+b=0$)
- y-intercept at $y=\dfrac{b}{d}$ (this is where $x=0$)

Knowing the position of the asymptotes gives you the domain and range of the function.

Worked example 5.15

Find the domain and range of the function $f(x)=\dfrac{3x-4}{2x+1}$.

The only value excluded from the domain is when the denominator is zero

$2x+1=0$

$\Rightarrow x=-\dfrac{1}{2}$

The domain is:

$x\in\mathbb{R}, x\neq-\dfrac{1}{2}$

Sketching the graph can show us the range. We need to find the horizontal asymptote: divide the coefficients of x

Horizontal asymptote:

$y=\dfrac{3}{2}$

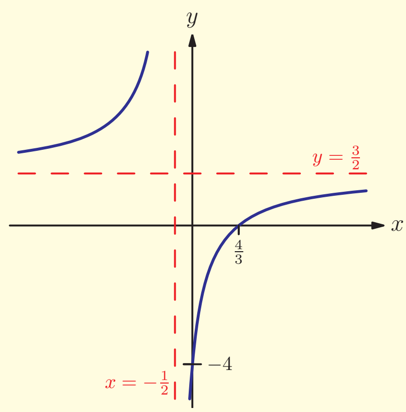
Find the intercepts to decide which quadrants the graph is in

Intercepts:

$x=0\Rightarrow y=-4$

$y=0\Rightarrow 3x-4=0\Rightarrow x=\dfrac{4}{3}$

continued . . .



The range is:

$y \in \mathbb{R}, y \neq \frac{3}{2}$

Is there any point learning how to sketch graphs when you have a graphical calculator (aside from the fact that there is a non-calculator paper!)? Do you sometimes need knowledge in order to acquire new knowledge?

Exercise 5F

1. Find the coordinates of all axis intercepts of the following rational functions:

(a) (i) $f(x) = \frac{3x+1}{x+3}$ (ii) $f(x) = \frac{2x+5}{x+1}$

(b) (i) $f(x) = \frac{2x-3}{2x+7}$ (ii) $f(x) = \frac{3x-5}{x+2}$

2. Find the equations of the asymptotes of the following graphs:

(a) (i) $y = \frac{4x+3}{x-1}$ (ii) $y = \frac{2x+1}{x-7}$

(b) (i) $y = \frac{3x+2}{2x-1}$ (ii) $y = \frac{4x+1}{3x-5}$

(c) (i) $y = \frac{3-x}{2x+5}$ (ii) $y = \frac{2x+1}{2-3x}$

(d) (i) $y = \frac{3}{x-2}$ (ii) $y = \frac{2}{2x+1}$

3. Sketch the graphs of the following rational functions, labelling all the axis intercepts and asymptotes.

(a) (i) $y = \frac{2x+1}{x-2}$ (ii) $y = \frac{3x+1}{x-3}$

(b) (i) $y = \frac{x-3}{4-x}$ (ii) $y = \frac{5-x}{x-2}$

(c) (i) $y = \frac{2}{x+3}$ (ii) $y = \frac{1}{x-2}$

(d) (i) $y = -\frac{3}{x}$ (ii) $y = -\frac{2}{x}$

4. Find the domain, range and the inverse function of the following rational functions:

(a) (i) $f(x) = \frac{3}{x}$ (ii) $f(x) = \frac{7}{x}$

(b) (i) $f(x) = \frac{2}{x-3}$ (ii) $f(x) = \frac{5}{x+1}$

(c) (i) $f(x) = \frac{2x+1}{3x-1}$ (ii) $f(x) = \frac{4x-5}{2x+1}$

(d) (i) $f(x) = \frac{5-2x}{x+2}$ (ii) $f(x) = \frac{3x-1}{4x-3}$

5. Find the equations of the asymptotes of the graph of $y = \frac{3x-1}{4-5x}$. *[3 marks]*

6. (a) Find the domain and range of the function $f(x) = \frac{1}{x+3}$.

(b) Find $f^{-1}(x)$. *[5 marks]*

7. Sketch the graph of $y = \frac{3x-1}{x-5}$. *[5 marks]*

8. A function is defined by $f : x \mapsto \frac{ax+3}{2x-8}, x \neq 4$, where $a \in \mathbb{R}$.

(a) Find, in terms of *a*, the range of *f*.

(b) Find the inverse function $f^{-1}(x)$.

(c) Find the value of *a* such that *f* is a self-inverse function. *[5 marks]*

Summary

- In this chapter we have looked at the difference between a function and a relation.
  - a **relation** is any set of paired inputs and outputs, while a **function** is a particular sort of relation where for each input there is only one output
  - we can distinguish between these using the **vertical line test**.
- We met two different types of function.
  - in a **one-to-one function**, each output comes from a unique input
  - in a **many-to-one function**, some outputs may come from more than one input
  - we distinguish between these using the **horizontal line test**.
- To fully define a function, as well as stating a rule, we also need to know its **domain** – the set of allowed inputs. Once we know the domain we can also find the range – the set of outputs resulting from that domain.
- A function can act upon the output of another function. The effect is called a **composite function** and we write $f(g(x))$ or $fg(x)$ or $f \circ g(x)$ for g followed by f.
- Reversing the effect of a function is achieved by applying an **inverse function**, $f^{-1}(x)$.
- Only one-to-one functions have inverse functions. The general method to find an inverse function is:
  - start with $y = f(x)$
  - rearrange to get x in terms of y
  - replace each appearance of y with an x.
- You need to know these properties of **inverse functions**
  - the graph of the inverse function is a reflection in the line $y = x$ of the graph of the original function
  - the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function
  - composing a function with its inverse gives the identity function:

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

- A **self-inverse function** has $f^{-1}(x) = f(x)$. This is equivalent to $f\left(f(x)\right) = x$.

- Rational functions of the form $f(x)=\dfrac{ax+b}{cx+d}$ have a graph called a **hyperbola** with the following properties:
  - vertical asymptote $x=-\dfrac{d}{c}$
  - horizontal asymptote $y=\dfrac{a}{c}$
  - x-intercept at $x=-\dfrac{b}{a}$
  - y-intercept at $y=\dfrac{b}{d}$
- A special case of a rational function is a **reciprocal function**, $f(x)=\dfrac{k}{x}$, which is an example of a self-inverse function.

Introductory problem revisited

> Think of any number. Add 3. Double your answer. Take away 6. Divide by the number you first thought of. Is your answer always prime? Why?

We can write the introductory problem as a function:

$$f(x)=\frac{2(x+3)-6}{x}$$

In most cases this simplifies to give 2, which is a prime number.

However, we should now be used to the idea that a function is about more than just a rule, it also needs a domain. The domain for this function cannot include zero, so it gives a prime number for any input other than zero, when the process fails.

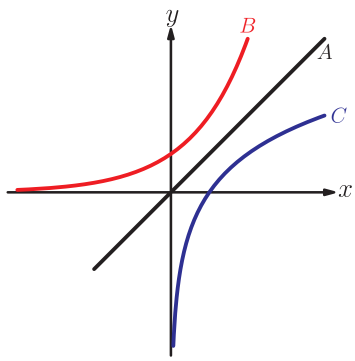
Mixed examination practice 5

Short questions

1. Find the inverses of the following functions:

(a) $f(x) = \log_3(x+3),\ x > 0$ (b) $g(x) = 3e^{x^3-1}$ *[5 marks]*

2. The diagram shows three graphs.



A is part of the graph of $y = x$.
B is part of the graph of $y = 2^x$.
C is the reflection of graph B in line A.
Write down:

(a) The equation of C in the form $y = f(x)$

(b) The coordinates of the point where C cuts the x-axis. *[5 marks]*

3. (a) Write down the equations of all asymptotes of the graph of $y = \frac{4x-3}{5-x}$.

(b) Find the inverse function of $f(x) = \frac{4x-3}{5-x}$. *[6 marks]*

4. The function f is given by $f(x) = x^2 - 6x + 10$, for $x \geq 3$.

(a) Write $f(x)$ in the form $(x-p)^2 + q$.

(b) Find the inverse function $f^{-1}(x)$.

(c) State the domain of $f^{-1}(x)$. *[6 marks]*

5. If $h(x) = x^2 - 6x + 2$:

(a) Write $h(x)$ in the form $(x-p)^2 + q$.

(b) Hence or otherwise find the range of $h(x)$.

(c) By using the largest possible domain of the form $x > k$ where, find the inverse function $h^{-1}(x)$. *[7 marks]*

6. The function $f(x)$ is defined by $f(x)=\dfrac{3-x}{x+1}, x\neq -1$.

(a) Find the range of f.

(b) Sketch the graph of $y=f(x)$.

(c) Find the inverse function of f in the form $f^{-1}(x)=\dfrac{ax+b}{cx+d}$.
State its domain and range. *[11 marks]*

7. A function is defined by:

$$f(x)=\begin{cases} 5-x, & x<0 \\ pe^{-x}, & x\geq 0 \end{cases}$$

(a) Given that $p=3$,

(i) Find the range of $f(x)$.

(ii) Find an expression for $f^{-1}(x)$ and state its domain.

(b) Find the value of p for which $f(x)$ is continuous. *[7 marks]*

8. The functions $f(x)$ and $g(x)$ are given by $f(x)=\sqrt{x-2}$ and $g(x)=x^2+x$.
The function $f\circ g(x)$ is defined for $x\in\mathbb{R}$ except for the interval $]a,b[$.

(a) Calculate the value of a and of b.

(b) Find the range of $f\circ g$. *[7 marks]*

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Long questions

1. If $f(x)=x^2+1,\ x>3$ and $g(x)=5-x$:

(a) evaluate $f(3)$.

(b) Find and simplify an expression for $gf(x)$.

(c) State the geometric relationship between the graphs of $y=f(x)$ and $y=f^{-1}(x)$.

(d) (i) Find an expression for $f^{-1}(x)$.

(ii) Find the range of $f^{-1}(x)$.

(iii) Find the domain of $f^{-1}(x)$.

(e) Solve the equation $f(x)=g(3x)$. *[10 marks]*

2. If $f(x)=2x+1$ and $g(x)=\dfrac{x+3}{x-1},\ x\neq 1$

(a) find and simplify

(i) $f(7)$ (ii) the range of $f(x)$

(iii) $fg(x)$ (iv) $ff(x)$

(b) Explain why $gf(x)$ does not exist.

(c) (i) Find the form of $g^{-1}(x)$.

(ii) State the domain of $g^{-1}(x)$.

(iii) State the range of $g^{-1}(x)$. *[9 marks]*

3. The functions f and g are defined over the domain of all real numbers, $g(x) = e^x$.

(a) Write $f(x) = x^2 + 4x + 9 \;\; x \in \mathbb{R}$ in the form $f(x) = (x + p)^2 + q$.

(b) Hence sketch the graph of $y = x^2 + 4x + 9$, labelling carefully all axes intercepts and the coordinates of the turning point.

(c) State the range of $f(x)$ and $g(x)$.

(d) Hence or otherwise find the range of $h(x) = e^{2x} + 4e^x + 9$. *[10 marks]*

4. Given that $(2x + 3)(4 - y) = 12$ for $x, y \in \mathbb{R}$:

(a) Write y in terms of x, giving your answer in the form $y = \dfrac{ax + b}{cx + d}$.

(b) Sketch the graph of y against x.

(c) Let $g(x) = 2x + k$ and $h(x) = \dfrac{8x}{2x + 3}$.

(i) Find $h(g(x))$.

(ii) Write down the equations of the asymptotes of the graph of $y = h(g(x))$.

(iii) Show that when $k = -\dfrac{19}{2}$, $h(g(x))$ is a self-inverse function. *[17 marks]*

5. (a) Show that if $g(x) = \dfrac{1}{x}$ then $gg(x) = x$.

(b) A function satisfies the identity $f(x) + 2f\left(\dfrac{1}{x}\right) = 2x + 1$.

By replacing all instances of x with $\dfrac{1}{x}$, find another identity that $f(x)$ satisfies.

(c) By solving these two identities simultaneously, express $f(x)$ in terms of x. *[10 marks]*

6 Transformations of graphs

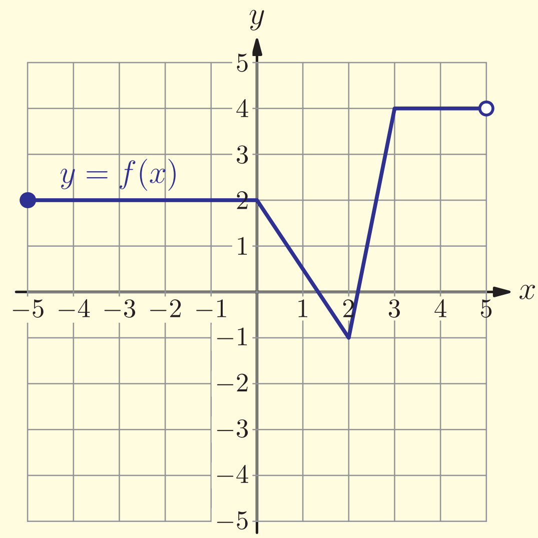
In this chapter you will learn:

- how some changes to functions affect their graphs
- how to use the modulus functions to transform graphs
- how to sketch complicated functions by considering them as transformations of simpler functions
- about graphs of reciprocal functions
- how the symmetries of a graph can be seen from its equation.

Introductory problem

Here is the graph of $y = f(x)$.

Sketch the graph of $y = \dfrac{1}{f(x)}$.

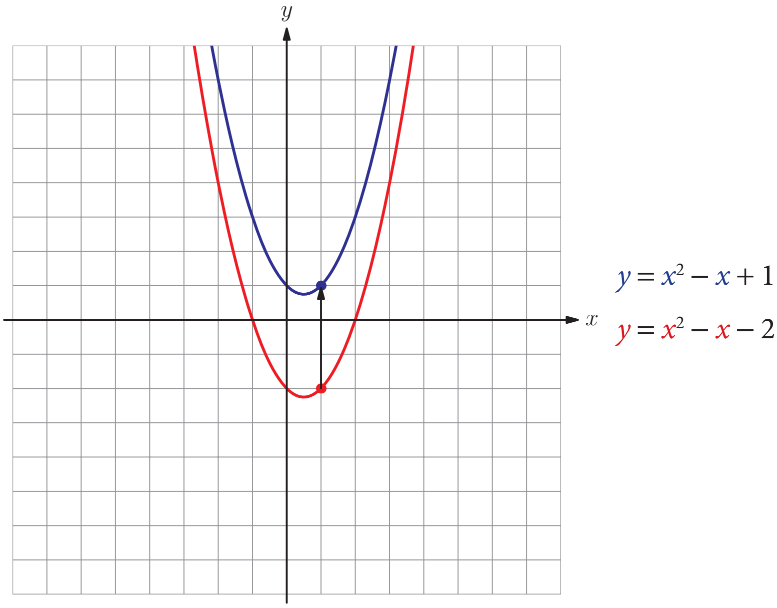


You have met various transformations which can be applied to two-dimensional shapes: translations, enlargements, reflections and rotations. In this chapter you will learn how you can apply translations, stretches and reflections to a graph by changing parts of its equation.

You can investigate some of these transformations for yourself; see Self-Discovery worksheet 2 'Changing functions and their graphs' on the CD-ROM. Here we will summarise and explain the results.

6A Translations

Compare the graphs of two functions that only differ by a constant:

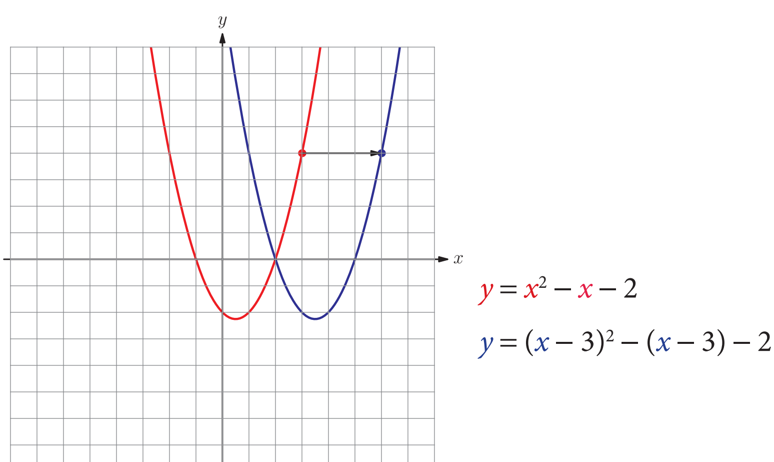


When the x-coordinates on the two graphs are the same (where $x = x$) the y-coordinates differ by 3 ($y = y + 3$). We can interpret this as meaning that at the same x-coordinates the blue graph is three units above the red graph – it has been translated vertically.

KEY POINT 6.1

> The graph $y = f(x) + c$ is the graph of $y = f(x)$ translated up by c units.
>
> If c is negative the graph is translated down.

In the next example, the blue function is obtained from the red function by replacing x by $x - 3$.



Mathematics is often thought to be a 'universal' language and it is not easy to find examples where our language influences mathematics. However, the educational researcher Bill Barton suggests that the Cartesian coordinate system we are using to visualise functions is a reflection of our own culture, with everything being measured relative to a fixed origin. He suggests that Maori mathematics, where it is considered rude to describe a position relative to just one person, leads to a different coordinate system which is described on Supplementary sheet 4 'Coordinate systems and graphs'.

*A translation can also be described using a vector: $\begin{pmatrix} a \\ b \end{pmatrix}$ means translate **a** units horizontally and **b** units vertically. You will learn more about vectors in chapter 13.*

If $x = x$ there is nothing useful we can say about the relationship between y and y.

However, a way of making $y = y$ is to have $x = x - 3$ or equivalently $x + 3 = x$. We can interpret this as meaning that when the two graphs have the same height, the blue graph is three units to the right of the red graph; it has been translated horizontally.

KEY POINT 6.2

The graph $y = f(x+d)$ is the graph of $y = f(x)$ translated left by d units. If d is negative the graph is translated right.

Worked example 6.1

The graph of $y = x^2 + 2x$ is translated 5 units to the left.

Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

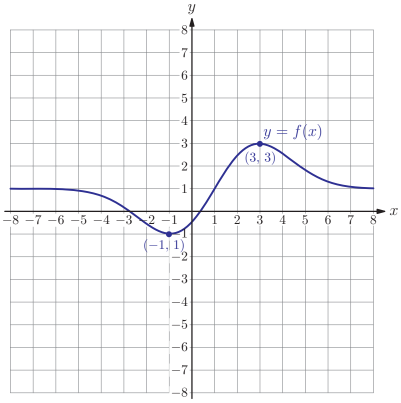
Relate the transformation to function notation

If $f(x) = x^2 + 2x$ then the new graph is $y = f(x+5)$

Replace all instances of x in the equation by $(x + 5)$

$y = (x+5)^2 + 2(x+5)$
$= x^2 + 12x + 35$

Exercise 6A



1. The graph shows a function $y = f(x)$. Sketch the graphs of the following functions, including the positions of the minimum and maximum points.
   (a) (i) $y = f(x) + 3$ (ii) $y = f(x) + 5$
   (b) (i) $y = f(x) - 7$ (ii) $y = f(x) - 0.5$
   (c) (i) $y = f(x + 2)$ (ii) $y = f(x + 4)$
   (d) (i) $y = f(x - 1.5)$ (ii) $y = f(x - 2)$
2. Find the equation of the graph after the given transformation is applied.
   (a) (i) $y = 3x^2$ after a translation of 3 units vertically up.
   (ii) $y = 9x^3$ after a translation of 7 units vertically down.
   (b) (i) $y = 7x^3 - 3x + 6$ after a translation of 2 units down.
   (ii) $y = 8x^2 - 7x + 1$ after a translation of 5 units up.

(c) (i) $y = 4x^2$ after a translation of 5 units to the right.

(ii) $y = 7x^2$ after a translation of 3 units to the left.

(d) (i) $y = 3x^3 - 5x^2 + 4$ after a translation of 4 units to the left.

(ii) $y = x^3 + 6x + 2$ after a translation of 3 units to the right.

3. Find the required translations.

(a) (i) Transform the graph $y = x^2 + 3x + 7$ to the graph $y = x^2 + 3x + 2$.

(ii) Transform the graph $y = x^3 - 5x$ to the graph $y = x^3 - 5x - 4$.

(b) (i) Transform the graph $y = x^2 + 2x + 7$ to the graph $y = (x+1)^2 + 2(x+1) + 7$.

(ii) Transform the graph $y = x^2 + 5x - 2$ to the graph $y = (x+5)^2 + 5(x+5) - 2$.

(c) (i) Transform the graph $y = e^x + x^2$ to the graph $y = e^{x-4} + (x-4)^2$.

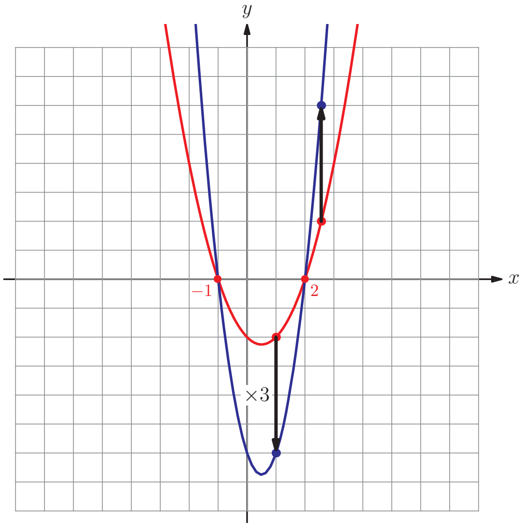
(ii) Transform the graph $y = \log(3x) - \sqrt{4x}$ to the graph $y = \log(3(x-5)) - \sqrt{4(x-5)}$.

(d) (i) Transform the graph $y = \ln(4x)$ to the graph $y = \ln(4x + 12)$.

(ii) Transform the graph $y = \sqrt{2x+1}$ to the graph $y = \sqrt{2x-3}$.

6B Stretches

With the following two graphs, one function is 3 times the other.



$y = x^2 - x - 2$

$y = 3(x^2 - x - 2)$

When the x-coordinates on the two graphs are the same (where $x = x$), the y-coordinate of the blue graph is three times as large ($y = 3y$). We can interpret this as meaning that, at the same x-coordinates, the blue graph is three times as high as than the red graph; it has been stretched vertically.

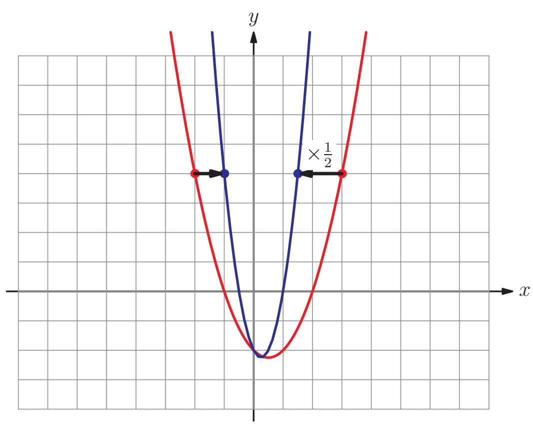
KEY POINT 6.3

The graph $y = pf(x)$, where $p > 0$, is the graph of $y = f(x)$ *stretched* vertically relative to the x-axis (away from) with scale factor p. If $0 < p < 1$, then $y = f(x)$ is *compressed* vertically relative to the x-axis (towards). If $p < 0$ the scale factor is negative $(-p)$ and it might be easier to think of the transformation as a stretch/compression by scale factor p followed by reflection in the x-axis.

In the next example, the second function is obtained by replacing x by $2x$.

$y = x^2 - x - 2$

$y = (2x)^2 - (2x) - 2$



If $x = x$ there is nothing useful we can say about the relationship between y and y. However, a way of making $y = y$ is to have $x = 2x$ or equivalently $\frac{x}{2} = x$. We can interpret this as meaning that when the two graphs have the same height, the distance from the y-axis of the blue graph is half the distance of the red graph; it has been stretched horizontally.

KEY POINT 6.4

The graph $y = f(qx)$ is the graph of $y = f(x)$ stretched horizontally relative to the y-axis by scale factor $\frac{1}{q}$. This can be considered a compression relative to the y-axis (towards) when $q > 0$. When $0 < q < 1$, it is considered a stretch relative to the y-axis (away from) and when $q < 0$, the scale factor is negative $\left(-\frac{1}{q}\right)$ and it is easier to think of the transformation as a stretch/compression by scale factor $\frac{1}{q}$ followed by a reflection in the y-axis.

Worked example 6.2

Describe a transformation which transforms the graph of $y = \ln x - 1$ to the graph of $y = \ln x^4 - 4$.

Try to relate the two equations using function notation. None of the transformations we know involve raising x to a power, so think of a different way to write $\ln x^4$

Let $f(x) = \ln x - 1$

$\ln x^4 - 4 = 4\ln x - 4$

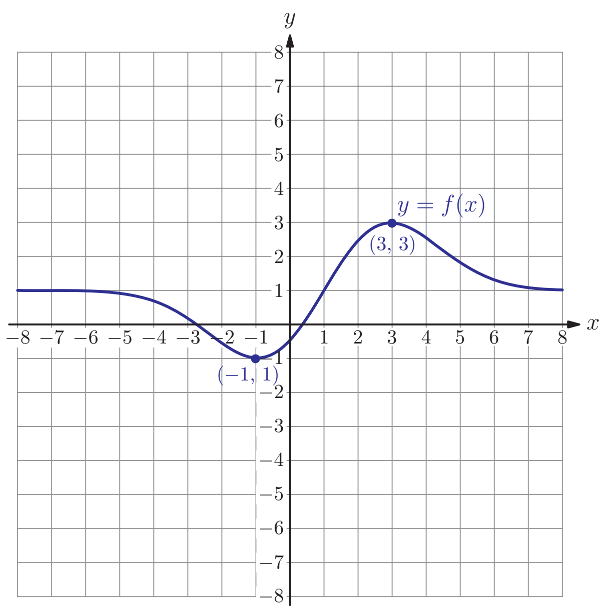
$= 4f(x)$

Relate the function notation to the transformation

It is a vertical stretch with scale factor 4

Exercise 6B

1. Alongside is the graph of $y = f(x)$. Sketch the graphs of the following functions, including the positions of the minimum and maximum points.



(a) (i) $y = 3f(x)$ (ii) $y = 5f(x)$

(b) (i) $y = \dfrac{f(x)}{4}$ (ii) $y = \dfrac{f(x)}{2}$

(c) (i) $y = f(2x)$ (ii) $y = f(6x)$

(d) (i) $y = f\left(\dfrac{2x}{3}\right)$ (ii) $y = f\left(\dfrac{5x}{6}\right)$

2. Find the equation of the graph after the given transformation is applied:

(a) (i) $y = 3x^2$ after a vertical stretch factor 7 relative to the x-axis

(ii) $y = 9x^3$ after a vertical stretch factor 2 relative to the x-axis

(b) (i) $y = 7x^3 - 3x + 6$ after a vertical stretch factor $\frac{1}{3}$ relative to the x-axis

(ii) $y = 8x^2 - 7x + 1$ after a vertical stretch factor $\frac{4}{5}$ relative to the x-axis

(c) (i) $y = 4x^2$ after a horizontal stretch factor 2 relative to the y-axis

(ii) $y = 7x^2$ after a horizontal stretch factor 5 relative to the y-axis

(d) (i) $y = 3x^3 - 5x^2 + 4$ after a horizontal stretch factor $\frac{1}{2}$ relative to the y-axis

(ii) $y = x^3 + 6x + 2$ after a horizontal stretch factor $\frac{2}{3}$ relative to the y-axis

3. Describe the following transformations:

(a) (i) Transforming the graph $y = x^2 + 3x + 7$ to the graph $y = 4x^2 + 12x + 28$

(ii) Transforming the graph $y = x^3 - 5x$ to the graph $y = 6x^3 - 30x$

(b) (i) Transforming the graph $y = x^2 + 2x + 7$ to the graph $y = (3x)^2 + 2(3x) + 7$

(ii) Transforming the graph $y = x^2 + 5x - 2$ to the graph $y = (4x)^2 + 5(4x) - 2$

(c) (i) Transforming the graph $y = e^x + x^2$ to the graph $y = e^{\frac{x}{2}} + \left(\frac{x}{2}\right)^2$

(ii) Transforming the graph $y = \log(3x) - \sqrt{4x}$ to the graph $y = \log\left(\frac{3x}{5}\right) - \sqrt{\frac{4x}{5}}$

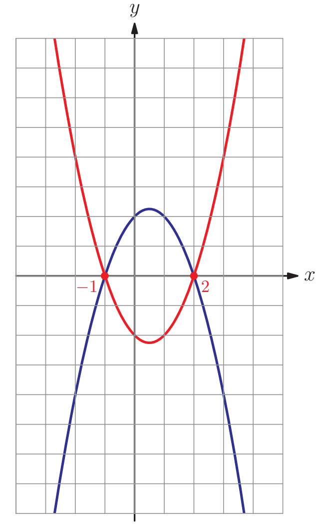
(d) (i) Transforming the graph $y = \ln(4x)$ to the graph $y = \ln(12x)$

(ii) Transforming the graph $y = \sqrt{2x+1}$ to the graph $y = \sqrt{x+1}$

6C Reflections

$y = x^2 - x - 2$

$y = -(x^2 - x - 2)$

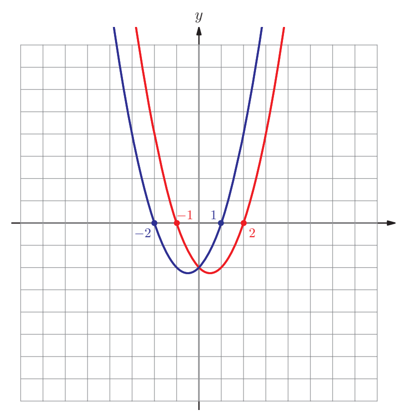


Compare these pairs of graphs and their equations.

When the x-coordinates on the two graphs are the same (where $x = x$) the y-coordinates are the negative equivalent of each other ($y = -y$). We can interpret this as meaning that, at the same x-coordinates, the blue graph is the same vertical distance relative to the x-axis but in the opposite direction to the red graph; it has been reflected vertically.

KEY POINT 6.5

> The graph $y = -f(x)$ is the graph of $y = f(x)$ reflected in the x-axis.



$y = x^2 - x - 2$
$y = (-x)^2 - (-x) - 2$

In the second example, x has been replaced by $-x$.

If $x = x$ there is nothing useful we can say about the relationship between y and y.

However, a way of making $y = y$ is to have $x = -x$.

We can interpret this as meaning that when the heights of the two graphs are the same, the blue graph is on the opposite side of the y-axis to the red graph – it has been reflected horizontally.

KEY POINT 6.6

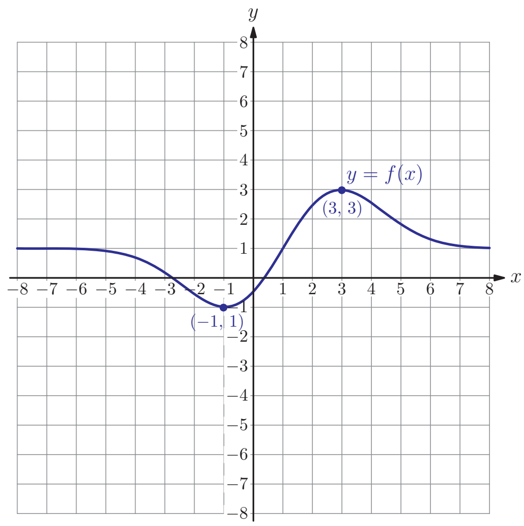
> The graph $y = f(-x)$ is the graph of $y = f(x)$ reflected in the y-axis.

Worked example 6.3

The graph of $y = f(x)$ has a single maximum point with coordinates (4, –3). Find the coordinates of the maximum point on the graph of $y = f(-x)$.

Relate the function notation to a transformation	The transformation taking $y = f(x)$ to $y = f(-x)$ is a reflection in the y-axis
Reflection in the y-axis leaves y-coordinates unchanged and changes x to $-x$	The maximum point is (–4, –3)

Exercise 6C



1. Alongside is the graph of $y = f(x)$. Sketch the graph of the following functions, including the position of the minimum and maximum points.
   (a) $y = -f(x)$
   (b) $y = f(-x)$

2. Find the equation of the graph after the given transformation is applied.
   (a) (i) $y = 3x^2$ after reflection in the x-axis
   (ii) $y = 9x^3$ after reflection in the x-axis
   (b) (i) $y = 7x^3 - 3x + 6$ after reflection in the x-axis
   (ii) $y = 8x^2 - 7x + 1$ after reflection in the x-axis
   (c) (i) $y = 4x^2$ after reflection in the y-axis
   (ii) $y = 7x^3$ after reflection in the y-axis
   (d) (i) $y = 3x^3 - 5x^2 + 4$ after reflection in the y-axis
   (ii) $y = x^3 + 6x + 2$ after reflection in the y-axis

3. Describe the following transformations:
   (a) (i) Transform the graph $y = x^2 + 3x + 7$ to the graph $y = -x^2 - 3x - 7$
   (ii) Transform the graph $y = x^3 - 5x$ to the graph $y = 5x - x^3$
   (b) (i) Transform the graph $y = x^2 + 2x + 7$ to the graph $y = x^2 - 2x + 7$
   (ii) Transform the graph $y = x^2 - 5x - 2$ to the graph $y = x^2 + 5x - 2$
   (c) (i) Transform the graph $y = e^x + x^2$ to the graph $y = e^{-x} + x^2$
   (ii) Transform the graph $y = \log(3x) - \sqrt{4x}$ to the graph $y = \sqrt{4x} - \log(3x)$
   (d) (i) Transform the graph $y = \ln(4x)$ to the graph $y = \ln(-4x)$
   (ii) Transform the graph $y = \sqrt{2x+1}$ to the graph $y = \sqrt{1-2x}$

6D Modulus transformations

The modulus (absolute value) leaves positive numbers unchanged but reverses the sign of negative numbers. In this section we consider the effect of applying modulus transformations to graphs.

You can think of the modulus function as giving the distance of a number away from the zero on the number line. This allows the modulus function to be applied to many other geometric situations. But the distance between two objects is not always easy to define – what length gives the 'distance' between two points on the surface of the Earth? This introduces the idea of a *metric* – a way of defining distances between objects. You will not be surprised by the *Euclidean Metric* for distances in flat space, but there are some amazing non-Euclidean metrics including the *Minkowski Metric* for finding the distance between two points in space-time in the theory of relativity.

The function $|f(x)|$ may be described as:

$$|f(x)| = \begin{cases} f(x) & \text{when } f(x) \geq 0 \\ -f(x) & \text{when } f(x) < 0 \end{cases}$$

See Prior learning G on the CD-ROM for an introduction to the modulus.

Hence the graph of $y = |f(x)|$ will be identical to that of $y = f(x)$ wherever $f(x)$ is positive, and will be the same as $y = -f(x)$ whenever $f(x)$ is negative.

KEY POINT 6.7

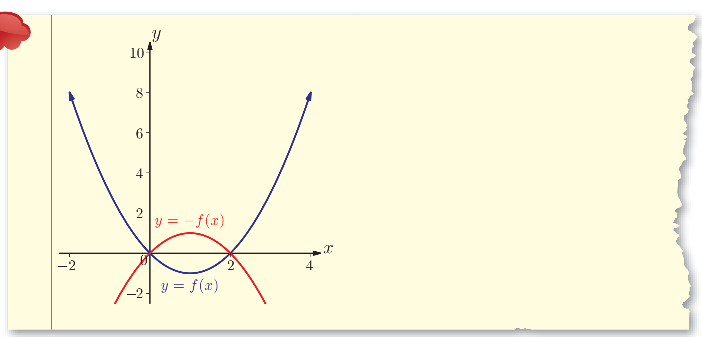
To draw $y = |f(x)|$, draw the graphs of both $y = f(x)$ and $y = -f(x)$, and take only those parts which are above the x-axis.

The idea of modulus will be extended to vectors in chapter 13 and complex numbers in chapter 15.

Worked example 6.4

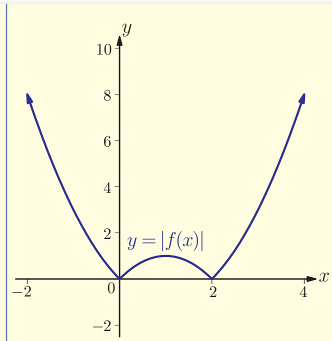
If $f(x) = x(x-2)$, draw the graph of $y = |f(x)|$.

Lightly draw both $y = f(x)$ and $y = -f(x)$



continued . . .

Restrict the values of y to non-negative



The function $f(|x|)$ will have the same output value for $-x$ as for x because $|x| = |-x|$. Thus:

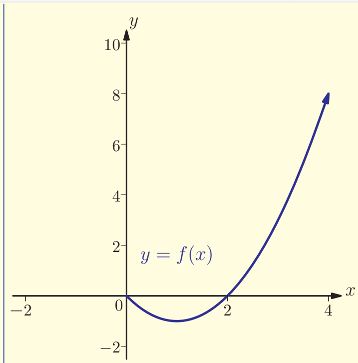
KEY POINT 6.8

The graph of $y = f(|x|)$ can be drawn directly from the graph of $y = f(x)$ by plotting $y = f(x)$ for $x \geq 0$, and then applying the mirror-image of that plot for negative values of x.

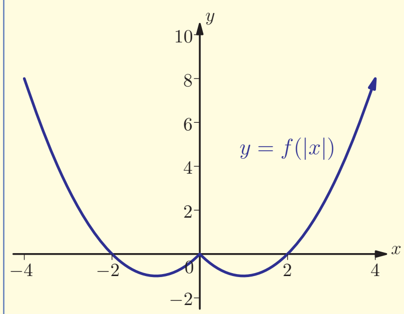
Worked example 6.5

If $f(x) = x(x-2)$, draw the graph of $y = f(|x|)$.

Draw the graph of $y = f(x)$ for $x \geq 0$



Reflect in the y-axis



You may have to solve equations involving the modulus function by finding the intersections of two graphs.

EXAM HINT

Your GDC can sketch modulus functions. See Calculator sheet 3 on the CD-ROM.

KEY POINT 6.9

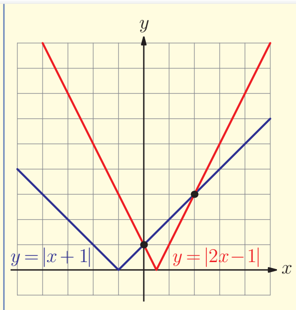
When solving an equation involving a modulus function, sketch the graph.

Remember that, when sketching a graph of the form $y = |f(x)|$ you reflect a part of the graph in the x-axis. When finding the coordinates of an intersection point you therefore need to decide whether it is on the reflected or the unreflected part of the graph. If it is on the unreflected part you simply remove the modulus signs and solve the equation. If it is on the reflected part you replace the modulus sign by the minus sign.

Worked example 6.6

Solve the equation $|x+1| = |2x-1|$.

Sketch the graphs of $y = |x+1|$ and $y = |2x-1|$



Describe the intersection points

There are two intersections – one with the unreflected blue line and the reflected red line:

$x + 1 = -(2x - 1)$ (1)

and one with the unreflected blue line and the unreflected red line:

$x + 1 = 2x - 1$ (2)

Solve the resulting equations

(1) $x + 1 = -2x + 1$

$\Leftrightarrow 3x = 0$

$\Leftrightarrow x = 0$

(2) $x + 1 = 2x - 1$

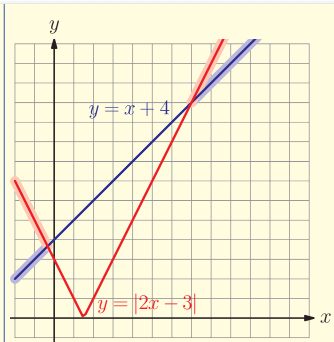
$\Leftrightarrow x = 2$

To solve inequalities you also sketch the graphs and find their intersections. You then decide on which parts of the graph the inequality is satisfied.

Worked example 6.7

Solve $|2x-3| > x+4$.

Sketch the graphs of $y = |2x-3|$ and $y = x+4$, highlighting where the equation is satisfied



Find where the two lines meet

Using GDC:
Lines meet when $x = -\frac{1}{3}$ and when $x = 7$

Describe the highlighted region in terms of x

$x < -\frac{1}{3}$ or $x > 7$

Exercise 6D

1. Sketch the following graphs, showing the axes intercepts.

(a) (i) $y = |2x-1|$ (ii) $y = |-3x+2|$

(b) (i) $y = |x|-2$ (ii) $y = |x|-2$

(c) (i) $y = |x^2+x-2|$ (ii) $y = |x^2-5x+6|$

(d) (i) $y = |x|^2-3|x|+2$ (ii) $y = |x|^2+|x|-6$

2. Sketch the following graphs, showing the axes intercepts and any asymptotes.

(a) (i) $y = |x^3-x^2-3|$ (ii) $y = |x^3+3x-1|$

(b) (i) $y = \dfrac{1}{1+|x|}$ (ii) $y = \sqrt{1+2|x|}$

(c) (i) $y = |\ln(x-2)|$ (ii) $y = e^{|x-1|}$

3. Solve these equations.

(a) (i) $|x| = 4$ (ii) $|x| = 18$

(b) (i) $|2x-4| = 4$ (ii) $|3x+1| = 2$

(c) (i) $|2x+4| = 4-x$ (ii) $|5-2x| = x+3$

(d) (i) $|3-2x| = |x+1|$ (ii) $|4+x| = |5-3x|$

4. Solve the following equations.

(a) (i) $|3x-4|=8-x$ (ii) $|5+2x|=3-2x$

(b) (i) $|6-x|=|5+x|$ (ii) $|4+3x|=|x|$

(c) (i) $|x+1|+|x-1|=x+4$ (ii) $|3x-1|=x+|2-x|$

(d) (i) $|x+1|=x$ (ii) $|x|-3=x^2$

5. Solve the following inequalities.

(a) (i) $|x|>5$ (ii) $|x|>2$

(b) (i) $|x|<3$ (ii) $|x|<10$

(c) (i) $|2x+1|>4$ (ii) $|3x-2|<3$

(d) (i) $|2x-5|<x+1$ (ii) $|5-3x|>2x$

(e) (i) $|x+4|>|2x|$ (ii) $|1+3x|<|x+3|$

6. Solve the inequality $|3x+1|>2x$. *[5 marks]*

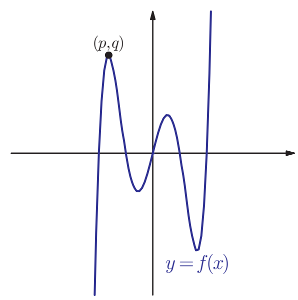
7. Solve the equation $|x^2-4|=8x$. *[5 marks]*

8. Solve the equation $|x^2-7x+10|=7x-10-x^2$. *[4 marks]*

9. Solve the equation $x|x|=4x$. *[4 marks]*

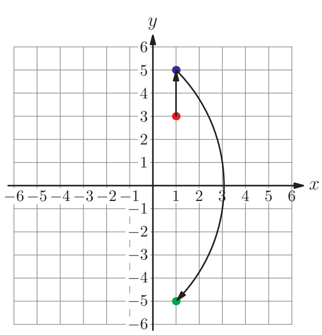
10. Solve the equation $|x+q^2|=|x-2q^2|$ giving your answer in terms of q. *[5 marks]*

11. Alongside is the graph of $y=f(x)$. Sketch the graph of $y=f(x)+|f(x)|$. *[4 marks]*

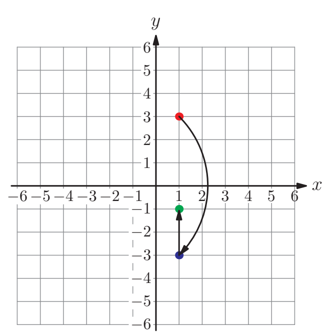


6E Consecutive transformations

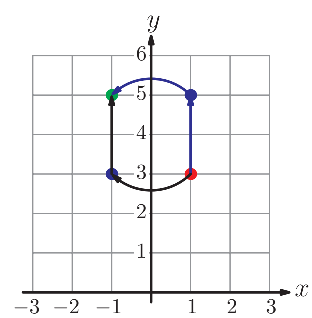
In this section we look at what happens when we apply two transformations in succession.



If the point $(1, 3)$ is translated two units up followed by a reflection in the x-axis (two vertical transformations) the new point is $(1, -5)$.



If $(1, 3)$ is reflected in the x-axis followed by a translation two units up (two vertical transformations) the new point is $(1, -1)$.



However, if the transformations were a translation two units up and a reflection in the y-axis (one vertical transformation and one horizontal transformation), irrespective of the order the result would be $(-1, 5)$.

In advanced mathematics, algebra is much more than using letters to represent numbers. Unknowns can include transformations and many other things.

As you can see in this section, the rules for transformations are different from the rules for numbers, but there are certain similarities too. The study of this more general form of algebra is called group theory, and it is studied further in Option 8: Sets, Relations and Groups. It has many applications from particle physics to painting polyhedra.

EXAM HINT

You may wish to simply remember that when resolving vertical transformations we follow the normal order of operations, while horizontal transformations are resolved in the opposite order.

KEY POINT 6.10

> When two horizontal or two vertical transformations are combined, the outcome depends on the order.
>
> When one vertical and one horizontal transformation are combined, the outcome does not depend on the order.

There is a very important rule to remember when resolving horizontal or vertical transformations:

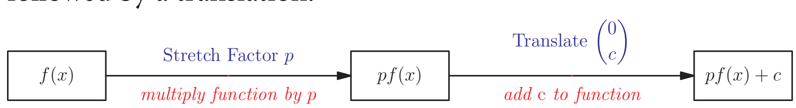
- vertical transformations follow the 'normal' order of operations as applied to arithmetic
- horizontal transformations are resolved in the opposite order to the 'normal' order of operations.

This is demonstrated below:

Let us combine two vertical transformations to transform the graph of $y = f(x)$ into the graph of:

$$y = pf(x) + c$$

We can achieve this by first multiplying $f(x)$ by p and then adding on c, so this form represents a stretch / reflection followed by a translation.

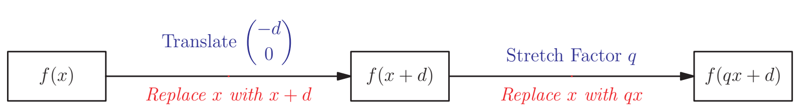


This follows the order of operations as you would expect.

If we combine two horizontal transformations, we can transform the graph of $y = f(x)$ into the graph of:

$$y = f(qx + d)$$

We can achieve this by first replacing x with $x + d$ and then replacing all occurrences of x by qx, so this represents a translation followed by a stretch / reflection.

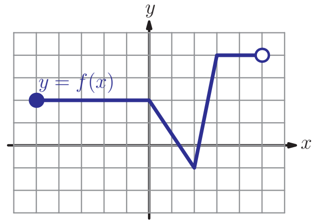


Following the normal order of operations, you would expect to resolve 'qx' before '$+d$' but you resolve the transformation in the opposite order.

In the first question in Exercise 6E you must consider what happens if you apply these transformations in the reverse order.

Worked example 6.8

Below is the graph of $y = f(x)$. Sketch the corresponding graph for $y = 3 - 2f(2x+1)$.



Break down the transformation into components

$(2x+1)$ is two horizontal transformations (changes x)

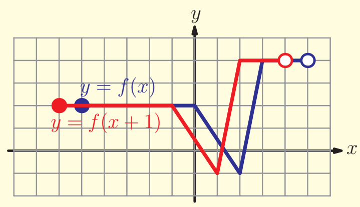
$3 - 2f(2x+1)$ is two vertical transformations (changes y)

Changing x: add 1, then multiply by 2

Changing y: multiply by -2, then add 3

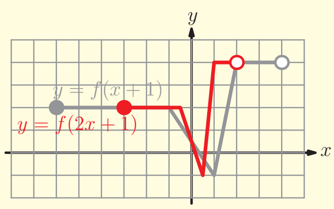
Replace x with $x + 1$
Change $y = f(x)$ to $y = f(x + 1)$

Horizontal Translation $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$



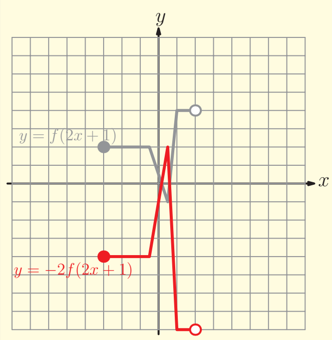
Replace x with $2x$
Change $y = f(x + 1)$ to $y = f(2x + 1)$

Horizontal Stretch, scale factor $\frac{1}{2}$, relative to $x = 0$



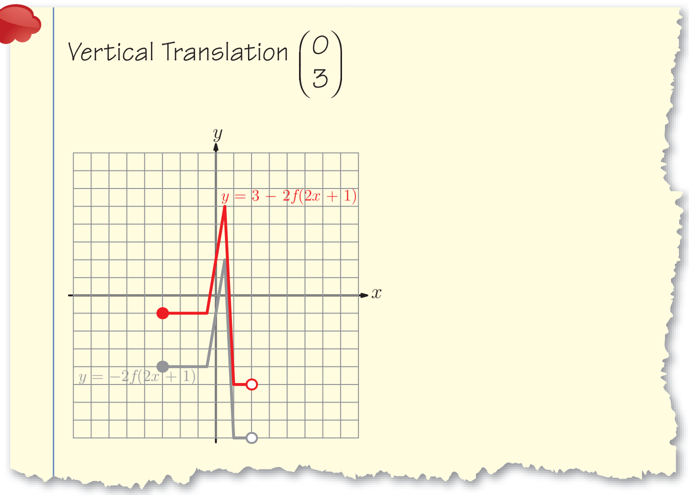
Multiply RHS by -2
Change $y = f(2x + 1)$ to $y = -2f(2x + 1)$

Vertical Reflection through $y = 0$ and
Vertical Stretch, scale factor 2, relative to line $y = 0$



continued . . .

Add 3 to RHS
Change $y = -2f(2x+1)$
to $y = 3 - 2f(2x+1)$

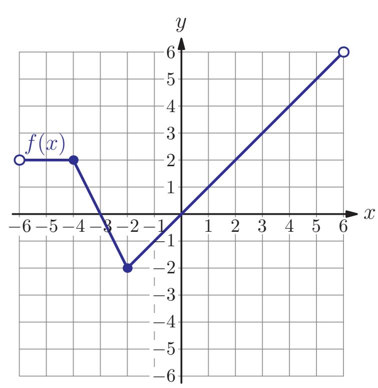


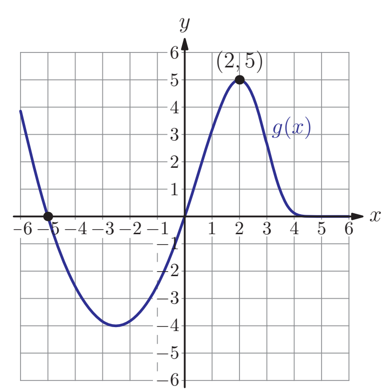
Exercise 6E

1. (a) The graph $y = f(x)$ is transformed by applying first a vertical translation c upwards, then a vertical stretch factor p relative to the x-axis. What is the equation of the resulting graph?

   (b) The graph $y = f(x)$ is transformed by applying first a horizontal stretch, factor q relative to the y-axis, then a horizontal translation d to the left. What is the equation of the resulting graph?

2. Here are the graphs of $y = f(x)$ and $y = g(x)$.





Sketch:

(a) (i) $2f(x) - 1$ (ii) $\frac{1}{2}g(x) + 3$

(b) (i) $4 - f(x)$ (ii) $2 - 2g(x)$

(c) (i) $3(f(x)-2)$ (ii) $\frac{1-g(x)}{2}$

(d) (i) $f\left(\frac{x}{2}-1\right)$ (ii) $g(2x+3)$

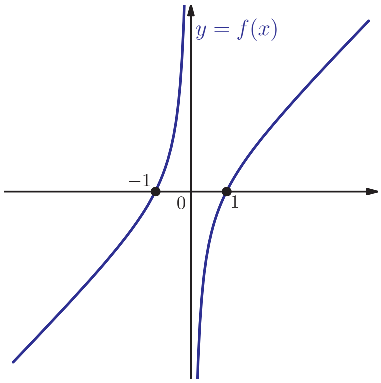
(e) (i) $f\left(\frac{4-x}{5}\right)$ (ii) $g\left(\frac{x-3}{2}\right)$

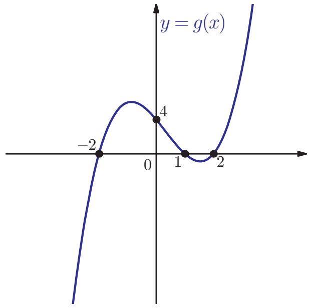
3. For the graphs of $f(x)$ and $g(x)$, sketch

(a) (i) $|f(x)|$ (ii) $|g(x)|$

(b) (i) $4+2|f(x)|$ (ii) $2|g(x)|-2$

(c) (i) $2|2+f(x)|$ (ii) $2|g(x)-1|$





4. For the functions $f(x)$ and $g(x)$ from question 1, sketch

(a) (i) $f(|x|)$ (ii) $g(|x|)$

(b) (i) $f(2+|x|)$ (ii) $g(|x|-1)$

(c) (i) $f(|x-2|+1)$ (ii) $g(|x+1|-1)$

5. If $f(x)=x^2$, express each of the following functions as $af(x)+b$ and hence describe the transformation mapping $f(x)$ to the given function.

(a) (i) $k(x)=2x^2-6$ (ii) $k(x)=5x^2+4$

(b) (i) $h(x)=5-3x^2$ (ii) $h(x)=4-8x^2$

6. If $f(x)=2x^2-4$, give the function $g(x)$ which shows the graph of $f(x)$ after:

(a) (i) translation $\begin{pmatrix}0\\2\end{pmatrix}$ followed by a vertical stretch of scale factor 3

(ii) translation $\begin{pmatrix}0\\6\end{pmatrix}$ followed by a vertical stretch of scale factor $\frac{1}{2}$

(b) (i) vertical stretch of scale factor $\frac{1}{2}$ followed by a translation $\begin{pmatrix}0\\6\end{pmatrix}$

(ii) vertical stretch of scale factor $\frac{7}{2}$ followed by a translation $\begin{pmatrix}0\\10\end{pmatrix}$

(c) (i) reflection through the horizontal axis

(ii) reflection through the horizontal axis followed by a translation $\begin{pmatrix}0\\2\end{pmatrix}$

(d) (i) reflection through the horizontal axis and vertical stretch of scale factor $\frac{1}{2}$ followed by a translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(ii) reflection through the horizontal axis followed by a translation $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ followed by a vertical stretch of scale factor $\frac{3}{2}$

7. If $f(x) = x^2$, express each of the following functions as $f(ax+b)$ and hence describe the transformation mapping $f(x)$ to the given function.

(a) (i) $g(x) = x^2 + 2x + 1$ (ii) $g(x) = x^2 - 6x + 9$

(b) (i) $h(x) = 4x^2$ (ii) $h(x) = \frac{x^2}{9}$

(c) (i) $k(x) = 2x^2 + 8x + 4$ (ii) $k(x) = 9x^2 - 6x + 1$

8. If $f(x) = 2x^2 - 4$, give the function g which shows the graph of $f(x)$ after:

(a) (i) translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ followed by a horizontal stretch of scale factor $\frac{1}{4}$

(ii) translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ followed by a vertical stretch of scale factor $\frac{1}{2}$

(b) (i) vertical stretch of scale factor $\frac{1}{2}$ followed by a translation $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$

(ii) vertical stretch of scale factor $\frac{2}{3}$ followed by a translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c) (i) translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ followed by a reflection through the vertical axis

(ii) reflection through the vertical axis followed by a translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

9. For each of the following functions $f(x)$ and $g(x)$, express $g(x)$ in the form $g(x) = af(x+b)+c$, for some values a, b and c, and hence describe a sequence of horizontal and vertical transformations which map $f(x)$ to $g(x)$.

(a) (i) $f(x) = x^2 \quad g(x) = 2x^2 + 4x$

(ii) $f(x) = x^2 \quad g(x) = 3x^2 - 24x + 8$

(b) (i) $f(x) = x^2 + 3 \quad g(x) = -6x + 8$

(ii) $f(x) = x^2 - 2 \quad g(x) = 2 + 8x - 4x^2$

10. (a) The graph of function $f(x) = ax + b$ is transformed by the following sequence:

- translation by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- reflection through $y = 0$
- horizontal stretch, scale factor $\frac{1}{3}$, relative to $x = 0$.

The resulting function is $g(x) = 4 - 15x$. Find a and b.

(b) The graph of function $f(x) = ax^2 + bx + c$ is transformed by the following sequence:

- reflection through $x = 0$
- translation by $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$
- horizontal stretch, scale factor 2, relative to $y = 0$.

The resulting function is $g(x) = 4x^2 + ax - 6$. Find a, b and c. *[10 marks]*

11. If $f(x) = 2^x + x$, give in simplest terms the formula for $h(x)$, which is obtained by transforming $f(x)$ by the following sequence of transformations:

- vertical stretch, scale factor 8, relative to $y = 0$
- translation by $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$
- horizontal stretch, scale factor $\frac{1}{2}$, relative to $x = 0$.

[6 marks]

12. Sketch the following graphs:

(a) $y = \ln x$

(b) $y = 3\ln(x+2)$

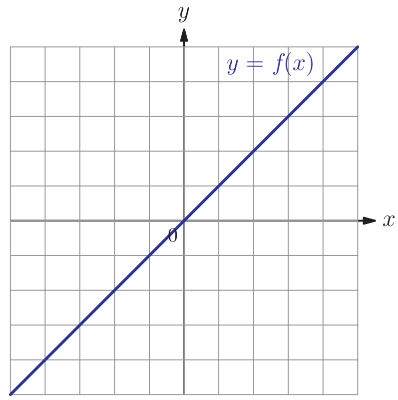
(c) $y = \ln(2x-1)$

In each case, indicate clearly the positions of the vertical asymptote and the x-intercept. *[6 marks]*

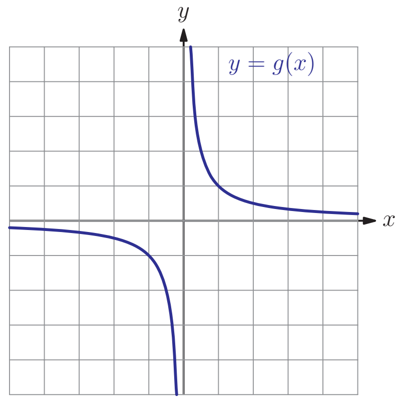
6F Reciprocal transformations

A reciprocal transformation turns the graph of $y = f(x)$ into the graph of $y = \frac{1}{f(x)}$.

The simplest reciprocal transformation turns $y = x$ into $y = \frac{1}{x}$.



$y = f(x)$



$y = g(x) = \frac{1}{f(x)}$

The table below summarises the relationship between the graphs of $f(x)$ and $g(x) = \frac{1}{f(x)}$

KEY POINT 6.11

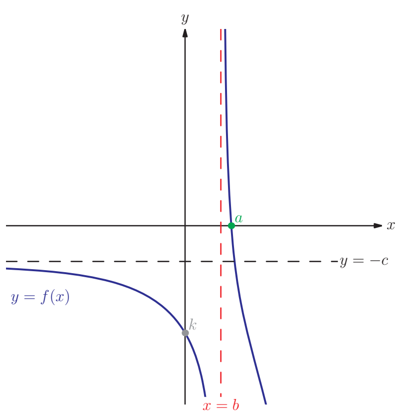
$f(x)$	$g(x) = \frac{1}{f(x)}$
$f(a) = 0$	The line $x = a$ is a vertical asymptote
The line $x = a$ is a vertical asymptote	$g(a) = 0$
$f(x) \to +\infty$	$g(x) \to 0$ (tends to zero from above)
$f(x) \to -\infty$	$g(x) \to 0$ (tends to zero from below)
$f(x) \to 0$	$g(x) \to +\infty$ or $-\infty$
$f(x)$ has a horizontal asymptote $y = b$	$g(x)$ has a horizontal asymptote $y = \frac{1}{b}$
$0 < f(x) < 1$	$g(x) > 1$
$f(x) = 1$	$g(x) = 1$
$f(x) > 1$	$0 < g(x) < 1$

$f(x)$	$g(x)=\dfrac{1}{f(x)}$
$-1<f(x)<0$	$g(x)<-1$
$f(x)=-1$	$g(x)=-1$
$f(x)<-1$	$-1<g(x)<0$
$(a, f(a))$ is a turning point	$(a, g(a))$ is the opposite turning point, if $g(a)$ is finite

Using these Key points as a guide, it is possible to make a rough sketch of the reciprocal for any given graph.

Worked example 6.9

The diagram shows the graph of an unspecified function $f(x)$, with a root at a and asymptotes $x=b$ and $y=-c$; sketch the corresponding graph of $g(x)=\dfrac{1}{f(x)}$.



What happens when x is large and negative?

As $x \to -\infty$, $f(x) \to -c \quad \therefore g(x) \to \left(-\dfrac{1}{c}\right)$

So $y = -\dfrac{1}{c}$ is a horizontal asymptote of $g(x)$

What happens when x is large and positive?

As $x \to +\infty$, $f(x) \to -\infty \therefore g(x) \to 0$

$y=0$ is a horizontal asymptote of $g(x)$

What happens when $y=f(x)$ has a root?

$f(a)=0$ so $x=a$ is a vertical asymptote of $g(x)$

What happens when $y=f(x)$ has a vertical asymptote?

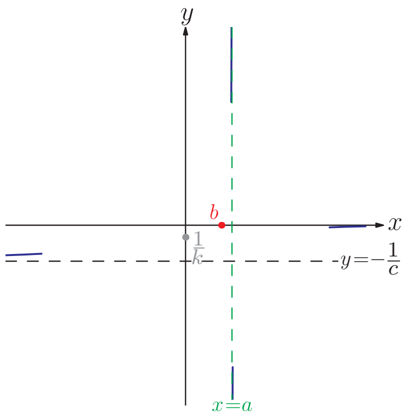
$x=b$ is a vertical asymptote of $f(x)$ so $g(b)=0$

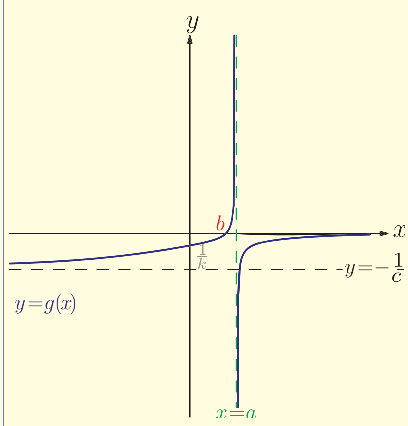
continued . . .

What happens when $y = f(x)$ has a turning point?

$f(x)$ has no turning points

Putting this information together:





EXAM HINT

If you are unsure about which side of an asymptote the graph lies, check a few points.

Exercise 6F

1. By first sketching the graph of $y = f(x)$, sketch the following graphs of the form $y = \dfrac{1}{f(x)}$, indicating the positions of axes intercepts and asymptotes.

(a) (i) $y = \dfrac{1}{x-3}$ (ii) $y = \dfrac{1}{x-5}$

(b) (i) $y = \dfrac{1}{x+1}$ (ii) $y = \dfrac{1}{x+3}$

(c) (i) $y = \dfrac{1}{x^2}$ (ii) $y = \dfrac{1}{(x-1)^2}$

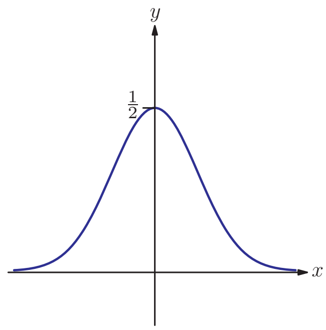
(d) (i) $y = \dfrac{1}{x^2-4}$ (ii) $y = \dfrac{1}{x^2+x-6}$

(e) (i) $y = \dfrac{1}{e^x+1}$ (ii) $y = \dfrac{1}{e^{x+2}}$

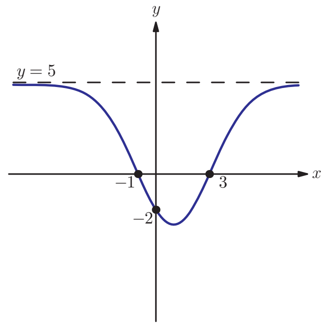
(f) (i) $y = \dfrac{1}{\ln x + 2}$ (ii) $y = \dfrac{1}{\ln(x+3)}$

2. The graph of $y = f(x)$ is shown. Sketch the graph of $y = \dfrac{1}{f(x)}$.

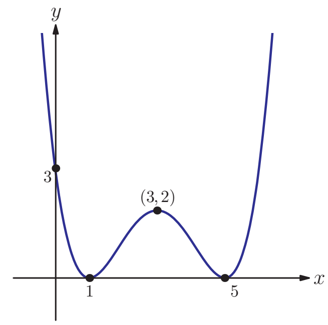
(a)



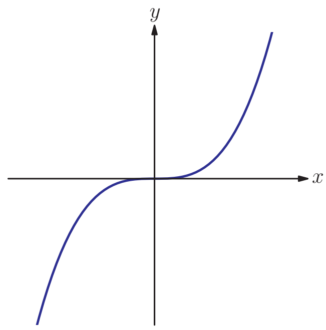
(b)



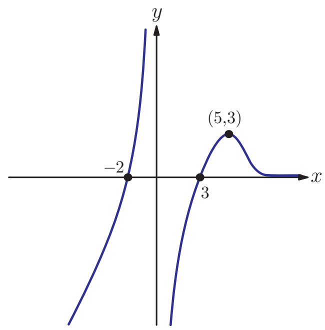
(c)



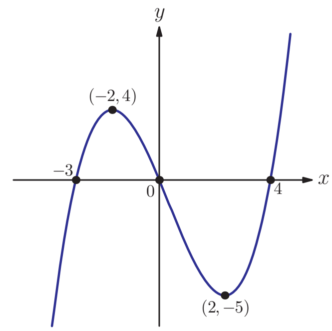
(d)



(e)



(f)

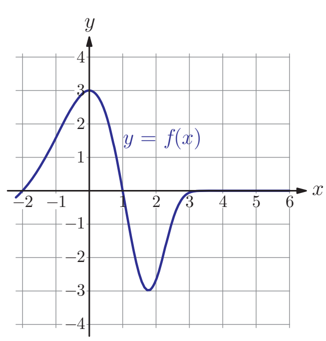


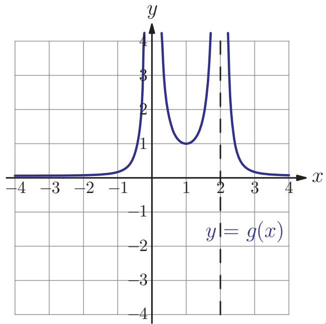
3. (a) The graph of $y = f(x)$ is transformed into the graph of $y = f(x-3)$ by a translation with vector $\begin{pmatrix} p \\ q \end{pmatrix}$. State the value of p and the value of q.

(b) Sketch the following graphs.

(i) $y = (x-3)^2$ (ii) $y = \dfrac{1}{(x-3)^2}$. *[6 marks]*

4. The graph of $y = f(x)$ is shown alongside. Sketch a copy of $y = f(x)$ and on the same diagram, sketch the graph of $y = \dfrac{1}{f(x)}$, indicating clearly the positions of any asymptotes and coordinates of any maximum and minimum points. *[6 marks]*





5. The graph alongside shows the function $y = g(x)$. On a separate diagram, sketch the graph of

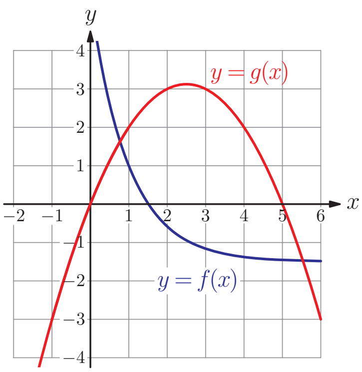
$$y = \frac{1}{g(x)} - 2$$

[6 marks]

6. State the equations of the vertical asymptotes of the graph

$$y = \frac{1}{x^2 e^{x^2} - 4x^2}.$$

[4 marks]



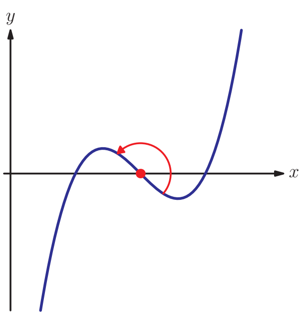
7. The graphs of the functions $y = f(x)$ and $y = g(x)$ are shown alongside.

Sketch a copy of $y = f(x)$ and $y = g(x)$ and, on the same diagram, sketch the graph of $y = \frac{f(x)}{g(x)}$.

[5 marks]

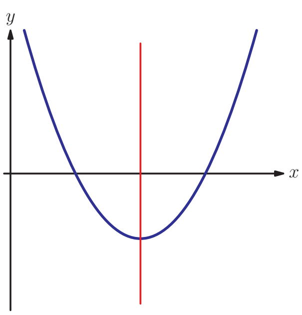
6G Symmetries of graphs and functions

There are several possible ways that a graph can be symmetrical. It can have:



- Two-fold rotational symmetry

  When the graph is rotated by a $180°$ about a given point, it becomes the same graph.

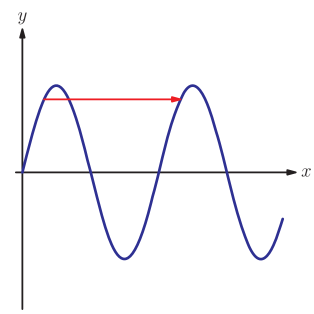


- Reflection symmetry

  When the graph is reflected in a given line it becomes the same graph.

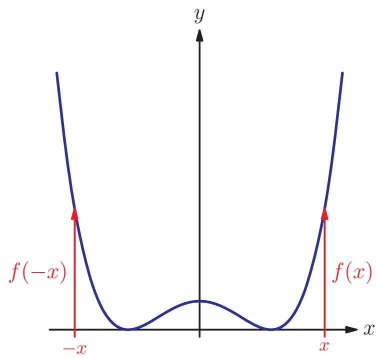
- Translational symmetry

  When the graph is shifted it becomes the same graph.



In this section we shall look at how the algebraic properties of the function are related to the symmetry properties of the graph.

When the graph has reflection symmetry in the y-axis, its function is called an **even function**. The height of the graph at every value of x must be the same as the height of the graph at $-x$:



In chapter 10 you will find that trigonometric functions have translational symmetry.

KEY POINT 6.12

For an even function $f(x)$:

$$f(x) = f(-x)$$

This means that $y = f(x)$ has reflection symmetry in the y-axis.

Although quite hard to imagine, there is also a symmetry where the graph becomes the same when it is enlarged. Graphs like this are called fractals.

Worked example 6.10

Consider the function $h(x) = e^x + e^{-x}$.

(a) Find $h(0)$.

(b) Prove that $h(x)$ is even.

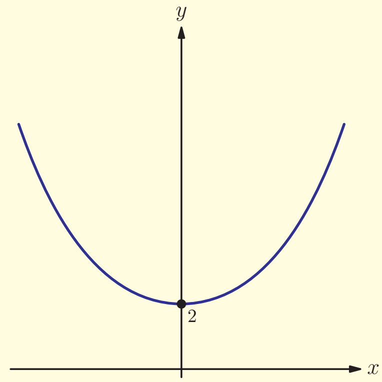
(c) Sketch the graph of $y = h(x)$.

(a) $h(0) = e^0 + e^0$
$= 2$

(b) $h(-x) = e^{-x} + e^{-(-x)}$
$= e^{-x} + e^x$
$= h(x)$

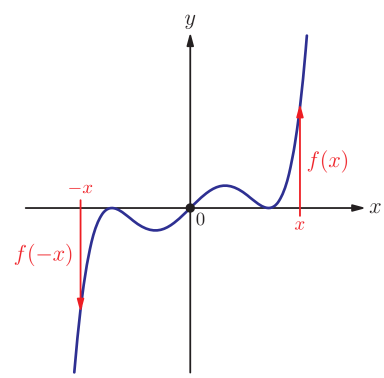
continued . . .

(c) When x is very large $h(x)$ also gets very large, so the graph should look like:



Although the graph above looks like a parabola, it is actually called a *catenary*. A rope hanging between two level points will form this shape.

When the graph has two-fold rotation symmetry about the origin its function is called an **odd function**. The height above the x-axis of the graph at every value of x must be the same as the height below the x-axis of the graph at x, therefore:



KEY POINT 6.13

For an odd function $f(x)$:

$$f(-x) = -f(x)$$

This means that $y = f(x)$ has two-fold rotation symmetry about the origin.

Worked example 6.11

$f(x)$ is an odd function and $g(x)$ is an even function. Prove that $h(x) = f(x)g(x)$ is an odd function.

Describe the symmetries of $f(x)$ and $g(x)$ mathematically

$f(-x) = -f(x)$

$g(-x) = g(x)$

If $h(x)$ is an odd function then we need to show that $h(-x) = -h(x)$. Let's start with $h(-x)$

$$h(-x) = f(-x)g(-x)$$
$$= -f(x)g(x)$$
$$= -h(x)$$

Therefore $h(x)$ is an odd function.

Exercise 6G

1. Decide if the following functions are odd, even or neither:

(a) (i) $y = 3x^4 + 1$ (ii) $y = x^6 - x^2$

(b) (i) $y = 4x^3 + x$ (ii) $y = 5x^5$

(c) (i) $y = 1 + x$ (ii) $y = x^2 - x$

(d) (i) $y = \frac{1}{x}$ (ii) $y = \frac{1}{x^2}$

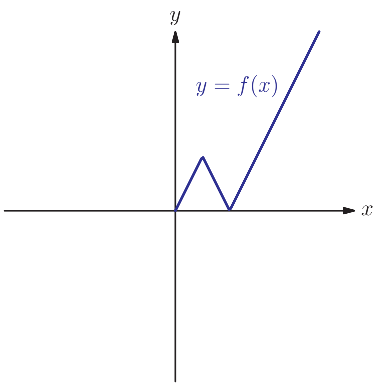
(e) (i) $y = e^x$ (ii) $y = \log x$

2. (a) Describe fully the symmetry of the graph of an odd function.

(b) Complete the following graph for negative values of x to make $f(x)$:

(i) an even function

(ii) an odd function. *[2 marks]*



3. Show that $f(x) = e^{x^2}(1 + x^4)$ is an even function. *[3 marks]*

4. Explain why an even function can never be a one-to-one function.

5. Prove that the sum of two even functions is an even function. *[4 marks]*

6. Prove that the product of two odd functions is an even function. *[4 marks]*

7. Prove that if $f(x) = ax^n$ is an odd function then n is odd. *[4 marks]*

8. $f(x)$ is an odd function and $g(x)$ is an even function. Prove that $gf(x)$ is an even function. *[4 marks]*

9. (a) State the line of symmetry of $f(x) = x^2 + 6x + 7$.

(b) If $f(x - a)$ is an even function find the value of a. *[5 marks]*

10. The function $f(x)$ satisfies $f(x) = f(10 - x)$. Describe fully the symmetry of the graph of $y = f(x)$. *[3 marks]*

11. The function $f(x)$ is a one-to-one function over all real numbers. The graph of $y = f(x)$ is symmetrical in the line $y = x$. Evaluate, with justification, $f \circ f(4)$. *[3 marks]*

12. (a) If $f(x)$ is any function show that $\frac{1}{2}\left(f(x) - f(-x)\right)$ is an odd function.

(b) Find a similar expression which will always be an even function.

(c) Hence or otherwise prove that all functions can be written as the sum of an odd and an even function.

(d) $f(x) = e^x$ can be written as $g(x) + h(x)$ where $g(x)$ is an odd function and $h(x)$ is an even function.

(i) Sketch $y = g(x)$, labelling all axes intercepts.

(ii) Sketch $y = h(x)$, labelling all axes intercepts. *[14 marks]*

13. Prove that any odd function which is a polynomial passes through the origin. Give an example to show that this is not necessarily true if the function is not a polynomial.

Summary

- These transformations must be learnt.

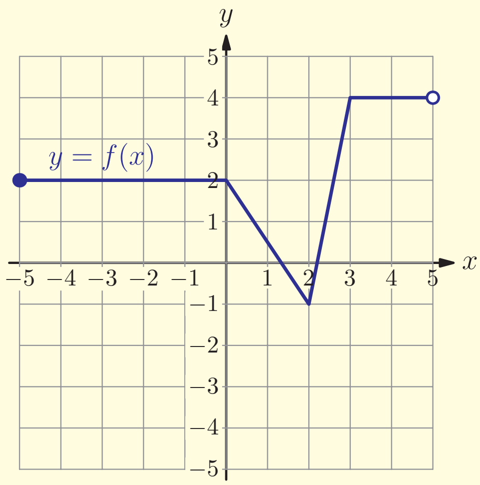
Transformation of $y=f(x)$	Transformation of graph
$y=f(x)+c$	Translation $\begin{pmatrix}0\\c\end{pmatrix}$
$y=f(x+d)$	Translation $\begin{pmatrix}-d\\0\end{pmatrix}$
$y=pf(x)$	Vertical stretch factor p relative to the x-axis: when $p>0$, stretches away from x-axis when $0<p<1$, stretches towards x-axis when $p<0$, stretch by p and then reflect in x-axis.
$y=f(qx)$	Horizontal stretch factor $\frac{1}{q}$ relative to the y-axis: when $q>0$, stretches towards y-axis when $0<q<1$, stretches away from y-axis when $q<0$ stretch by $\frac{1}{q}$ then reflect in y-axis
$y=-f(x)$	Reflection in the x-axis
$y=f(-x)$	Reflection in the y-axis
$y=\lvert f(x)\rvert$	The part of the graph which is below the x-axis is reflected in the x-axis.
$y=f(\lvert x\rvert)$	The part of the graph which is to the left of the y-axis disappears. The part of the graph which is to the right of the y-axis is reflected in the y-axis.

- The order in which transformations occur may affect the outcome.
- To sketch reciprocal functions we consider the position of zeros, asymptotes and minimum/maximum points on the graph and connect this information together.
- To solve equations and inequalities involving the modulus function, sketch the graph and find intersection points.
- If $f(x)$ is an odd function:
  - $f(-x)=-f(x)$
  - The graph $y=f(x)$ has order two rotational symmetry about the origin.
- If $f(x)$ is an even function:
  - $f(-x)=f(x)$
  - The graph $y=f(x)$ has reflection symmetry in the y-axis.

Introductory problem revisited

Here is the graph of $y = f(x)$.

Sketch the graph of $y = \dfrac{1}{f(x)}$.



We look for the significant features of the graph.

Where $f(x)$ is constant, $\dfrac{1}{f(x)}$ is also constant.

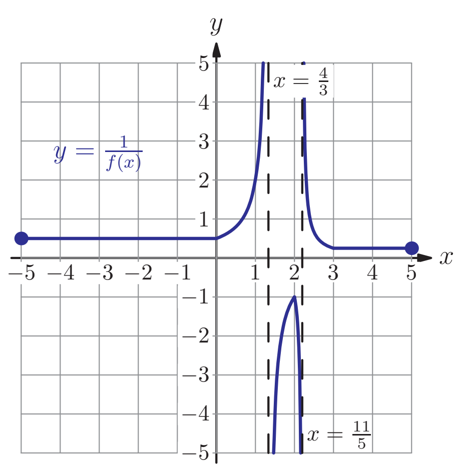
So $\dfrac{1}{f(x)} = \dfrac{1}{2}$ for $x \le 0$ and $\dfrac{1}{f(x)} = \dfrac{1}{4}$ for $x \ge 3$.

$f(x)$ has two zeros, so $\dfrac{1}{f(x)}$ has two vertical asymptotes.

$f(x)$ is negative between the zeros and positive on either side, so the same is true for $\dfrac{1}{f(x)}$.
$f(x)$ has a minimum point with $x = 2, y = -1$.

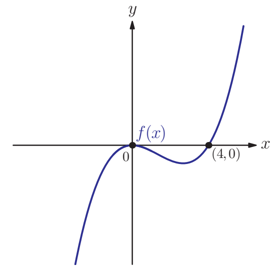
Hence $\dfrac{1}{f(x)}$ has a maximum point with $x = 2, y = \dfrac{1}{-1} = -1$.

We can now sketch the graph:



Mixed examination practice 6

Short questions



1. The graph of $y = f(x)$ is shown.

Sketch on separate diagrams the graphs of

(a) $y = 3f(x-2)$

(b) $\dfrac{1}{f(x)}$

Indicate clearly the positions of any x-intercepts and asymptotes. *[6 marks]*

2. The graph of $y = x^3 - 1$ is transformed by applying a translation with vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ followed by a vertical stretch with scale factor 2. Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$. *[4 marks]*

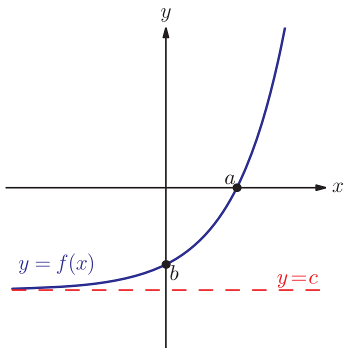
3. Solve the inequality $|2x-1| < x$. *[6 marks]*

4. The diagram shows the graph of $y = f(x)$.

On separate diagrams sketch the following graphs, labelling appropriately.

(a) $y = |f(x)|$

(b) $y = f(|x|) - 1$ *[5 marks]*



5. (a) Sketch the graph of $y = -\dfrac{3}{x}$.

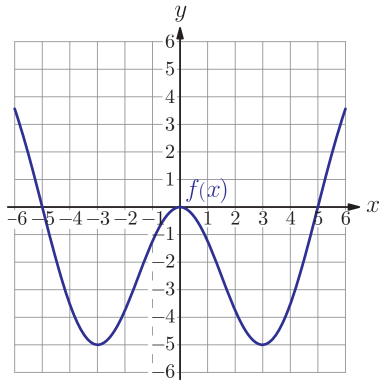
(b) Describe two transformations which transform the graph of $y = \dfrac{1}{x}$ to the graph of $y = -\dfrac{3}{x}$.

(c) Let $f(x) = -\dfrac{3}{x}, x \neq 0$. Write down an equation for $f^{-1}(x)$. *[4 marks]*

6. The graph of $y = f(x)$ is shown.

(a) On the same diagram sketch the graph of $y = \frac{1}{f(x)}$.

(b) State the coordinates of the maximum points. *[5 marks]*



7. Find two transformations whose composition transforms the graph of $y = (x-1)^2$ to the graph of $y = 3(x+2)^2$. *[4 marks]*

8. (a) Describe two transformations whose composition transforms the graph of $y = f(x)$ to the graph of $y = 3f\left(\frac{x}{2}\right)$.

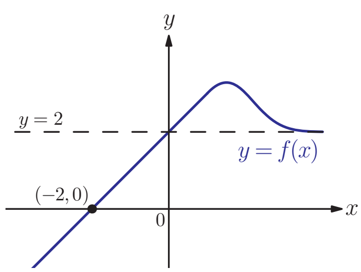
(b) Sketch the graph of $y = 3\ln\left(\frac{x}{2}\right)$.

(c) Sketch the graph of $y = 3\ln\left(\frac{x}{2}+1\right)$ marking clearly the positions of any asymptotes and x-intercepts. *[7 marks]*

9. The diagram shows a part of the graph of $y = f(x)$

On separate diagrams sketch the graphs of

(a) $y = \frac{1}{f(x)}$ (b) $y = xf(x)$ *[6 marks]*



10. For which values of the real number x is $|x+k| = |x| + k$, where k is a positive real number? *[4 marks]*

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Long questions

1 (a) Describe two transformations which transform the graph of $y = x^2$ to the graph of $y = 3x^2 - 12x + 12$.

(b) Describe two transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = x^2$.

(c) Hence describe a sequence of transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = 3x^2 - 12x + 12$.

(d) Sketch the graph of $y = \dfrac{1}{3x^2 - 12x + 12}$. *[12 marks]*

2. Given that $f(x) = \dfrac{3x-5}{x-2}$

(a) Write down the equation of the horizontal asymptote of the graph of $y = f(x)$.

(b) Find the value of constants p and q such that $f(x) = p + \dfrac{q}{x-2}$.

(c) Hence describe a single transformation which transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = f(x)$.

(d) Find an expression for $f^{-1}(x)$ and state its domain.

(e) Describe the transformation which transforms the graph of $y = f(x)$ to the graph of $y = f^{-1}(x)$. *[11 marks]*

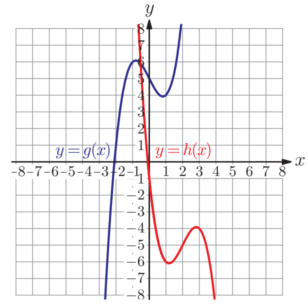
3. (a) Describe a transformation which transforms the graph of $y = f(x)$ to the graph of $y = f(x+2)$.

(b) Sketch on the same diagram the graphs of

(i) $y = \ln(x+2)$ (ii) $y = \dfrac{1}{\ln(x+2)}$.

Mark clearly any asymptotes and x-intercepts on your sketches.

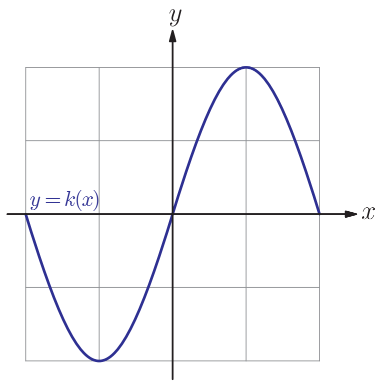
(c) The graph of the function $y = g(x)$ has been translated and then reflected in the x-axis to produce the graph of $y = h(x)$.



(i) State the translation vector.

(ii) If $g(x) = x^3 - 2x + 5$, find constants a, b, c and d such that $h(x) = ax^3 + bx^2 + cx + d$.

(d) The diagram below shows the graph of $y = k(x)$.



On the same diagram, sketch the graph of $y = (k(x))^2$. *[14 marks]*

4. $f(x) = x^2 - 7x + 10$ $\qquad$ $g(x) = x^2 - 7|x| + 10$

(a) Sketch the graph of $y = f(x)$.

(b) Show that $g(x) = f(|x|)$.

(c) Sketch the graph of $y = g(x)$.

(d) Solve the equation $g(x) = x^2$.

(e) Solve the equation $g(x) = -2$. *[12 marks]*

5. If $f(x) = 3x^2 + bx + 10$ and the graph $y = f(x)$ has a line of symmetry when $x = 3$

(a) find b.

(b) If $f(x) = f(d - x)$ for all x, find the value of d.

(c) $g(x) = f(x + p) + q$ and $g(x)$ is an even function which passes through the origin. Find p and q. *[14 marks]*

(d) Find the set values which satisfy $g(x) = g(|x|)$.

6. (a) Sketch the graph of $y = e^x - 2$, including the coordinates of all axes intercepts.

(b) On separate axes sketch the graphs of

(i) $y = |e^x - 2|$

(ii) $y = e^{|x|} - 2$

(c) Hence solve the equation $e^{|x|} - 2 = |e^x - 2|$. *[10 marks]*

In this chapter you will learn:

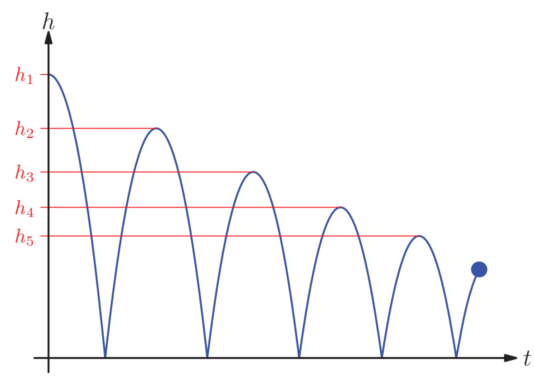
- how to describe sequences mathematically
- a way to describe sums of sequences
- about sequences with a constant difference between terms
- about finite sums of sequences with a constant difference between terms
- about sequences with a constant ratio between terms
- about finite sums of sequences with a constant ratio between terms
- about infinite sums of sequences with a constant ratio between terms
- how to apply sequences to real life problems.

7 Sequences and series

Introductory problem

A mortgage of $100 000 is fixed at 5% compound interest. It needs to be paid off over 25 years by annual instalments. Interest is added at the end of each year, just before the payment is made. How much should be paid each year?

If you drop a tennis ball, it will bounce a little lower each time it hits the ground. The heights to which the ball bounces form a **sequence**. Although the study of sequences may just seem to be the maths of number patterns, it also has a remarkable number of applications in the real world, from calculating mortgages to estimating the harvests on farms.



7A General sequences

A sequence is a list of numbers in a specified order. You may recognise a pattern in each of the following examples:

1, 3, 5, 7, 9, 11, ...

1, 4, 9, 16, 25, ...

100, 50, 25, 12.5, ...

To study sequences further, it is useful to have a notation to describe them.

KEY POINT 7.1

u_n is the value of the nth term of a sequence.

EXAM HINT

u_n is a conventional symbol for a sequence, but there is nothing special about the letters used. We could also have a sequence t_x or a_h. The important thing is that the letter with a subscript represents a value and the subscript represents where the term is in the sequence.

So in the sequence 1, 3, 5, 7, 9, 11, ... above, we could say that $u_1 = 1$, $u_2 = 3$, $u_5 = 9$.

The whole of a sequence u_1, u_2, u_3, … is sometimes written $\{u_n\}$.

We are mainly interested in sequences with well-defined mathematical rules. There are two types: **recursive definitions** and **deductive** rules.

Recursive definitions link new terms to previous terms in the sequence. For example, if each term is three times the previous term we would write $u_{n+1} = 3u_n$.

Worked example 7.1

A sequence is defined by $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 1$ and $u_2 = 1$.
What is the fifth term of this sequence?

The sequence is defined inductively, so we have to work our way up to u_5
To find u_3 we set $n = 2$

$$u_3 = u_2 + u_1 = 1 + 1 = 2$$

To find u_4 we set $n = 3$

$$u_4 = u_3 + u_2 = 2 + 1 = 3$$

To find u_5 we set $n = 4$

$$u_5 = u_4 + u_3 = 3 + 2 = 5$$

You may recognise this as the famous Fibonacci Sequence, based on a model Leonardo Fibonacci made for the breeding of rabbits. This has many applications from the arrangement of seeds in pine cones to a proof of the infinity of prime numbers.

There is also a beautiful link to the golden ratio: $\frac{1 \pm \sqrt{5}}{2}$

Deductive rules link the value of the term to where it is in the sequence. For example, if each term is the square of its position in the sequence then we would write $u_n = n^2$.

Worked example 7.2

A sequence is defined by $u_n = 2n - 1$. List the first four terms of this sequence.

With a deductive rule, we can find the first four terms by setting $n = 1, 2, 3, 4$

$u_1 = 2 \times 1 - 1 = 1$
$u_2 = 2 \times 2 - 1 = 3$
$u_3 = 2 \times 3 - 1 = 5$
$u_4 = 2 \times 4 - 1 = 7$

EXAM HINT

There are several alternative names used for deductive and recursive definitions.

An recursive definition may also be referred to as 'term-to-term rule', 'recurrence relation' or 'recursive definition'.

A deductive rule may be referred to as 'position-to-term rule', 'nth term rule' or simply 'the formula' of the sequence.

Exercise 7A

1. Write out the first five terms of the following sequences, using the inductive definitions.

(a) (i) $u_{n+1} = u_n + 5,\ u_1 = 3.1$ (ii) $u_{n+1} = u_n - 3.8,\ u_1 = 10$

(b) (i) $u_{n+1} = 3u_n + 1,\ u_1 = 0$ (ii) $u_{n+1} = 9u_n - 10,\ u_1 = 1$

(c) (i) $u_{n+2} = u_{n+1} \times u_n,\ u_1 = 2, u_2 = 3$

(ii) $u_{n+2} = u_{n+1} \div u_n,\ u_1 = 2u_2 = 1$

(d) (i) $u_{n+2} = u_n + 5,\ u_1 = 3u_2 = 4$

(ii) $u_{n+2} = 2u_n + 1,\ u_1 = -3u_2 = 3$

(e) (i) $u_{n+1} = u_n + 4,\ u_4 = 12$ (ii) $u_{n+1} = u_n - 2,\ u_6 = 3$

2. Write out the first five terms of the following sequences, using the deductive definitions.

(a) (i) $u_n = 3n + 2$ (ii) $u_n = 1.5n - 6$

(b) (i) $u_n = n^3 - 1$ (ii) $u_n = 5n^2$

(c) (i) $u_n = 3^n$ (ii) $u_n = 8 \times (0.5)^n$

(d) (i) $u_n = n^n$ (ii) $u_n = \sin(90n°)$

3. Give an inductive definition for each of these sequences.

(a) (i) 7, 10, 13, 16, … (ii) 1, 0.2, −0.6, −1.4, …

(b) (i) 3, 6, 12, 24, … (ii) 12, 18, 27, 40.5, …

(c) (i) 1, 3, 6, 10, … (ii) 1, 2, 6, 24, …

4. Give a deductive definition for each of the following sequences.

(a) (i) 2, 4, 6, 8, … (ii) 1, 3, 5, 7, …

(b) (i) 2, 4, 8, 16, … (ii) 5, 25, 125, 625, …

(c) (i) 1, 4, 9, 16, … (ii) 1, 8, 27, 64, …

(d) (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ (ii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16} \ldots$

5. A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1} - 1$ where $n \in \mathbb{Z}$.

(a) Find u_2, u_3 and u_4.

(b) (i) Based on your answer to (a), suggest a formula for u_n in terms of n.

(ii) Verify that your answer to part (b)(i) satisfies the equation $u_{n+1} = 3u_n - 2u_{n-1}$. *[6 marks]*

7B General series and sigma notation

If 10% interest is paid on money in a bank account each year, the amounts paid form a sequence. While it is good to know how much is paid in each year, you may be even more interested to know how much will be paid in altogether.

This is an example of a situation where we may want to sum a sequence. The sum of a sequence up to a certain point is called a **series**, and we often use the symbol S_n to denote the sum of the first n terms of a sequence.

Worked example 7.3

Adding up consecutive odd numbers starting at 1 forms a series.

Let S_n denote the sum of the first n terms. List the first five terms of the sequence S_n and suggest a rule for it.

Start by examining the first few terms

$S_1 = 1$

$S_2 = 1 + 3 = 4$

$S_3 = 1 + 3 + 5 = 9$

$S_4 = 1 + 3 + 5 + 7 = 16$

$S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Do we recognise these numbers?

It seems that $S_n = n^2$

Defining such sums by saying 'Add up a defined sequence from a given start point to a given end point' is too wordy and imprecise for mathematicians.

Exactly the same thing is written in a shorter (although not necessarily simpler) way using **sigma notation**:

KEY POINT 7.2

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \cdots + f(n)$$

Greek capital sigma means 'add up'

This is the last value taken by r, where counting ends

r is a placeholder; it shows what changes with each new term

This is the first value taken by r; where counting starts

EXAM HINT

Do not be intimidated by this complicated-looking notation.

If you struggle with it, try writing out the first few terms.

If there is only one variable in the expression being summed, it is acceptable to miss out the '$r =$' above and below the sigma.

In the example we use both the letters n and r as unknowns – but they are not the same type of unknown.

If we replaced r by any other letter (apart from f or n) then the expression on the right would be unchanged. r is called a dummy variable. If we replaced n by any other letter then the expression would change.

Worked example 7.4

$T_n = \sum_{2}^{n} r^2$ Find the value of T_4.

Put the starting value, $r = 2$ into the expression to be summed	$T_4 = 2^2 + \ldots$
We've not reached the end value, so put in $r = 3$	$T_4 = 2^2 + 3^2 + \ldots$
We've not reached the end value, so put in $r = 4$	$T_4 = 2^2 + 3^2 + 4^2$
We've reached the end value, so evaluate	$T_4 = 4 + 9 + 16 = 29$

Worked example 7.5

Write the series $T_4 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ using sigma notation.

We must write in terms of the dummy variable r what eachterm of the sequence looks like

General term $= \frac{1}{r}$

What is the first value of r?

Starts when $r = 2$

What is the final value of r?

Ends when $r = 6$

Summarise in sigma notation

Series $= \sum_{2}^{6} \frac{1}{r}$

Exercise 7B

1. Evaluate the following expressions.

(a) (i) $\sum_{2}^{4} 3r$ (ii) $\sum_{5}^{7} (2r+1)$

(b) (i) $\sum_{3}^{6} (2^r - 1)$ (ii) $\sum_{-1}^{4} 1.5^r$

(c) (i) $\sum_{a=1}^{a=4} b\ (a+1)$ (ii) $\sum_{q=-3}^{q=2} pq^2$

2. Write the following expressions in sigma notation. Be aware that there is more than one correct answer.

(a) (i) $2 + 3 + 4 \ldots + 43$ (ii) $6 + 8 + 10 \ldots + 60$

(b) (i) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots + \frac{1}{128}$ (ii) $2 + \frac{2}{3} + \frac{2}{9} \ldots + \frac{2}{243}$

(c) (i) $14a + 21a + 28a \ldots + 70a$ (ii) $0 + 1 + 2^b + 3^b \ldots + 19^b,\ (b \neq 0)$

7C Arithmetic sequences

We will now focus on one particular type of sequence: one where there is a constant difference, known as the common difference, between consecutive terms.

This is called an **arithmetic sequence** (or an **arithmetic progression**). The standard notation for the difference between terms is d, so arithmetic sequences obey the recursive definition $u_{n+1} = u_n + d$.

Knowing the common difference is not enough to fully define the sequence. There are many different sequences with common difference 2, for example:

1, 3, 5, 7, 9, 11, … and 106, 108, 110, 112, 114, …

To fully define the sequence we also need the first term. Conventionally this is given the symbol u_1.

So the sequence 106, 108, 110, 112, 114, … is defined by:

$$u_1 = 106, d = 2.$$

Worked example 7.6

What is the fourth term of an arithmetic sequence with $u_1 = 300$, $d = -5$?

Use the recursive definition to find the first four terms $u_{n+1} = u_n + d$

$u_1 = 300$
$u_2 = u_1 - 5 = 295$
$u_3 = u_2 - 5 = 290$
$u_4 = u_3 - 5 = 285$

In the above example it did not take long to find the first four terms. But what if you had been asked to find the hundredth term? To do this efficiently we must move from the inductive definition of arithmetic sequences to the deductive definition.

We need to think about how arithmetic sequences are built up. To get to the nth term we start at the first term and add on the common difference $n - 1$ times. This suggests a formula:

KEY POINT 7.3

$$u_n = u_1 + (n - 1)d$$

Worked example 7.7

The fifth term of an arithmetic sequence is 7 and the eighth term is 16. What is the 100th term?

Write down the information given and relate it to u_1 and d to give an expression for the fifth term in terms of u_1 and d

$u_5 = u_1 + 4d$

But we are told that $u_5 = 7$

$7 = u_1 + 4d$ (1)

Repeat for the eighth term

$16 = u_1 + 7d$ (2)

continued . . .

Solve simultaneously (2) – (1)

$$9 = 3d$$
$$\Leftrightarrow d = 3$$
$$\therefore u_1 = -5$$

Write down the general term and use it to answer the question

$$u_n = -5 + (n-1) \times 3$$
$$\therefore u_{100} = -5 + 99 \times 3 = 292$$

EXAM HINT

Many exam-style questions on sequences and series involve writing the given information in the form of simultaneous equations and then solving them.

Worked example 7.8

An arithmetic progression has first term 5 and common difference 7. What is the term number corresponding to the value 355?

The question is asking for n when $u_n = 355$. Write this as an equation

$$355 = u_1 + (n-1)d = 5 + 7(n-1)$$

Solve this equation

$$350 = 7(n-1)$$
$$\Leftrightarrow 50 = n - 1$$
$$\Leftrightarrow n = 51$$

So 355 is the 51st term.

EXAM HINT

'Arithmetic progression' is just another way of saying 'arithmetic sequence'.

Make sure you know all the alternative expressions for the same thing.

Exercise 7C

1. Using Key point 7.3, find the general formula for each arithmetic sequence given the following information.

   (a) (i) First term 9, common difference 3

   (ii) First term 57, common difference 0.2

(b) (i) First term 12, common difference −1

(ii) First term 18, common difference $\frac{1}{2}$

(c) (i) First term 1, second term 4

(ii) First term 9, second term 19

(d) (i) First term 4, second term 0

(ii) First term 27, second term 20

(e) (i) Third term 5, eighth term 60

(ii) Fifth term 8, eighth term 38

2. How many terms are there in the following sequences?

(a) (i) 1, 3, 5, …, 65

(ii) 18, 13, 8, …, −122

(b) (i) First term 8, common difference 9, last term 899

(ii) First term 0, ninth term 16, last term 450

3. An arithmetic sequence has 5 and 13 as its first two terms.

(a) Write down, in terms of n, an expression for the nth term, u_n.

(b) Find the number of terms of the sequence which are less than 400. *[8 marks]*

4. The 10th term of an arithmetic sequence is 61 and the 13th term is 79. Find the value of the 20th term. *[4 marks]*

5. The 8th term of an arithmetic sequence is 74 and the 15th term is 137. Which term has the value 227? *[4 marks]*

6. The heights of the rungs in a ladder form an arithmetic sequence. The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground. If the top rung is 350 cm above the ground, how many rungs does the ladder have? *[5 marks]*

7. The first four terms of an arithmetic sequence are 2, $a - b$, $2a + b + 7$ and $a - 3b$, where a and b are constants. Find a and b. *[5 marks]*

8. A book starts at page 1 and is numbered on every page.

(a) Show that the first eleven pages contain thirteen digits.

(b) If the total number of digits used is 1260, how many pages are in the book? *[8 marks]*

7D Arithmetic series

When you add up terms of an arithmetic sequence you get an arithmetic series. There is a formula for the sum of an arithmetic series, the proof of which is not required in the IB. See Fill-in proof 4 'Arithmetic series and the story of Gauss' on the CD-ROM if you are interested.

There are two different forms for the formula.

KEY POINT 7.4

If you know the first and last terms:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

If you know the first term and the common difference:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Worked example 7.9

Find the sum of the first 30 terms of an arithmetic progression with first term 8 and common difference 0.5.

We have all the information we need to use the second formula

$$S_{30} = \frac{30}{2}(2 \times 8 + (30-1) \times 0.5) = 457.5$$

Sometimes you have to interpret the question carefully to be sure that it is about an arithmetic sequence.

Worked example 7.10

Find the sum of all the multiples of 3 between 100 and 1000.

Write out the first few terms to see what is happening

Sum $= 102 + 105 + 108 + \ldots + 999$

This is an arithmetic series with $u_1 = 102$ and $d = 3$

To use either sum formula, we also need to know how many terms are in this sequence

We do this by setting $u_n = 999$

$$999 = 102 + 3(n-1)$$

$$\Leftrightarrow 897 = 3(n-1)$$

$$\Leftrightarrow n = 300$$

Use the first sum formula

$$S_{300} = \frac{300}{2}(102 + 999) = 165\,150$$

You must be able to work backwards too; given information which includes the sum of the series, you may be asked to find out how many terms are in the series. Remember that the number of terms can only be a positive integer.

Worked example 7.11

An arithmetic sequence has first term 5 and common difference 10.

If the sum of all the terms is 720, how many terms are in the sequence?

We need to find n and it is the only unknown in the second sum formula

$$720 = \frac{n}{2}(2\times 5 + (n-1)\times 10)$$
$$= \frac{n}{2}(10 + 10n - 10)$$
$$= 5n^2$$

Solve this equation

$$n^2 = 144$$
$$n = \pm 12$$

But n must be a positive integer, so $n = 12$

Exercise 7D

1. Find the sum of the following arithmetic sequences:
   (a) (i) 12, 33, 54, … (17 terms)
   (ii) −100, −85, −70, … (23 terms)
   (b) (i) 3, 15, …, 459
   (ii) 2, 11, …, 650
   (c) (i) 28, 23, …, −52
   (ii) 100, 97, …,40
   (d) (i) 15, 15.5, …, 29.5
   (ii) $\frac{1}{12}, \frac{1}{6}, \ldots, 1.5$
2. An arithmetic sequence has first term 4 and common difference 8.

   How many terms are required to get a sum of:
   (a) (i) 676 (ii) 4096 (iii) 11236
   (b) $x^2, x > 0$
3. The second term of an arithmetic sequence is 7. The sum of the first four terms of the sequence is 12. Find the first term, a, and the common difference, d, of the sequence. *[5 marks]*

4. Consider the arithmetic series $2 + 5 + 8 + \ldots$:

(a) Find an expression for S_n, the sum of the first n terms.

(b) Find the value of n for which $S_n = 1365$. *[5 marks]*

5. Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ... *[6 marks]*

6. The second term of an arithmetic sequence is 6. The sum of the first four terms of the arithmetic sequence is 8. Find the first term, a, and the common difference, d, of the sequence. *[6 marks]*

7. Consider the arithmetic series $-6 + 1 + 8 + 15 + \ldots$

Find the least number of terms so that the sum of the series is greater than 10 000. *[6 marks]*

8. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n. *[6 marks]*

9. A circular disc is cut into twelve sectors whose angles are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector. Find the size of the angle of the smallest sector. *[6 marks]*

10. The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$.

If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence. *[7 marks]*

11. What is the sum of all three-digit numbers which are multiples of 14 but not 21? *[8 marks]*

7E Geometric sequences

Geometric sequences have a constant ratio, called the *common ratio, r*, between terms:

$$u_{n+1} = r \times u_n$$

So examples of geometric sequences might be:

$1, 2, 4, 8, 16, \ldots$ $(r = 2)$

$100, 50, 25, 12.5, 6.25, \ldots$ $(r = \frac{1}{2})$

$1, -3, 9, -27, 81, \ldots$ $(r = -3)$

As with arithmetic sequences, we also need to know the first term to fully define a geometric sequence. Again this is normally given the symbol u_1.

To get immediately to the deductive rule, we can see that to get to the nth term you start at the first term and multiply by the common ratio $n - 1$ times.

KEY POINT 7.5

$$u_n = u_1 r^{n-1}$$

Worked example 7.12

The seventh term of a geometric sequence is 13. The ninth term is 52.

What values could the common ratio take?

Write an expression for the seventh term in terms of u_1 and r

$$u_7 = u_1 r^6$$

But $u_7 = 13$

$$13 = u_1 r^6 \quad (1)$$

Repeat the same process for the ninth term

$$52 = u_1 r^8 \quad (2)$$

Solve the two equations simultaneously. Divide to eliminate u_1

$(2) \div (1)$

$$4 = r^2$$

$$\Leftrightarrow r = \pm 2$$

EXAM HINT

Notice that the question asked for values rather than value. This is a big hint that there is more than one solution.

When questions on geometric sequences ask what term satisfies a particular condition, you will usually use logarithms to solve an equation. Be careful when dealing with logarithms and inequalities; if you divide by the logarithm of a number less than 1 then you must flip the inequality.

Worked example 7.13

A geometric sequence has first term 10 and common ratio $\frac{1}{3}$. What is the first term that is less than 10^{-6}?

Express the condition as an inequality

$$10 \times \left(\frac{1}{3}\right)^{n-1} < 10^{-6}$$

$$\Leftrightarrow \left(\frac{1}{3}\right)^{n-1} < 10^{-7}$$

The unknown is in the power so we solve it using logarithms

$$\Leftrightarrow \log\left(\frac{1}{3}\right)^{n-1} < \log 10^{-7}$$

→

continued . . .

Use the logarithm law $\log_a x^p = p\log_a x$

$\Leftrightarrow\ (n-1)\log\left(\frac{1}{3}\right) < \log 10^{-7}$

$\log\left(\frac{1}{3}\right) < 0$ so reverse the inequality when dividing

$\Leftrightarrow\ (n-1) > \frac{\log 10^{-7}}{\log\left(\frac{1}{3}\right)}$

$\Leftrightarrow\ n > 15.7$ (3SF)

But n is a whole number so the first term satisfying the condition is 16

A particular problem is when the common ratio is negative, as we cannot take the log of a negative number. However, we can get around this problem using the fact that a negative number raised to an even power is always positive.

Inequalities are covered in Prior learning Section L.

Worked example 7.14

A geometric sequence has first term 2 and common ratio –3. What term has the value –4374?

Write the information given as an equation

$-4374 = 2\times(-3)^{n-1}$

Multiply both sides by –3 to make the left hand side positive

$\Leftrightarrow 13\,122 = 2\times(-3)^n$

Since the LHS is positive the RHS must also be positive, so n must be even and we can replace $(-3)^n$ with $(3)^n$

Since both sides must be positive:

$13\,122 = 2\times(3)^n$

Solve the equation using logarithms

$6561 = 3^n$

$\log 6561 = n\log 3$

$n = \frac{\log 6561}{\log 3} = 8$

In Worked example 7.14, you might have tried taking logarithms at the first line. Although this would have meant logs of negative numbers, using the rules of logs leads to the correct answer. This suggests that there may be some interpretation of logs of negative numbers, and when we meet complex numbers there will indeed be an interpretation. So does the logarithm of a negative number 'exist'? To some extent in mathematics if a concept is useful, that is enough to justify its introduction.

Exercise 7E

1. Find an expression for the nth term of these geometric sequences:

(a) (i) 6, 12, 24, … (ii) 12, 18, 27, …

(b) (i) 20, 5, 1.25, … (ii) $1, \frac{1}{2}, \frac{1}{4}, \dots$

(c) (i) 1, −2, 4, … (ii) 5, −5, 5, …

(d) (i) $a, ax, ax^2, \dots$ (ii) $3, 6x, 12x^2, \dots$

2. How many terms are there in the following geometric sequences?

(a) (i) 6, 12, 24, …, 24576 (ii) 20, 50, …, 4882.8125

(b) (i) $1, -3, \dots, -19683$ (ii) 2, −4, 8, …, −1024

(c) (i) $\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{1024}$ (ii) $3, 2, \frac{4}{3}, \dots, \frac{128}{729}$

3. How many terms are needed in the following geometric sequences to get within 10^{-9} of zero?

(a) (i) $5, 1, \frac{1}{5}, \dots$ (ii) 0.6, 0.3, 0.15, …

(b) (i) 4, −2, 1, … (ii) −125, 25, −5, …

4. The second term of a geometric sequence is 6 and the fifth term is 162. Find the tenth term. *[5 marks]*

5. The third term of a geometric sequence is 112 and the sixth term is 7168.

Which term takes the value 1 835 008? *[5 marks]*

6. Which is the first term of the sequence $\frac{2}{5}, \frac{4}{25}, \dots, \frac{2^n}{5^n}$ that is less than 10^{-6}? *[6 marks]*

7. The difference between the fourth and the third term of a geometric sequence is $\frac{75}{8}$ times the first term. Find all possible values of the common ratio. *[6 marks]*

8. The third term of a geometric progression is 12 and the fifth term is 48. Find the two possible values of the eighth term. *[6 marks]*

9. The first three terms of a geometric sequence are a, $a + 14$ and $9a$. Find all possible values of a. *[6 marks]*

10. The three terms a, 1, b are in arithmetic progression. The three terms 1, a, b are in geometric progression. Find the value of a and b given that $a \neq b$. *[7 marks]*

11. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. *[7 marks]*

7F Geometric series

As with arithmetic series there is a complicated formula for the sum of geometric sequences. See Fill-in proof 5 'Self-similarity and geometric series', on the CD-ROM for the derivation.

KEY POINT 7.6

$$S_n = \frac{u_1(1-r^n)}{1-r} \quad (r \neq 1)$$

or equivalently

$$S_n = \frac{u_1(r^n-1)}{r-1} \quad (r \neq 1)$$

We generally use the first of these formulae when the common ratio is less than one and the second when the common ratio is greater than one. This avoids working with negative numbers.

Worked example 7.15

Find the exact value of the sum of the first 6 terms of the geometric sequence with first term 8 and common ratio $\frac{1}{2}$.

$r < 1$, so use the first sum formula

$$S_6 = \frac{8\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$= \frac{8\left(1-\frac{1}{64}\right)}{\frac{1}{2}}$$

$$= 16\left(\frac{63}{64}\right)$$

$$= \frac{63}{4}$$

We may be given information about the sum and have to deduce other information.

Worked example 7.16

How many terms are needed for the sum of the geometric series 3 + 6 + 12 + 24 + … to exceed 10 000?

State the values of u_1 and r

$u_1 = 3$
$r = 2$

$r > 1$, so use the second sum formula and express the condition as an inequality

$$S_n = \frac{3(2^n - 1)}{2 - 1} > 10\,000$$

The unknown n is in the power, so use logarithms to solve the inequality

$$3(2^n - 1) > 10\,000$$

$$2^n > \frac{10\,003}{3}$$

$$\Leftrightarrow \log 2^n > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow n \log 2 > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow \quad n > \log\left(\frac{10\,003}{3}\right) \div \log 2$$

$$n > 11.7 \text{ (3SF)}$$

But n must be a whole number so 12 terms are needed.

Exercise 7F

Geometric series get very large very quickly. A mathematical legend involving the supposed inventor of chess, Sissa Ibn Dahir, illustrates how poor our intuition is with large numbers. It is explored on Supplementary sheet 6 'The chess legend and extreme numbers'.

1. Find the sums of the following geometric series. (There may be more than one possible answer!)

(a) (i) 7, 35, 175, … (10 terms)

(ii) 1152, 576, 288, … (12 terms)

(b) (i) 16, 24, 36, …, 182.25

(ii) 1, 1.1, 1.21, …, 1.771651

(c) (i) First term 8, common ratio −3, last term 52 488

(ii) First term −6, common ratio −3, last term 13 122

(d) (i) Third term 24, fifth term 6, 12 terms

(ii) Ninth term 50, thirteenth term 0.08, last term 0.0032

2. Find the possible values of the common ratio if the:

(a) (i) first term is 11 and the sum of the first 12 terms is 2 922 920

(ii) first term is 1 and the sum of the first 6 terms is 1.249 94

(b) (i) first term 12 and the sum of the first 6 terms is –79 980

(ii) first term is 10 and the sum of the first 4 terms is 1.

3. The nth term, u_n, of a geometric sequence is given by $u_n = 3 \times 5^{n+2}, n \in \mathbb{Z}^+$.

(a) Find the common ratio, r.

(b) Hence or otherwise, find S_n, the sum of the first n terms of this sequence. *[5 marks]*

4. The sum of the first three terms of a geometric sequence is $23\frac{3}{4}$ and the sum of the first four terms is $40\frac{5}{8}$. Find the first term and the common ratio. *[6 marks]*

5. (a) A geometric sequence has first term 1 and common ratio x. Express the sum of the first four terms as a polynomial in x.

(b) Factorise $x^6 - 1$ into a linear factor and a polynomial of order 5. *[6 marks]*

7G Infinite geometric series

If we keep adding together terms of any arithmetic sequence the answer grows (or decreases) without limit. The series is **divergent**.

Sometimes this happens with geometric series too, but there are cases where the sum gets closer and closer to a finite number. The series is **convergent**.

Not all geometric series converge. To decide which ones do, we need to use the formula for a geometric series from Key point 7.6.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

and look at the r^n term. We are interested in what happens to this as n gets very large.

When you raise most numbers to a large power the result gets bigger and bigger, except when r is a number between –1 and 1. In this case, r^n gets smaller as n increases – in fact it tends to 0.

Many other sequences and series show interesting long-term behaviour. For example if the series $\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}$ continues for ever, the result is π. Series like these are investigated in Supplementary sheet 5 'Long-term behaviour of sequences and series' on the CD-ROM.

EXAM HINT

The condition that $|r|<1$ is just as important as the formula.

We can summarise:

KEY POINT 7.7

As n increases the sum of a geometric series converges to:

$$S_\infty = \frac{u_1}{1-r} \text{ if } |r|<1$$

This is called the **sum to infinity** of the series.

When $r = 1$ the geometric sequence definitely diverges. When $r = -1$ it is uncertain whether the sequence converges or not. It might converge to 0, to u_1 or to $\frac{u_1}{2}$ depending upon how the terms are grouped. This is an example of a situation where mathematics is open to interpretation.

Worked example 7.17

The sum to infinity of a geometric series is 5. The second term is $-\frac{6}{5}$. Find the common ratio.

Express the information given as equations in terms of u_1 and r

$S_\infty = \frac{u_1}{1-r} = 5$ (1)

$u_2 = u_1 r = -\frac{6}{5}$ (2)

Solve the equations simultaneously

From (2) $u_1 = -\frac{6}{5r}$

Substituting into (1)

$-\frac{6}{5r(1-r)} = 5$

$\Leftrightarrow \quad -6 = 25(r - r^2)$

$\Leftrightarrow \quad 0 = 25r^2 - 25r - 6 = (5r-6)(5r+1)$

Therefore $r = \frac{6}{5}$ or $r = -\frac{1}{5}$

Watch out for a trick! Check that the series converges

But since the sum to infinity exists, $|r|<1$ so

$r = -\frac{1}{5}$

Remember that some questions may focus on the condition for the sequence to converge as well as the value that it converges to.

Worked example 7.18

The geometric series $(2-x)+(2-x)^2+(2-x)^3+\dots$ converges. What values can x take?

Identify r

$r=(2-x)$

Use the fact that the series converges

Since the series converges $|2-x|<1$

Solve the inequality

$-1<2-x<1$

$\Leftrightarrow -3<-x<-1$

Therefore $1<x<3$

Exercise 7G

1. Find the value of these infinite geometric series, or state that they are divergent.
   (a) (i) $9+3+1+\frac{1}{3}+\dots$
   (ii) $56+8+1\frac{1}{7}+\dots$
   (b) (i) $0.3+0.03+0.003+\dots$
   (ii) $0.78+0.0078+0.000078+\dots$
   (c) (i) $0.01+0.02+0.04+\dots$
   (ii) $\frac{19}{10000}+\frac{19}{1000}+\frac{19}{100}+\dots$
   (d) (i) $10-2+0.4-\dots$
   (ii) $6-4+\frac{8}{3}-\dots$
   (e) (i) $10-40+160-\dots$
   (ii) $4.2-3.36+2.688-\dots$
2. Find the values of x which allow these geometric series to converge.
   (a) (i) $9+9x+9x^2+\dots$
   (ii) $-2-2x-2x^2-\dots$
   (b) (i) $1+3x+9x^2+\dots$
   (ii) $1+10x+100x^2+\dots$
   (c) (i) $-2-10x-50x^2-\dots$
   (ii) $8+24x+72x^2+\dots$

(d) (i) $40+10x+2.5x^2+\dots$

(ii) $144+12x+x^2+\dots$

(e) (i) $243-81x+27x^2-\dots$

(ii) $1-\frac{5}{4}x+\frac{25}{16}x^2-\dots$

(f) (i) $3-\frac{6}{x}+\frac{12}{x^2}-.$

(ii) $18-\frac{9}{x}+\frac{1}{x^2}-\dots$

(g) (i) $5+5(3-2x)+5(3-2x)^2+\dots$

(ii) $7+\frac{7(2-x)}{2}+\frac{7(2-x)^2}{4}+\dots$

(h) (i) $1+\left(3-\frac{2}{x}\right)+\left(3-\frac{2}{x}\right)^2+\dots$

(ii) $1+\frac{1+x}{x}+\frac{(1+x)^2}{x^2}+\dots$

(i) (i) $7+7x^2+7x^4+\dots$

(ii) $12-48x^3+192x^6-\dots$

3. Find the sum to infinity of the geometric sequence
$-18, 12, -8, \dots$ *[4 marks]*

4. The first and fourth terms of a geometric series are 18 and $-\frac{2}{3}$ respectively. Find:

(a) the sum of the first n terms of the series

(b) the sum to infinity of the series. *[5 marks]*

5. A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

(a) the common ratio

(b) the first term. *[5 marks]*

6. The sum to infinity of a geometric sequence is 32. The sum of the first four terms is 30 and all the terms are positive.

Find the difference between the sum to infinity and the sum of the first eight terms. *[5 marks]*

7. Consider the infinite geometric series:

$$1+\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^2+\dots$$

(a) For what values of x does the series converge?

(b) Find the sum of the series if $x=1.2$. *[6 marks]*

8. The sum of an infinite geometric sequence is 13.5, and the sum of the first three terms is 13. Find the first term. *[6 marks]*

9. An infinite geometric series is given by $\sum_{k=1}^{\infty} 2(4-3x)^k$.

(a) Find the values of x for which the series has a finite sum.

(b) When $x = 1.2$, find the minimum number of terms needed to give a sum which is greater than 1.328. *[7 marks]*

10. The common ratio of the terms in a geometric series is $2x$.

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40. *[6 marks]*

11. $f(x) = 1 + 2x + 4x^2 + 8x^3 \ldots$ is an infinitely long expression. Evaluate:

(a) $f\left(\frac{1}{3}\right)$ (b) $f\left(\frac{2}{3}\right)$ *[6 marks]*

7H Mixed questions

Be very careful when dealing with sequences and series questions.

It is vital that you

- identify whether it is a geometric or an arithmetic sequence
- identify whether it is asking for a term in the sequence or the sum of terms in the sequence
- interpret the information given in the question into equations.

One frequently examined topic is **compound interest**. This is about savings or loans, where the interest added is a percentage of the current amount. As long as no other money is added or removed, the value of the investment will follow a geometric sequence.

If the compound interest rate is $p\%$ then this is equivalent to a ratio of $r = 1 + \frac{p}{100}$.

Exercise 7H

1. Philippa invests £1000 at 3% compound interest for 6 years.

(a) How much interest does she get paid in the 6th year?

(b) How much does she get back after 6 years? *[6 marks]*

2. Lars starts a job on an annual salary of \$32 000 and is promised an annual increase of \$1500.

(a) How much will his 20th year's salary be?

(b) After how many complete years will he have earned a total of \$1 million? *[6 marks]*

3. A sum of \$5000 is invested at a compound interest rate of 6.3% per annum.

(a) Write down an expression for the value of the investment after n full years.

(b) What will be the value of the investment at the end of 5 years?

(c) The value of the investment will exceed \$10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n. *[8 marks]*

4. Each consecutive row of seats in a theatre has 200 more seats than the previous row. There are 50 seats in the front row and the designer wants the theatre capacity to be at least 8000.

(a) How many rows are required?

(b) Assuming the spacings between rows are equal, what percentage of people are seated in the front half of the theatre? *[7 marks]*

5. A sum of \$100 is invested.

(a) If the interest is compounded annually at a rate of 5% per year, find the total value V of the investment after 20 years.

(b) If the interest is compounded monthly at a rate of $\frac{5}{12}$ % per month, find the minimum number of months for the value of the investment to exceed V. *[6 marks]*

6. A marathon is a 26 mile race. In his training program, a marathon a runner runs 1 mile on his first day of training and each day increases his distance by $\frac{1}{4}$ of a mile.

(a) After how many days has he run for a total of 26 miles?

(b) On which day does he first run over 26 miles? [6 *marks*]

7. A ball is dropped vertically from 2 m in the air. Each time it bounces up to a height of 80% of its previous height.

(a) How high does it bounce on the fourth bounce?

(b) How far has it travelled when it hits the ground for the ninth time? *[7 marks]*

8. Samantha puts \$1000 into a bank account at the beginning of each year, starting in 2010.

At the end of each year 4% interest is added to the account.

(a) Show that at the beginning of 2012 there is $\$1000+\$1000\times 1.04+\$1000\times(1.04)^2$ in the account.

(b) Find an expression for the amount in the account at the beginning of year n.

(c) When Samantha has a total of at least \$50 000 in her account at the beginning of a year she will buy a house. In which year will this happen? *[7 marks]*

Summary

- Sequences can be described using either **recursive definitions**– from term-to-term – or **deductive rules** – finding the nth term.
- A **series** is a sum of terms in a sequence and it can be described neatly using sigma notation:

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \ldots + f(n)$$

- One very important type of sequence is an **arithmetic sequence** which has a constant difference, d, between terms. The relevant formulae are given in the Formula booklet:
  - If you know the first term, u_1, the general term is:

$$u_n = u_1 + (n-1)d$$

  - If you know the first and last terms the sum of all n terms in the sequence is:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

  - If you know the first term and the common difference:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

- Another very important type of sequence is a **geometric sequence** which has a constant ratio, r, between terms. The following are also given in the Formula booklet:
  - If you know the first term, u_1, the general term is:

$$u_n = u_1 r^{n-1}$$

  - The sum of the first n terms is:

$$S_n = \frac{u_1(1-r^n)}{1-r} \text{ or } \frac{u_1(r^n-1)}{r-1} (r \neq 1)$$

- A series can be **convergent** (the sum gets closer and closer to a single value) or **divergent** (keeps increasing or decreasing without limit)
  - If $|r|<1$ the sum to infinity of a geometric sequence is given by:

$$S_\infty = \frac{u_1}{1-r}$$

Introductory problem revisited

A mortgage of \$100 000 is fixed at 5% compound interest. It needs to be paid off over 25 years by annual instalments. Interest is added at the end of each year, just before the payment is made. How much should be paid each year?

Imagine that you have two separate bank accounts. One is overdrawn and interest is added annually to the debt. You make regular payments to the second account, and this account earns interest each year at the same rate.

The first payment you make will have interest paid on it 24 times. The second payment will have 23 interest payments and so on.

The amount in the debt account after 25 years will be $100\,000 \times 1.05^{25}$.

If the annual payment is \$$x$, the amount in the credit account will be:

$$x \times 1.05^{24} + x \times 1.05^{23} + x \times 1.05^{22} + \ldots + x \times 1.05^{1} + x$$

But this is a finite geometric series with 25 terms, first term x and common ratio 1.05 so it can be simplified to:

$$\frac{x(1.05^{25}-1)}{1.05-1}$$

If the debt is to be paid off, the amount in the credit account must equal the amount in the debit account, so:

$$100\,000 \times 1.05^{25} = \frac{x(1.05^{25}-1)}{1.05-1}$$

Solving this gives $x = 7095.25$ and the annual payments must be \$7095.25.

Mixed examination practice 7

Short questions

1. The fourth term of an arithmetic sequence is 9.6 and the ninth term is 15.6.
Find the sum of the first nine terms. *[5 marks]*

2. The sum of the first n terms of a series is given by:

$$S_n = 2n^2 - n, \text{ where } n \in \mathbb{Z}^+.$$

(a) Find the first three terms of the series.

(b) Find an expression for the nth term of the series, giving your answer in terms of n. *[6 marks]*

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3. Which is the first term of this sequence which is less than 10^{-6}?

$$\frac{1}{3}, \frac{1}{9} \dots \frac{1}{3^n}$$ *[5 marks]*

4. The fifth term of an arithmetic sequence is three times larger than the second term. Find the ratio: $\dfrac{\text{common difference}}{\text{first term}}$ *[6 marks]*

5. A geometric sequence and an arithmetic sequence both start with a first term of 1. The third term of the arithmetic sequence is the same as the second term of the geometric sequence. The fourth term of the arithmetic sequence is the same as the third term of the geometric sequence. Find the possible values of the common difference of the arithmetic sequence. *[7 marks]*

6. Evaluate $\displaystyle\sum_{i=0}^{i=\infty} \frac{(2^i + 4^i)}{6^i}$. *[6 marks]*

7. Find the sum of all the integers between 300 and 600 which are divisible by 7. *[7 marks]*

8. Find an expression for the sum of the first 23 terms of the series

$$\ln\frac{a^3}{\sqrt{b}} + \ln\frac{a^3}{b} + \ln\frac{a^3}{b\sqrt{b}} + \ln\frac{a^3}{b^2} + \dots$$

giving your answer in the form $\ln\dfrac{a^m}{b^n}$, where $m, n \in \mathbb{Z}$. *[7 marks]*

Long questions

1. Kenny is offered two investment plans, each requiring an initial investment of \$10 000:

Plan A offers a fixed return of \$800 per year.

Plan B offers a return of 5% each year, reinvested in the plan.

(a) Find an expression for the amount in plan A after n years.

(b) Find an expression for the amount in plan B after n years.

(c) Over what period of time is plan A better than plan B? *[10 marks]*

2. Ben builds a pyramid out of toy bricks. The top row contains one brick, the second row contains three bricks and each row after that contains two more bricks than the previous row.

(a) How many bricks are in the nth row?

(b) If a total of 36 bricks are used how many rows are there?

(c) In Ben's largest ever pyramid he noticed that the total number of bricks was four more than four times the number of bricks in the bottom row. What is the total number of bricks? *[10 marks]*

3. A pupil writes '1' on the first line of a page, then the next two integers '2, 3' on the second line of the page then the next three integers '4, 5, 6' on the third line. He continues this pattern.

(a) How many integers are on the nth line?

(b) What is the last integer on the nth line?

(c) What is the first integer on the nth line?

(d) Show that the sum of all the integers on the nth line is $\frac{n}{2}(n^2+1)$.

(e) The sum of all the integers on the last line of the page is 16 400. How many lines are on the page? *[10 marks]*

4. Selma has a mortgage of £150 000. At the end of each year 6% interest is added before Selma pays £10 000.

(a) Show that at the end of the third year the amount owing is

$$£150000\times(1.06)^3-10000\times(1.06)^2-10000\times1.06-10000$$

(b) Find an expression for how much is owed at the end of the nth year.

(c) After how many years will the mortgage be paid off? *[10 marks]*

8 Binomial expansion

In this chapter you will learn:

- how to expand an expression of the form $(a + b)^n$ for positive integer n
- how to find individual terms in the expansion of $(a + b)^n$ for positive integer n
- how to use partial expansions of $(a + bx)^n$ to find approximate values.

Introductory problem

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

A **binomial** expression is one which contains two terms, for example, $a + b$.

Expanding a power of such a binomial expression could be performed by expanding brackets; for example $(a+b)^7$ could be found by calculating, at length,

$$(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)(a+b).$$

This is time-consuming, but happily there is a much faster approach.

The Ancient Babylonians made an unexpected use of expanding brackets – it helped them multiply numbers in their base 60 number system. This is explored in Supplementary sheet 7 'Babylonian multiplication' on the CD-ROM.

8A Introducing the binomial theorem

To see how to expand an expression of the form $(a + b)^n$ for integer n rapidly, consider the following expansions of $(a + b)^n$, done using the slow method of repeatedly multiplying out brackets.

$$
\begin{aligned}
(a+b)^0 &= 1 \\
&= 1a^0b^0 \\
(a+b)^1 &= a+b \\
&= 1a^1b^0 + 1a^0b^1 \\
(a+b)^2 &= a^2+2ab+b^2 \\
&= 1a^2b^0 + 2a^1b^1 + 1a^0b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
&= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4
\end{aligned}
$$

The red numbers form a famous mathematical construction called Pascal's triangle. Each value not on the edge is formed by adding up the value directly above and the value to its left. There are many amazing patterns in Pascal's triangle – for example, if you highlight all the even numbers you generate an ever-repeating pattern called a fractal.

See chapter 1, Section D for more combinations $\binom{n}{r}$.

See Calculator skills sheet 3 on the CD-ROM for a reminder of how to use your calculator to find $\binom{n}{r}$.

We can see several patterns in this structure.

- The powers of a and b always total n (in the 3rd row, $3+0=2+1=1+2=0+3=3$).
- Each power of a from 0 up to n is present in one of the terms, with the corresponding complementary power of b.
- Each **term** has a **coefficient** (given in red), and the pattern of coefficients in each line is symmetrical.

In this section we shall focus on the coefficients. The pattern of numbers may seem familiar as they are all numbers which were found when calculating combinations $\binom{n}{r}$. In this context, $\binom{n}{r}$ is called a **binomial coefficient**.

KEY POINT 8.1

Binomial coefficient

The binomial coefficient, $\binom{n}{r}$, is the coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$.

Worked example 8.1

Find the coefficient of x^5y^3 in the expansion of $(x+y)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a=x$, $b=y$, $n=8$, $r=3$

The relevant term is $\binom{8}{3}(x)^5(y)^3$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{8}{3}=56$

$(x)^5=x^5$

$(y)^3=y^3$

Combine the elements to calculate the coefficient

The term is $56x^5y^3$

The coefficient is 56

EXAM HINT

Questions might ask you to give either the whole term or just the coefficient. Make sure that you answer the question!

Exercise 8A

1. (a) Find the coefficient of xy^3 in the expansion of $(x+y)^4$.
   (b) Find the coefficient of x^3y^4 in the expansion of $(x+y)^7$.
   (c) Find the coefficient of ab^6 in the expansion of $(a+b)^7$.
   (d) Find the coefficient of a^5b^3 in the expansion of $(a+b)^8$.

2. (a) Find the term in x^5y^7 in the expansion of $(x+y)^{12}$.
   (b) Find the term in a^7b^9 in the expansion of $(a+b)^{16}$.
   (c) Find the term in c^3d^2 in the expansion of $(c+d)^5$.
   (d) Find the term in a^2b^7 in the expansion of $(a+b)^9$.
   (e) Find the term in x^2y^4 in the expansion of $(x+y)^6$.

8B Applying the binomial theorem

In Section 8A, you saw the general pattern for expanding powers of a binomial expression $(a+b)$. When expanding powers of more complicated expressions, you will still use this method, but may substitute more complicated expressions for a and b.

Worked example 8.2

Find the term in x^6y^4 in the expansion of $(x+3y^2)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a=x$, $b=3y^2$, $n=8$, $r=2$

The relevant term is $\binom{8}{2}(x)^6(3y^2)^2$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{8}{2}=28$

$(x)^6=x^6$

$(3y^2)^2=9y^4$

The term is $28\times x^6\times 9y^4=252x^6y^4$

Many examination questions ask you to focus on just one term, but you should also be able to find the entire expansion. To do this you repeat the process for each term for every possible value of r (from 0 up to n) and add together the results. This result is quoted in the Formula booklet.

EXAM HINT

Take care to apply the power not only to the algebraic part but also to its coefficient. In Worked example 8.2, $(3y^2)^2 = 9y^4$, not $3y^4$.

KEY POINT 8.2

Binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

Worked example 8.3

Use the Binomial theorem to expand and simplify $(2x-3y)^5$.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2x$, $b = -3y$, $n = 5$
Coefficients: 1, 5, 10, 10, 5, 1

The expansion is
$1(2x)^5 + 5(2x)^4(-3y)^1 + 10(2x)^3(-3y)^2$
$+ 10(2x)^2(-3y)^3 + 5(2x)^1(-3y)^4 + 1(-3y)^5$

Apply the powers to the bracketed terms and multiply through

$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

EXAM HINT

A common mistake is to assume that the powers of each variable correspond to the value of r in the expansion.

A question may ask for a binomial expansion where both of the terms in the binomial expression contain the same variable. You can use the rules of exponents to determine which term of the expansion is needed.

Worked example 8.4

Find the coefficient of x^5 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$.

Start with the form of a general term and simplify using the rules of exponents

Each term will be of the form
$\binom{7}{r}(2x^2)^{7-r}(-x^{-1})^r$
$= \binom{7}{r}(2)^{7-r}x^{14-2r}(-1)^r x^{-r}$
$= \binom{7}{r}(2)^{7-r}(-1)^r x^{14-3r}$

continued . . .

We need the term in x^5, so equate that to the power of x in the general term

Require that $14 - 3r = 5$

$\Leftrightarrow \quad 3r = 9$

$\Leftrightarrow \quad r = 3$

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2x^2$, $b = x^{-1}$, $n = 7$, $r = 3$

The relevant term is $\binom{7}{3}(2x^2)^4(-x^{-1})^3$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{7}{3} = 35$

$(2x^2)^4 = 16x^8$

$(-x^{-1})^3 = -x^{-3}$

Combine the elements to calculate the coefficient

The term is $35 \times 16x^8 \times (-x^{-3}) = -560x^5$

The coefficient is -560

EXAM HINT

Don't forget that if there is a negative sign it must stay part of the coefficient of the term it is acting upon. Lots of people fall into this trap!

Applying this process in reverse is always quite tricky, since in general you cannot 'undo' the $\binom{n}{r}$ operation. So, you must use the formula for $\binom{n}{r}$, seen in chapter 1, to rewrite it as a polynomial $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Remind yourself of Key point 1.6 in chapter 1.

Worked example 8.5

The coefficient of x^2 in the expansion of $(1+3x)^n$ is 189. Find n if $n > 0$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 1$, $b = 3x$, $r = 2$

Required term is

$\binom{n}{2}(1)^{n-2}(3x)^2$

continued . . .

Simplify the coefficient algebraically and apply the powers to the bracketed terms

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

$$(1)^{n-2} = 1$$

$$(3x)^2 = 9x^2$$

Combine these and equate to the given information

$$\frac{9n(n-1)}{2}x^2 \equiv 189x^2$$

Compare coefficients

$$\frac{9n(n-1)}{2} = 189$$

$$9n^2 - 9n = 378$$

$$n^2 - n - 42 = 0$$

$$(n-7)(n+6) = 0$$

$n = 7$ or $n = -6$

but $n > 0$ $\therefore n = 7$

The binomial expansion can be generalised to situations where n is negative or a fraction. This general binomial expansion results in an infinite polynomial.

Exercise 8B

EXAM HINT

The constant coefficient is the term in x^0. It may also be described as the term independent of x.

1. (a) Find the coefficient of xy^3 in:

(i) $(2x+3y)^4$ (ii) $(5x+y)^4$

(b) Find the term in x^3y^4 in:

(i) $(x-2y)^7$ (ii) $(y-2x)^7$

(c) Find the coefficient of a^2b^3 in:

(i) $\left(2a - \frac{1}{2}b\right)^5$ (ii) $(17a+3b)^5$

2. (a) Find the coefficient of x^2 in the expansion of:

(i) $\left(x + \frac{1}{x}\right)^8$ (ii) $\left(2x + \frac{1}{\sqrt{x}}\right)^5$

(b) Find the constant coefficient in the expansion of:

(i) $(x - 2x^{-2})^9$ (ii) $(x^3 - 2x^{-1})^4$

3. (a) Fully expand and simplify:

(i) $(2-x)^5$ (ii) $(3+x)^6$

(b) (i) Find the first three terms in descending powers of x of $(3x+y)^5$.

(ii) Find the first three terms in ascending powers of d of $(2c-d)^4$.

(c) Fully expand and simplify:

(i) $(2x^2-3x)^3$ (ii) $(2x^{-1}+5y)^3$

(d) Fully expand and simplify:

(i) $\left(2z^2+\frac{3}{z}\right)^4$ (ii) $\left(3xy+\frac{5x}{y}\right)^3$

4. Write in polynomial form:

(a) $\binom{n}{1}$ (b) $\binom{n}{2}$ (c) $\binom{n}{3}$

5. Which term in the expansion of $(x-2y)^5$ has coefficient:

(a) 80 (b) −80 *[6 marks]*

6. Find the coefficient of x^2y^6 in $(3x+2y^2)^5$. *[5 marks]*

7. Find the term in x^5 in $\left(x^2-\frac{3}{x}\right)^7$. *[6 marks]*

8. Find the term that is independent of x in the expansion of $\left(2x-\frac{5}{x^2}\right)^{12}$. *[6 marks]*

9. The expansion of $(1+3x)^n$ starts with $1+42x\ldots$ Find the value of n. *[4 marks]*

10. The coefficient of x^2 in $(1+2x)^n$ is 264. Find the value of n given that $n\in\mathbb{N}$. *[6 marks]*

11. The coefficient of x^3 in $(1-5x)^n$ is −10 500. Find the value of n given that $n\in\mathbb{N}$. *[5 marks]*

12. The coefficient of x^2 in $(3+2x)^n$ is 20 412. Find the value of n given that $n\in\mathbb{N}$. *[6 marks]*

8C Products of binomial expansions

We may need to work with a product of a binomial and another expression. It is possible to do this by working with the entire expansion.

Worked example 8.6

Use the Binomial theorem to expand and simplify $(5-3x)(2-x)^4$.

Expand $(2-x)^4$. Coefficients are: 1, 4, 6, 4, 1

The expansion is

$(5-3x)\left[1(2)^4+4(2)^3(-x)+6(2)^2(-x)^2+4(2)^1(-x)^3+1(-x)^4\right]$

Multiply each term in the second bracket by 5 and then by $-3x$

$=5\left[16-32x+24x^2-8x^3+x^4\right]$

$\quad -3x\left[16-32x+24x^2-8x^3+x^4\right]$

$=80-208x+216x^2-112x^3+29x^4-3x^5$

This method is quite long-winded. In examinations you will usually only be asked to find a small number of terms from such an expression.

Worked example 8.7

Find the coefficient of a^4b^3 in the expansion of $(a-5b)(a+b)^6$.

Split the product into two parts, and treat each separately

$a(a+b)^6-5b(a+b)^6$

Decide which term from each expansion is needed to make a^4b^3

For a^4b^3, we need a^3b^3 from the first bracket ($a\times a^3b^3=a^4b^3$) and a^4b^2 from the second ($b\times a^4b^2=a^4b^3$)

Write down the required terms in the form $\binom{n}{r}(a)^{n-r}(b)^r$

a^4b^3 will arise from

$a\times\binom{6}{3}(a)^3(b)^3-5b\times\binom{6}{2}(a)^4(b)^2$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{6}{3}=20$ $\qquad$ $\binom{6}{2}=15$

$(a)^3=a^3$ $\qquad$ $(a)^4=a^4$

$(b)^3=b^3$ $\qquad$ $(b)^2=b^2$

Combine the elements to calculate the coefficient

The term is $20a^4b^3-75a^4b^3=-55a^4b^3$

The coefficient is -55

As you get more practised, you may not need to split the problem into several parts explicitly; always consider all possible ways that you can multiply to get the required term.

Worked example 8.8

Find the coefficient of x^4 in the expansion of $(1+3x-x^2)(2+x)^5$.

Split the product into three parts, and treat each separately

$1(2+x)^5 + 3x(2+x)^5 - x^2(2+x)^5$

Decide which term from each expansion is needed to make x^4

Write down the required terms in the form $\binom{n}{r}(a)^{n-r}(b)^r$

x^4 will arise from

$1\times\binom{5}{4}(2)^1 x^4 + 3x\times\binom{5}{3}(2)^2 x^3 - x^2\times\binom{5}{2}(2)^3 x^2$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{5}{4}=5$ $\quad\binom{5}{3}=10$ $\quad\binom{5}{2}=10$

$(2)^1=2$ $\quad(2)^2=4$ $\quad(2)^3=8$

Combine the elements to calculate the coefficient

The term is

$1\times5\times2x^4 + 3x\times10\times4x^3 - x^2\times10\times8x^2$

$=10x^4+120x^4-80x^4$

$=50x^4$

The coefficient is 50

Exercise 8C

1. (a) (i) Find the coefficient of x^2y^5 in the expansion of $(x-y)(x+y)^6$.

   (ii) Find the coefficient of x^5 in the expansion of $(1+3x)(1+x)^7$.

   (b) (i) Find the coefficient of x^6 in the expansion of $(1-x^2)(1+x)^5$.

   (ii) Find the coefficient of x^6 in the expansion of $(1-x^2)(1+x)^7$.

2. (a) (i) Find the coefficient of x^2y in the expansion of $(1+x)^3(1+y)^5$.

(ii) Find the coefficient of xy^3 in the expansion of $(1+x)^4(1+y)^5$.

3. (i) Find the coefficient of c^4d^{11} in the expansion of $(2c+5d)(c+d)^{14}$.

(ii) Find the coefficient of a^3b^{15} in the expansion of $(3a-b)(a+b)^{17}$.

4. (a) (i) Find the first three terms in descending powers of x of $(3x+7)(x^2-2x)^4$.

(ii) Find the first three terms in descending powers of x of $(x-x^2)(x-3)^5$.

(b) (i) Find the first three terms in ascending powers of x of $(x-1)^4(x+1)^5$.

(ii) Find the first three terms in ascending powers of x of $(x+2)^4(2-x)^3$.

5. Find the first 4 terms in the expansion of $(y+3y^2)^6$ in ascending powers of y. *[6 marks]*

6. Find the first three non-zero terms of the expansion of $(1-x)^{10}(1+x)^{10}$ in ascending powers of x. *[6 marks]*

7. Find the first 4 terms in the expansion of $(1-2x+x^2)^{10}$ in ascending powers of x. *[6 marks]*

8. Given that $(1+x)^3(1+mx)^4 \equiv 1+nx+93x^2+\ldots+m^4x^7$, find the possible values of m and n. *[8 marks]*

9. Given that $(1+kx)^4(1+x)^n \equiv 1+13x+74x^2+\ldots+k^4x^{n+4}$, find the possible positive integer values of k and n. *[8 marks]*

8D Binomial expansions as approximations

Calculators and computers use binomial expansions to work out powers and roots.

One of the main applications of the binomial expansion is in calculating approximate values of powers and roots. When x is a very small value, high powers of x become increasingly small, and so they have little impact on the value of the total sum, even when multiplied by the binomial coefficient. For this reason, using only the first few terms gives a good approximation to the total value of the sum.

KEY POINT 8.3

> If the value of x is much less than one, large powers of x will be extremely small.

Worked example 8.9

Find the first 3 terms in ascending powers of x of the expansion of $(2-x)^5$.
By setting $x = 0.01$, use your answer to find an approximate value of 1.99^5.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2, b = -x, n = 5$

The first 3 terms are
$1(2)^5 + 5(2)^4(-x)^1 + 10(2)^3(-x)^2$

Apply the powers to the bracketed terms and multiply through

$= 32 - 80x + 80x^2$

Calculate the powers of the appropriate value of x and thus the value of each term

$x^0 = 1$ $\quad 32x^0 = 32$
$x^1 = 0.01$ $\quad -80x^1 = -0.8$
$x^2 = 0.0001$ $\quad 80x^2 = 0.008$

Total the values of the terms

Hence $1.99^5 \simeq 31.208$

Exercise 8D

1. (a) Find the first 4 terms in the expansion of $(1+5x)^7$.
By setting $x = 0.01$, find an approximation for 1.05^7, leaving your answer correct to 6 significant figures.

(b) Find the first 3 terms in the expansion of $(2+3x)^6$.
By setting $x = 0.001$, find an approximation for 2.003^6, leaving your answer correct to 6 significant figures.

2. (a) Find the first 3 terms in the expansion of $(3-5x)^4$.
(b) Using a suitable value of x, use your answer to find a 6 significant figure approximation for 2.995^4. *[7 marks]*

3. (a) Find the first 3 terms in the expansion of $(2+5x)^7$.
(b) Using a suitable value of x, use your answer to find a 6 significant figure approximation for 2.005^7. *[7 marks]*

4. (a) Find the first 3 terms in the expansion of $(2+3x)^7$.

(b) Hence find an approximation to:

(i) 2.3^7 (ii) 2.03^7

(c) Which of your answers in part (b) provides a more accurate approximation? Justify your answer. *[6 marks]*

Summary

- A **binomial** expression is one that contains two terms, e.g., $a + b$.
- The **binomial coefficient**, $\binom{n}{r}$, is the coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$.
- The expansion of $(a+b)^n$ can be accomplished directly using the **binomial theorem**:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

- The coefficient of individual terms of a binomial expansion can be found by considering the powers of the algebraic components and using the formula for the rth term:

$$\binom{n}{r}(a)^{n-r}(b)^r$$

- By reversing this process and using the polynomial form of $\binom{n}{r}\frac{n!}{r!(n-r)!}$, you can also find the value of n if you have a term of the binomial expansion.
- Approximations for powers and roots can be made using the first few terms of a binomial expansion $(a+bx)^n$, valid when bx is very much less than one, meaning that terms with higher powers are increasingly small.

Introductory problem revisited

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

We recognise that $(1.002)^{10}$ can be calculated by evaluation of the binomial expansion $(1 + 2x)^{10}$ with $x = 0.001$.

To ensure accuracy to 8 decimal places, we need to include terms at least to x^3, but can safely disregard terms in x^4 and greater powers, since the magnitudes of the coefficients mean these are too small to affect the first 8 decimal places.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 1, b = 2x, n = 10$

The first 4 terms are
$1(1)^{10} + 10(1)^9(2x)^1 + 45(1)^8(2x)^2 + 120(1)^7(2x)^3$

Apply the powers to the bracketed terms and multiply through

The first 4 terms are
$1 + 20x + 180x^2 + 960x^3$

Calculate the powers of the appropriate value of x and thus the value of each term

$x^0 = 1$ $1 = 1$
$x^1 = 0.001$ $20x^1 = 0.02$
$x^2 = 0.000\,001$ $180x^2 = 0.000\,18$
$x^3 = 0.000\,000\,001$ $960x^3 = 0.000\,000\,96$

Total the values of the terms

Hence $1.002^{10} \simeq 1.020\,180\,96$

From calculator, $1.002^{10} = 1.020\,180\,963\,368\,08\ldots$
So approximation error is $3.30 \times 10^{-9} = 0.000\,000\,33\%$.

Mixed examination practice 8

Short questions

1. Find the coefficient of x^5 in the expansion of $(2-x)^{12}$. *[5 marks]*

2. $a = 2-\sqrt{2}$. Using the binomial theorem or otherwise, express a^5 in the form $m+n\sqrt{2}$. *[5 marks]*

3. (a) Find the expansion of $(2+x)^5$, giving your answer in ascending powers of x.

(b) By letting $x = 0.01$ or otherwise, find the exact value of 2.01^5. *[7 marks]*

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4. Determine the first 3 terms in the expansion of $(1-2x)^3(3+4x)^5$. *[7 marks]*

5. Fully expand and simplify $\left(x^2 - \frac{2}{x}\right)^4$. *[6 marks]*

6. The coefficient of x in the expansion of $\left(x+\frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$. Find the possible values of a. *[3 marks]*

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7. Given that $(1+x)^6(1+mx)^5 \equiv 1+nx+415x^2+\ldots+m^5x^{11}$, find the possible values of m and n. *[8 marks]*

Long questions

1. (a) Sketch the graph of $y=(x+2)^3$.

(b) Find the binomial expansion of $(x+2)^3$.

(c) Find the exact value of 2.001^3.

(d) Solve the equation $x^3+6x^2+12x+16=0$. *[12 marks]*

2. $f(x)=(1+x)^5$ and $g(x)=(2+x)^4$.

(a) Write down the vertical asymptote and axes intercepts of the graph $y=\frac{f(x)}{g(x)}$.

(b) Write down binomial expansions for $f(x)$ and $g(x)$.

(c) (i) Show that $\frac{f(x)}{g(x)} = x-k+\frac{ax^3+50x^2+85x+49}{g(x)}$, where k and a are constants to be found.

(ii) Hence explain why the graph of $y = \frac{f(x)}{g(x)}$ approaches a straight line when x is large, and write down the equation of this straight line.

(d) Sketch the curve $y = \frac{f(x)}{g(x)}$ for $-10 \le x \le 10$. *[12 marks]*

3. (a) Write $\left(1+\sqrt{2}\right)^3$ in the form $p + q\sqrt{2}$ where $p, q \in \mathbb{Z}$.

(b) Write down the general term in the binomial expansion of $\left(1+\sqrt{2}\right)^n$.

(c) Hence show that $\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ is always an integer.

(d) What is the smallest value of n such that $\left(1+\sqrt{2}\right)^n$ is within 10^{-9} of a whole number? *[12 marks]*

4. The expansion of $(a+x)^n$ where $n \in \mathbb{N}$ has the form:

$$a^n + \ldots + \alpha x^r + \beta x^{r+1} + \gamma x^{r+2} + \ldots + x^n$$

(a) Show that the ratio of $\frac{\alpha}{\beta}$ is $\frac{r+1}{n-r}a$.

(b) If $a = 1$ show that the expansion will contain two consecutive terms with the same coefficient as long as n is odd.

(c) Using the result of part (a) deduce an expression for $\frac{\beta}{\gamma}$

(d) Prove that there are no values for a such that in the expansion of $(a+x)^n$, $n \in \mathbb{N}$, three consecutive terms have the same coefficient. *[16 marks]*

9 Circular measure and trigonometric functions

In this chapter you will learn:

- about different units for measuring angles, and how measuring angles is related to distance travelled around a circle
- the definitions of sine, cosine and tangent functions, their basic properties and their graphs
- how to calculate certain special values of trigonometric functions
- how to apply your knowledge of transformations of graphs to sketch more complicated trigonometric functions
- how to use trigonometric functions to model periodic phenomena
- about inverses of trigonometric functions.

Introductory problem

The original Ferris Wheel was constructed in 1893 in Chicago. It was just over 80 m tall and could complete one full revolution in 9 minutes. During each revolution, how long did the passengers spend more than 5 m above ground?

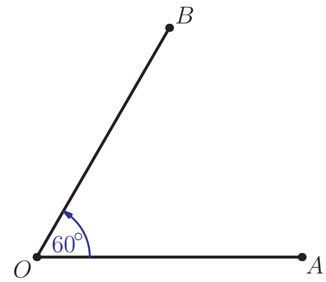
Measuring angles is related to measuring lengths around the perimeter of the circle. This observation leads to the introduction of a new unit for measuring angles, the **radian**, which will be a very useful unit of measurement in advanced mathematics.

Motion in a circle is just one example of periodic motion, which repeats after a fixed time interval. Other examples include oscillation of a particle attached to the end of a spring or vibration of a guitar string. There are also periodic phenomena where a pattern is repeated in space rather than in time; for example, the shape of a water wave. All these can be modelled using **trigonometric functions**.

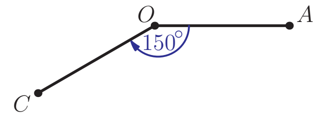
9A Radian measure

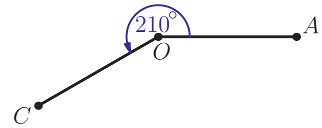
An angle measures the amount of rotation between two straight lines. You are already familiar with measuring angles in **degrees**, where a full turn measures 360° and that there are two directions (or senses) of rotation; clockwise and anti-clockwise. It is conventional in mathematics to measure angles anti-clockwise.

In the first diagram alongside, the line OA rotates 60° anti-clockwise into position OB. Therefore $A\hat{O}B = 60°$.



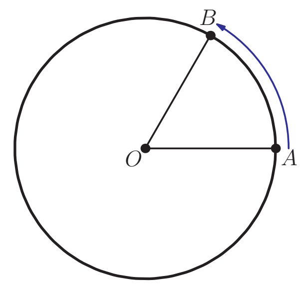
In the second diagram the line OA rotates 150° clockwise into position OC. We measure clockwise rotations with negative angles. Therefore $A\hat{O}C = -150°$. Note that OA can also move to position OC by rotating 210° anti-clockwise, so also $A\hat{O}C = 210°$, as shown in the third diagram. In this book, and in exam questions, it will always be made clear which of the two angles is required.





The measure of 360° for a full turn may seem arbitrary and there are other ways of measuring sizes of angles. In advanced mathematics, the most useful unit for measuring angles is the **radian**. This measure relates the size of the angle to the distance moved by a point around a circle.

Consider a circle with centre O and radius 1 (this is the **unit circle**), and two points, A and B, on its circumference. As the line OA rotates into position OB, point A moves the distance equal to the length of the arc AB. The measure of the angle AOB in radians is equal to this arc length.



If point A makes a full rotation around the circle, it will cover a distance equal to the length of the circumference of the circle. As the radius of the circle is 1, the length of the circumference is 2π. Hence a full turn measures 2π radians. We can deduce the sizes of other common angles in radians; for example, a right angle is one quarter of a full turn, so it measures $2\pi \div 4 = \frac{\pi}{2}$ radians.

Common angles, measured in radians, are often expressed as fractions of π but we can also use decimal approximations. Thus a right angle measures approximately 1.57 radians.

The fact that a full turn measures 2π radians can be used to convert any angle measurement from degrees to radians, and vice versa.

You may wonder why there are 360 degrees in a full turn. It came from ancient astrologers who noticed that the stars appeared to rotate in the sky, returning to their original position after around 360 days. One day's rotation corresponds to one degree. We now know that there are 365.24 days in a year, so these ancient astrologers were quite accurate – you might want to think how you could measure how many days there are in a year.

Worked example 9.1

(a) Convert 75° to radians. (b) Convert 2.5 radians to degrees.

What fraction of a full turn is 75°?

(a) $\frac{75}{360} = \frac{5}{24}$

Calculate the same fraction of 2π

$\frac{5}{24} \times 2\pi = \frac{5\pi}{12}$

$\therefore 75° = \frac{5\pi}{12}$ radians

This is the *exact* answer. We can also find the decimal equivalent, to 3 significant figures.

$75° = 1.31$ radians (3SF)

What fraction of a full turn is 2.5 radians?

(b) $\frac{2.5}{2\pi} (\approx 0.3979)$

Calculate the same fraction of 360°

$\frac{2.5}{2\pi} \times 360 = 143.24\ldots$

2.5 radians = 143° (3SF)

KEY POINT 9.1

Full turn = 360° = 2π radians

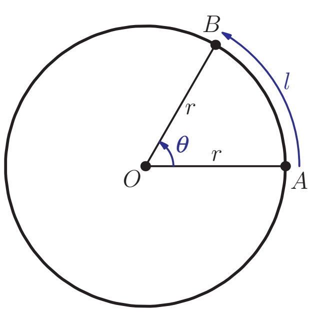
Half turn = 180° = π radians

To convert from degrees to radians, divide by 180 and multiply by π.

To convert from radians to degrees, divide by π and multiply by 180.

There are many different measures of angle. One historical attempt was gradians, which split a right angle into 100 units. Does this mean that the facts you have learnt about angles – ideas such as 180° in a triangle – are purely consequences of definitions and have no link to truth?

In our definition of the radian measure we have used the unit circle. However, we can think about a point moving around a circle of any radius.



As the line OA rotates into position OB, point A covers the distance equal to the length of the arc AB. The ratio of this arc length l to the circumference of the circle is $\frac{l}{2\pi r}$.

The measure of $A\hat{O}B$ in radians is in the same ratio to the full turn, so $\frac{\theta}{2\pi} = \frac{l}{2\pi r}$.

It follows that the measure of the angle in radians is $\theta = \frac{l}{r}$; the measure of the angle is the ratio of the length of the arc to the radius of the circle. In particular, the angle of 1 radian corresponds to an arc whose length is equal to the radius of the circle.

When the radius of the circle is 1, the size of the angle is numerically equal to the length of the arc. However, as the size of the angle is a ratio of two lengths, it has no units. We say that the radian is a *dimensionless unit*. For this reason, when writing the size of an angle in radians, we will simply write, for example, $\theta = 1.31$.

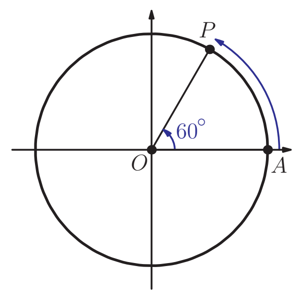
EXAM HINT

Some books write $\theta = 1.31$ rad or $\theta = 1.31^c$, but the IB® does not use this notation.

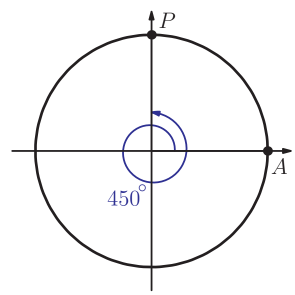
All this may sound a little complicated and you may wonder why we cannot just use degrees to measure angles. You will see in chapter 11 that the formulae for calculating lengths and areas for parts of circles are much simpler when radians are used, but an even stronger case for using radians will be presented when you study calculus.

Radians will be used whenever differentiating and integrating trigonometric functions (chapters 16 to 20).

One advantage of thinking of angles as measuring the amount of rotation around the unit circle is that we can show an angle of any size by marking the corresponding point on the unit circle. We have seen that the convention in mathematics is to measure positive angles anti-clockwise. Another convention is that we start measuring from a point which is on the positive x-axis. On the diagram alongside, the starting point is labelled A, and the point P corresponds to the angle of 60°. In other words, to get from the starting point to point P, we need to rotate 60°, or one-sixth of the full turn, around the circle.



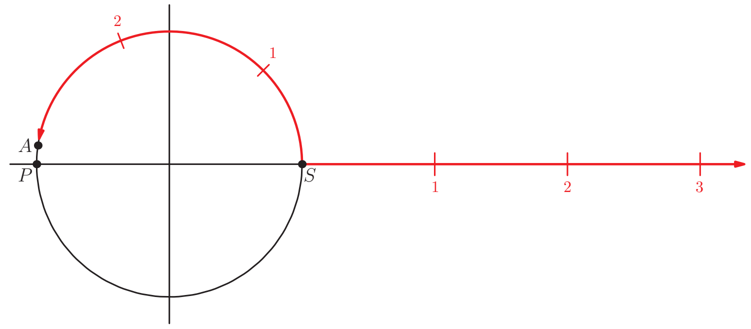
We can also represent negative angles (by rotating clockwise) and angles larger than 360° (by rotating through more than a full turn). The second diagram shows point P representing the angle of 450°, or one and a quarter turns.

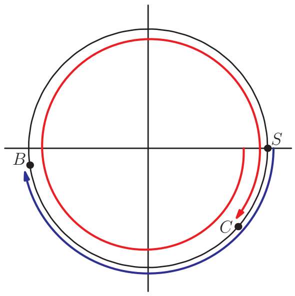


We can apply this idea to representing numbers; instead of representing numbers by points on a number line, we can represent them by points on the unit circle. To do this, we imagine wrapping the number line around a circle of radius 1, starting by placing zero on the positive x-axis (on the diagram on the next page, point S corresponds to number 0), and going anti-clockwise. As the length of the circumference of the circle is 2π, the numbers 2π, 4π, and so on are also represented by point S. Numbers π, 3π, and so on are represented by point P. Number 3, which is just less than π, is represented by point A as shown in the diagram on the next page.

There is a three-dimensional analogue of angle called solid angle, the unit of which is steradians. This unit measures the fraction of the surface area of a sphere covered. There are many aspects of trigonometry which can be transferred to these solid angles.

Looking at the number line in this way should also suggest why radians are a more natural measure of angle than degrees. While the radian is rarely used outside of the mathematical community, within mathematics it is the unit of choice.





We can also represent negative numbers in this way, by wrapping the negative part of the number line clockwise around the circle. For example, number −3 is represented by point *B* on the diagram alongside and number −7 by point *C* (7 is a bit bigger than 2π, which means wrapping once and a bit around the circle).

Worked example 9.2

(a) Mark on the unit circle the points corresponding to the following angles, measured in degrees:

A: 135° *B*: 270° *C*: −120° *D*: 765°

(b) Mark on the unit circle the points corresponding to the following angles, measured in radians:

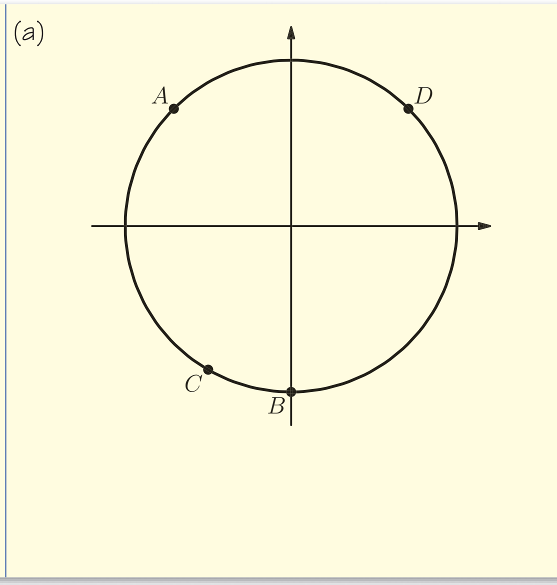
A: π *B*: $-\frac{\pi}{2}$ *C*: $\frac{5\pi}{2}$ *D*: $\frac{13\pi}{3}$

135 = 90 + 45, so point *A* represents quarter of a turn plus another eighth of a turn

270 = 3 × 90, so point *B* represents rotation through three right angles

120 = 360 ÷ 3, so point *C* represents a third of the full turn, but clockwise (because of the minus sign)

765 = 2 × 360 + 45, so point *D* represents two full turns plus one half of a right angle



continued . . .

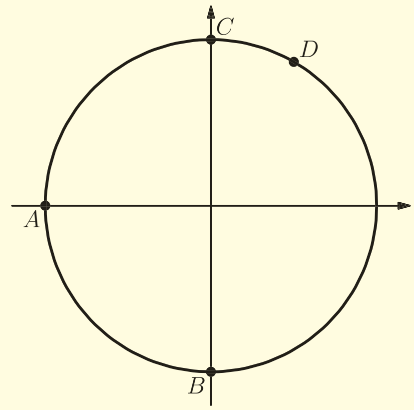
π radians is one half of a full turn so point A represents half a turn

$\frac{\pi}{2}$ is one quarter of a full turn, so point B represents a quarter of a turn in the clockwise direction (because of the minus sign)

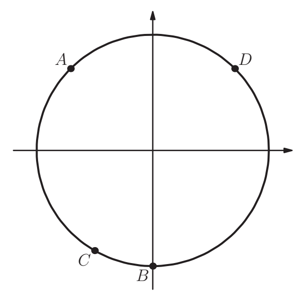
$\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$, so point C represents a full turn followed by another quarter of a turn

$\frac{13\pi}{3} = 4\pi + \frac{\pi}{3}$, so point D represents two full turns followed by another 6th of a turn

(b)



It is useful to describe where points are on the unit circle using quadrants. A **quadrant** is one quarter of the circle, and conventionally quadrants are labelled going anti-clockwise. So in the example (a) above, point A is in the second quadrant, point C in the third quadrant and point D in the first quadrant.



Exercise 9A

1. Draw a unit circle for each part and mark the points corresponding to the following angles.

(a) (i) 60° (ii) 150°

(b) (i) –120° (ii) –90°

(c) (i) 495° (ii) 390°

2. Draw a unit circle for each part and mark the points corresponding to the following angles.

(a) (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{3}$

(b) (i) $\frac{4\pi}{3}$ (ii) $\frac{3\pi}{4}$

(c) (i) $-\frac{\pi}{3}$ (ii) $-\frac{\pi}{6}$

(d) (i) -2π (ii) -4π

3. Express the following angles in radians, giving your answer in terms of π.

(a) (i) 135° (ii) 45°

(b) (i) 90° (ii) 270°

(c) (i) 120° (ii) 150°

(d) (i) 50° (ii) 80°

4. Express the following angles in radians, correct to 3 decimal places.

(a) (i) 320° (ii) 20°

(b) (i) 270° (ii) 90°

(c) (i) 65° (ii) 145°

(d) (i) 100° (ii) 83°

5. Express the following angles in degrees.

(a) (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{4}$

(b) (i) $\frac{5\pi}{6}$ (ii) $\frac{2\pi}{3}$

(c) (i) $\frac{3\pi}{2}$ (ii) $\frac{5\pi}{3}$

(d) (i) 1.22 (ii) 4.63

6. (a) (i) Express 42° in radians to 3 decimal places.

(ii) Express 168° in radians correct to 3 decimal places.

(b) (i) Express 202.5° in radians in terms of π.

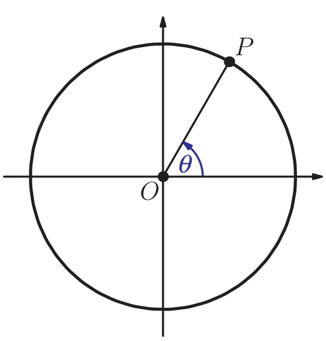
(ii) Express 67.5° in radians in terms of π.

(c) (i) Express $\frac{5\pi}{12}$ radians in degrees.

(ii) Express $\frac{11\pi}{20}$ radians in degrees.

(d) (i) Express 1.62 radians in degrees to one decimal place.

(ii) Express 2.73 radians in degrees to one decimal place.



7. The diagram shows point P on the unit circle corresponding to angle θ (measured in degrees).

Copy the diagram and mark the points corresponding to the following angles.

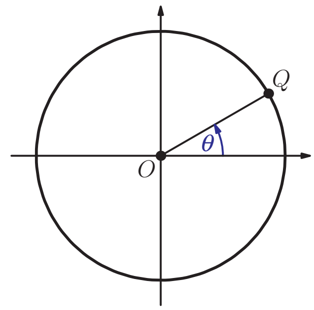
(a) (i) $180° - \theta$ (ii) $180° + \theta$

(b) (i) $\theta + 180°$ (ii) $\theta + 90°$

(c) (i) $90° - \theta$ (ii) $270° - \theta$

(d) (i) $\theta - 360°$ (ii) $\theta + 360°$

8. The diagram shows point Q on the unit circle corresponding to angle θ (measured in radians).



Copy the diagram and mark the points corresponding to the following angles.

(a) (i) $2\pi - \theta$ (ii) $\pi - \theta$

(b) (i) $\theta + \pi$ (ii) $-\pi - \theta$

(c) (i) $\frac{\pi}{2} + \theta$ (ii) $\frac{\pi}{2} - \theta$

(d) (i) $\theta - 2\pi$ (ii) $\theta + 2\pi$

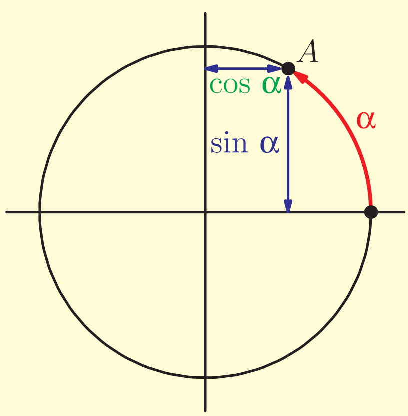
9B Definitions and graphs of sine and cosine functions

We will now define trigonometric functions. Starting with a real number α, first mark a point A on the unit circle representing the number α. The sine and cosine of this number is defined in terms of distance to the axes.

KEY POINT 9.2

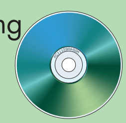
The **sine** of the number α, written $\sin\alpha$, is the distance of the point A from the horizontal axis.

The **cosine** of the number α, written $\cos\alpha$, is the distance of the point A from the vertical axis.



You have previously seen sine and cosine defined using right-angled triangles.

See Prior learning Section U on the CD-ROM



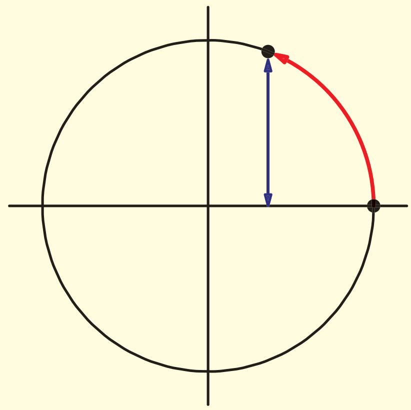
The definition given here is consistent with this but it allows us to define sine and cosine for values beyond 90°. This leads to the question about what makes something a definition. Is it the one that came first historically, the one that is more understandable or the one that is easier to generalise?

Worked example 9.3

By marking the corresponding points on the unit circle, estimate the value of:

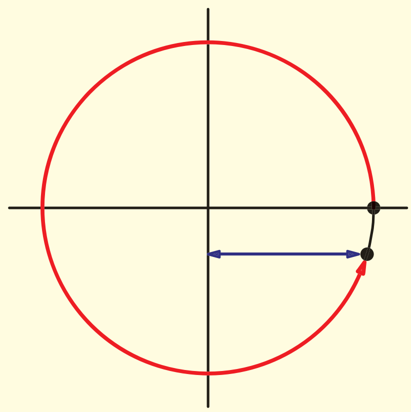
(a) $\sin 1.2$ (b) $\cos 6$ (c) $\cos 2.4$

(a) $\frac{\pi}{2} \approx 1.6$, so the point is in the first quadrant, close to the top



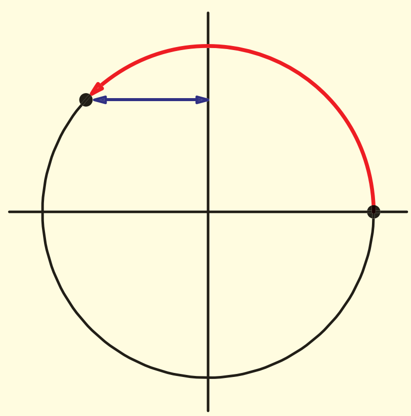
$\sin 1.2 \approx 0.9$

(b) $2\pi \approx 6.2$, so the point is just below the horizontal axis



$\cos 6 \approx 0.95$

(c) $\frac{\pi}{2} \approx 1.6$ and $\pi \approx 3.1$ so the point is around the middle of the second quadrant



$\cos 2.4 \approx -0.7$

Notice that in the last example cos 2.4 was a negative number. This is because the point corresponding to number 2.4 is to the left of the vertical axis, so we take the distance to be negative.

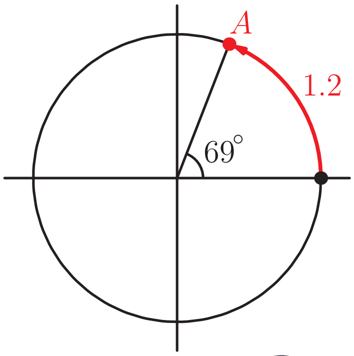
You may be wondering why we had to change the calculator into radian mode when we are working with real numbers rather than angles. Remember how we defined the radian measure: the size of the angle in radians is equal to the distance travelled by a point around the unit circle as it rotates through that angle. So measuring an angle in radians is the same as measuring the distance 'wrapped' around the circle, which corresponds to real numbers.

However, we can also think of a point on the unit circle as corresponding to an angle measured in degrees. For example, point A on the diagram corresponds to the real number 1.2, to an angle of 1.2 radians and to an angle of $69°$. If you change your calculator into degree mode and calculate sin $69°$, you will get the same answer as when you worked out sin 1.2 in radian mode. Although in applications the sine and cosine functions are frequently used in situations with angles, you should be aware that they are functions which can operate on any real number.

We can use the symmetry of the circle to see how the values of sine and cosine functions of different numbers are related to each other.

EXAM HINT

You can find the values of sine and cosine functions using your calculator. Make sure your calculator is in radian mode.



EXAM HINT

You should always use radians unless explicitly told to use degrees.

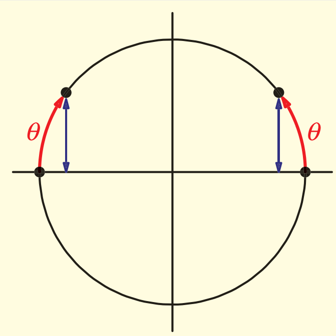
Worked example 9.4

Given that $\sin\theta = 0.6$, find the values of:

(a) $\sin(\pi - \theta)$ (b) $\sin(\theta + \pi)$

Mark the points corresponding to θ and $\pi - \theta$ on the circle

(a)



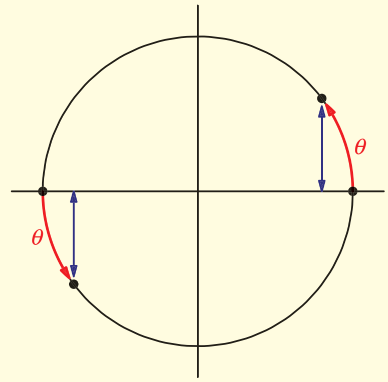
continued . . .

Compare the distances from the horizontal axis

The points are the same distance from the horizontal axis, so $\sin(\pi - \theta) = 0.6$

Mark the points corresponding to θ and $\pi + \theta$ on the circle

(b)

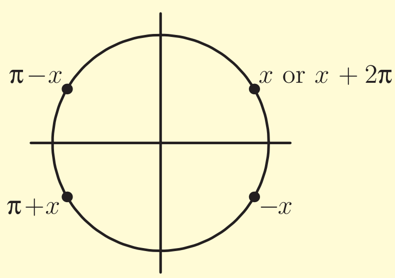


Compare the distances from the horizontal axis

The points are the same distance from the horizontal axis, but the second one is below the axis, so $\sin(\theta + \pi) = -0.6$

The above example illustrates some of the properties of the sine function. Similar properties can be derived for the cosine function. It may be useful to remember those results, summarised below. They can all be derived using circle diagrams.

KEY POINT 9.3



$$\sin x = \sin(\pi - x) = \sin(x + 2\pi)$$

$$\sin(\pi + x) = \sin(-x) = -\sin x$$

$$\cos x = \cos(-x) = \cos(x + 2\pi)$$

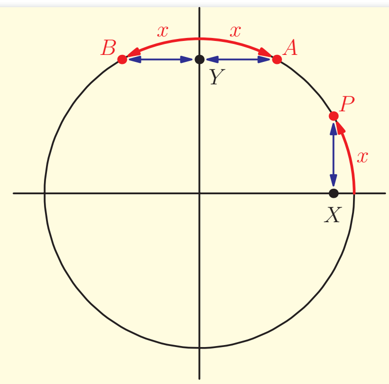
$$\cos(\pi - x) = \cos(\pi + x) = -\cos x$$

Worked example 9.5

Given that $\sin x = 0.4$, find the value of:

(a) $\cos\left(\frac{\pi}{2} - x\right)$ (b) $\cos\left(x + \frac{\pi}{2}\right)$

Label the points corresponding to x, $\frac{\pi}{2} - x$ and $x + \frac{\pi}{2}$ on the circle



$\left(\frac{\pi}{2} - x\right)$ is represented by point A, and $AY = PX$

(a) $\cos\left(\frac{\pi}{2} - x\right) = 0.4$

$\left(x + \frac{\pi}{2}\right)$ is represented by point B, and $BY = PX$, but B is to the left of the vertical axis

(b) $\cos\left(x + \frac{\pi}{2}\right) = -0.4$

The above example illustrates a relationship between sine and cosine functions. It may be a good idea to remember, or know how to derive, the following results:

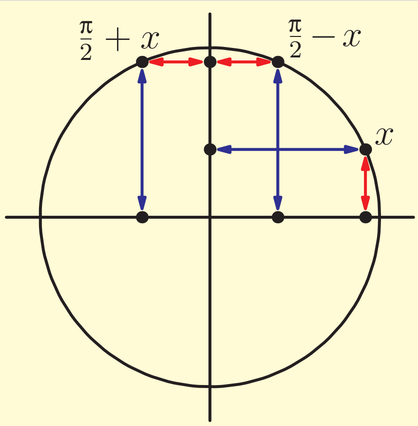
KEY POINT 9.4

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$



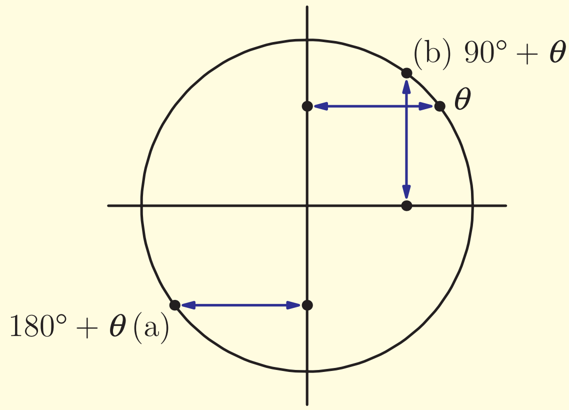
Similar results can be derived when θ represents an angle measured in degrees.

Worked example 9.6

Given that θ is an angle measured in degrees such that $\cos\theta = 0.8$ find the value of

(a) $\cos(180° + \theta)$ (b) $\sin(90° - \theta)$

Mark the points corresponding to angles θ, $180° + \theta$ and $90° - \theta$ on the circle



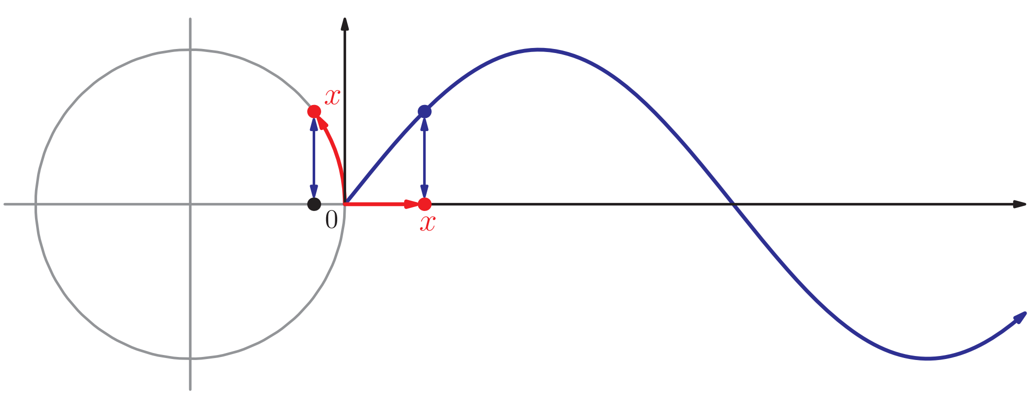
The distance from the vertical axis is 0.8, but the point is to the left of the axis

(a) $\cos(180° + \theta) = -0.8$

By reflection symmetry in the diagonal line, the distance from the *horizontal* axis is 0.8

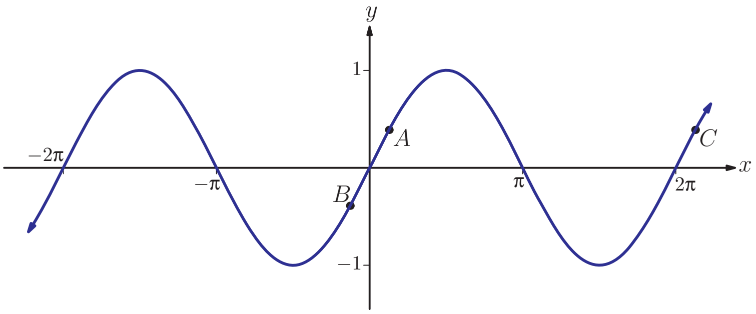
(b) $\sin(90° + \theta) = 0.8$

We have now defined the sine function for all real numbers, so we can draw its graph. To do this, we go back to thinking about the real number line wrapped around the unit circle. Each real number corresponds to a point on the circle, and the value of the sine function is the distance of the point from the horizontal axis.



All of the properties of the sine function discussed above can be seen from the graph. For example, increasing x by 2π corresponds to making a full turn around the circle and returning to the same point. Therefore $\sin(x+2\pi)=\sin x$. We say that the sine function is **periodic** with **period** 2π. By considering points A and B on the graph below, we can see that $\sin(-x)=-\sin x$. We can also see that the minimum possible value of $\sin x$ is -1 and the maximum value is 1. We say that the sine function has **amplitude** 1.

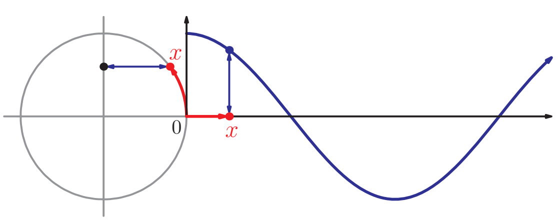
You may wonder how your calculator finds the values of sine and cosine functions. This is explored in Supplementary sheet 8 'How does your calculator work out sin and cos?' on the CD-ROM.

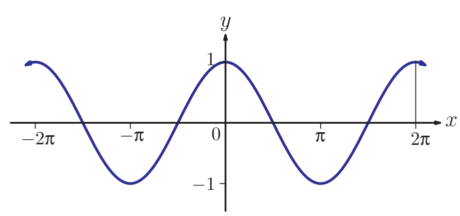


KEY POINT 9.5

> A function is periodic if it repeats regularly. The distance between repeats is called the period.
>
> The amplitude is half of the distance between the maximum and minimum value of a function.

To draw the graph of the cosine function we look at the distance of the point representing real number x from the vertical axis.





We can again see many of the properties discussed above on this graph; for example, $\cos(-x)=\cos x$ and $\cos(\pi-x)=-\cos x$. The period and the amplitude are the same as for the sine function. In fact, the graphs of the sine and the cosine functions are related to each other: the graph of $y=\cos x$ is obtained from the graph of $y=\sin x$ by translating it $\frac{\pi}{2}$ units to the left. This again corresponds to one of the properties listed above:

$$\cos x=\sin\left(x+\frac{\pi}{2}\right).$$

You can see from the graphs that $\sin x$ is an odd function and $\cos x$ is an even function. Both graphs have translational symmetry. We discussed these symmetries in Section 6G.

Trigonometric functions can be used to define **polar coordinates**. This alternative to the Cartesian coordinate system makes it easier to write equations of certain graphs. Such equations produce some beautiful curves, such as the cardioid and the polar rose.

The key properties of sine and cosine functions are summarised below.

KEY POINT 9.6

The sine and cosine functions are periodic with period 2π:

$$\sin x = \sin(x \pm 2\pi) = \sin(x \pm 4\pi) = \dots$$

$$\cos x = \cos(x \pm 2\pi) = \cos(x \pm 4\pi) = \dots$$

The sine and cosine functions have amplitude 1:

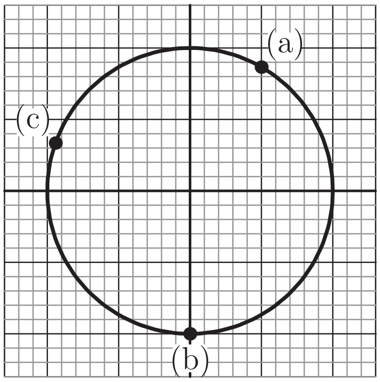
$$-1 \le \sin x \le 1 \text{ and } -1 \le \cos x \le 1$$

The cosine graph is obtained by translating the sine graph $\frac{\pi}{2}$ units to the left:

$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \text{ and } \sin x = \cos\left(x - \frac{\pi}{2}\right)$$

Exercise 9B

1. Write down the approximate values of $\sin x$ and $\cos x$ for the number x corresponding to each of the points marked on the diagram



2. Use the unit circle to find the value of:
   (a) (i) $\sin\frac{\pi}{2}$ (ii) $\sin 2\pi$
   (b) (i) $\cos 0$ (ii) $\cos(-\pi)$
   (c) (i) $\sin\left(-\frac{\pi}{2}\right)$ (ii) $\cos\frac{5\pi}{2}$

3. Use the unit circle to find the value of:
   (a) (i) $\cos 90°$ (ii) $\cos 180°$
   (b) (i) $\sin 270°$ (ii) $\sin 90°$
   (c) (i) $\sin 270°$ (ii) $\cos 450°$

4. Given that $\cos\frac{\pi}{5} = 0.809$ find the value of:

(a) $\cos\frac{4\pi}{5}$ (b) $\cos\frac{21\pi}{5}$

(c) $\cos\frac{9\pi}{5}$ (d) $\cos\frac{6\pi}{5}$

5. Given that $\sin\frac{2\pi}{3} = 0.866$ find the value of:

(a) $\sin\left(\frac{-2\pi}{3}\right)$ (b) $\sin\frac{4\pi}{3}$

(c) $\sin\frac{10\pi}{3}$ (d) $\sin\frac{\pi}{3}$

6. Given that $\cos 40° = 0.766$ find the value of:

(a) $\cos 400°$ (b) $\cos 320°$

(c) $\cos(-220°)$ (d) $\cos 140°$

7. Given that $\sin 130° = 0.766$ find the value of:

(a) $\sin 490°$ (b) $\sin 50°$

(c) $\sin(-130°)$ (d) $\sin 230°$

8. Sketch the graph of $y = \sin x$ for:

(a) (i) $0° \leq x \leq 180°$ (ii) $90° \leq x \leq 360°$

(b) (i) $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (ii) $-\pi \leq x \leq 2\pi$

9. Sketch the graph of $y = \cos x$ for:

(a) (i) $-180° \leq x \leq 180°$ (ii) $0° \leq x \leq 270°$

(b) (i) $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ (ii) $-\pi \leq x \leq 2\pi$

10. (a) On the unit circle, mark the points representing $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

(b) Given that $\sin\frac{\pi}{6} = 0.5$, find the value of:

(i) $\cos\frac{\pi}{3}$ (ii) $\cos\frac{2\pi}{3}$

11. Use your calculator to evaluate the following, giving your answers to 3 significant figures.

(a) (i) $\cos 1.25$ (ii) $\sin 0.68$

(b) (i) $\cos(-0.72)$ (ii) $\sin(-2.35)$

12. Use your calculator to evaluate the following, giving your answers to 3 significant figures.

(a) (i) $\sin 42°$ (ii) $\cos 168°$

(b) (i) $\sin(-50°)$ (ii) $\cos(-227°)$ *[4 marks]*

13. Evaluate $\cos(\pi + x) + \cos(\pi - x)$.

14. Simplify the following expression:

$$\sin x + \sin\left(x + \frac{\pi}{2}\right) + \sin(x + \pi) + \sin\left(x + \frac{3\pi}{2}\right) + \sin(x + 2\pi)$$

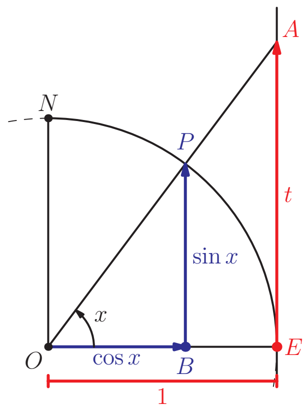
[5 marks]

9C Definition and graph of the tangent function

We now define another trigonometric function – the **tangent** function. It is defined as the ratio between the sine and the cosine functions.

KEY POINT 9.7

$$\tan x = \frac{\sin x}{\cos x}$$

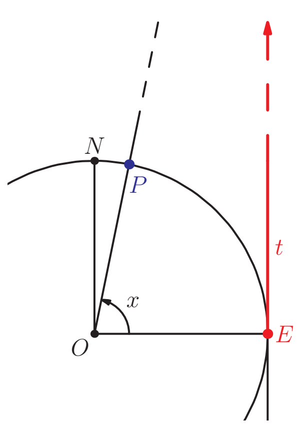


You may wonder why the function is called the tangent function. Consider the diagram alongside:

A tangent is drawn from the point E, where the unit circle meets the positive x-axis. The line at angle x is continued until it hits the tangent. Since OPB and OAE are similar triangles, we get that:

$$\frac{t}{1} = \frac{\sin x}{\cos x}$$
$$\therefore t = \tan x$$

This can be used to see several important properties of the tangent function.



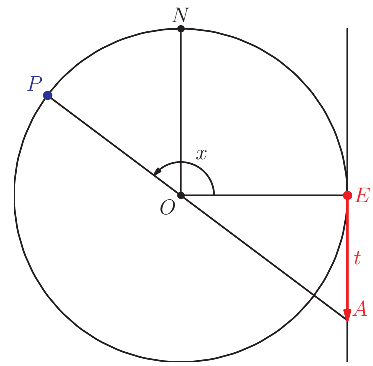
As P moves from E to N, the distance along the tangent increases. When P is close to N, the tangent gets extremely large. If P is actually at N the line ON is parallel to the tangent at E so there is no intersection, and $\tan x$ is undefined. This corresponds in the definition to the situation when $\cos x$ is zero, which is not allowed. In fact, the tangent function is undefined for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$ These lines are vertical asymptotes of the graph of $y = \tan x$.

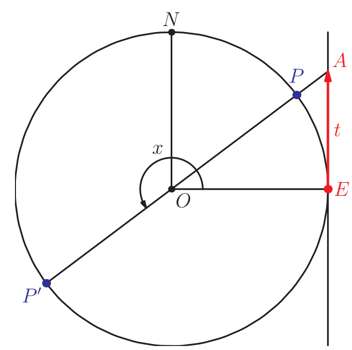
When P moves beyond N the intersection is now below the x-axis and the value of $\tan x$ is negative.

If P' is on the opposite side of the circle to P, it still intersects the tangent at the same point. This means that $\tan x$ repeats every π radians or 180°.

It is equal to zero when $x = 0, \pi, 2\pi \ldots$

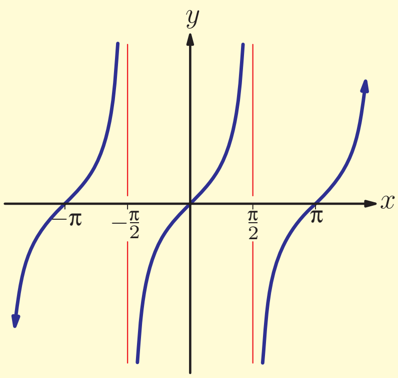
This information can be used to sketch the graph of $y = \tan x$.





KEY POINT 9.8

The graph of the tangent function:



Asymptotes are discussed in Section 4C.

We should remember that the points on the unit circle can also represent angles measured in degrees.

Worked example 9.7

Sketch the graph of $y = \tan x$ for $-90° < x < 270°$.

Find the values for which $\tan x$ is not defined

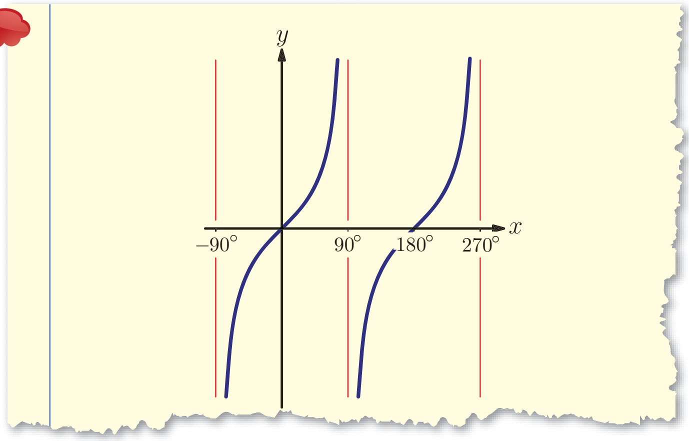
$\tan x$ undefined: $\cos x = 0$ when $x = -90°, 90°, 270°$.

When is the function positive / negative? When is it zero?

$\tan x$ is:
negative in the fourth quadrant,
positive in the first quadrant,
negative in the second quadrant, ...
$\tan x = 0$: $x = 0°$ and $180°$

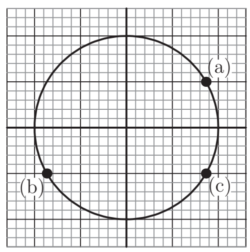
continued . . .

Start by marking the asymptotes and zeros



Exercise 9C

1. By estimating the values of sin θ and cos θ, find the approximate value of tan θ for the points shown on the diagram.



2. Sketch the graph of $y = \tan x$ for:

   (a) (i) $0° \le x \le 360°$ (ii) $-90° \le x \le 270°$

   (b) (i) $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$ (ii) $-\pi \le x \le \pi$

3. Use your calculator to evaluate the following, giving your answers to 2 decimal places.

   (a) (i) tan 1.2 (ii) tan 4.7

   (b) (i) tan (−0.65) (ii) tan (−7.3)

4. Use your calculator to evaluate the following, giving your answers to 3 significant figures.

   (a) (i) tan 32° (ii) tan 168°

   (b) (i) tan (−540°) (ii) tan (−128°)

5. Use the properties of sine and cosine to express the following in terms of tan x.

(a) $\tan(\pi - x)$ (b) $\tan\left(x + \frac{\pi}{2}\right)$

(c) $\tan(x + \pi)$ (d) $\tan(x + 3\pi)$

6. Use the properties of sine and cosine to express the following in terms of $\tan\theta°$.

(a) $\tan(-\theta°)$ (b) $\tan(-360° - \theta°)$

(c) $\tan(-90° - \theta°)$ (d) $\tan(-180° - \theta°)$

7. Sketch the graph of:

(a) $y = 2\sin x - \tan x$ for $-\frac{\pi}{2} \le x \le 2\pi$

(b) $y = 3\cos x + \tan x$ for $-\pi \le x \le \pi$

8. Find the zeros of the following functions:

(a) $y = 2\tan x° + \sin x°$ for $0 \le x \le 360$

(b) $y = 3\cos x° - \tan x°$ for $-180 \le x \le 360$

9. Find the coordinates of maximum and minimum points on the following graphs:

(a) $y = 3\sin x - \tan x$ for $0 \le x \le 2\pi$

(b) $y = \cos x - 2\sin x$ for $-\pi \le x \le \pi$

10. Find approximate solutions of the following equations, giving your answers correct to 3 significant figures.

(a) $\cos x - \tan x = 3,\ x \in]0, 2\pi[$

(b) $\sin x + \cos x = 1,\ x \in [0, 2\pi]$

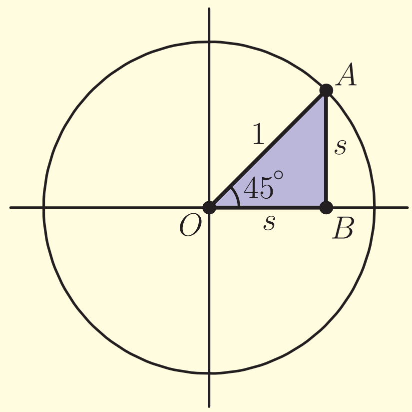
9D Exact values of trigonometric functions

Although values of trigonometric functions are generally difficult to find without a calculator, there are a few numbers for which exact values can be found. The method relies on properties of some special right-angled triangles.

Worked example 9.8

Find the exact values of $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$ and $\tan\frac{\pi}{4}$.

Mark the point corresponding to $\frac{\pi}{4}$ on the unit circle



Look at the triangle *OAB*. It is a right angle triangle, with the angle at *O* equal to 45° (because $\frac{\pi}{4}$ is one eighth of a full turn)

$s^2 + s^2 = 1$, so $s = \frac{1}{\sqrt{2}}$

$\therefore \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

We can now use the definition of $\tan x$

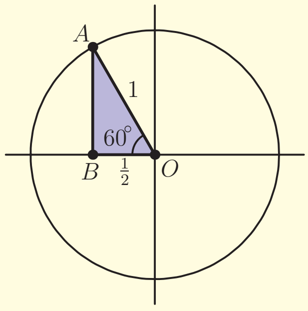
$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = 1$$

The other special right-angled triangle is made by cutting an equilateral triangle in half.

Worked example 9.9

Find the exact values of the three trigonometric functions of $\frac{2\pi}{3}$.

Mark the point corresponding to $\frac{2\pi}{3}$ on the unit circle. This is one third of a full turn, so angle *AOB* is 60°



continued . . .

Triangle AOB is half of an equilateral triangle with side 1. OB is equal to half the side of the triangle

$OB = \frac{1}{2}$, point A is to the left of the vertical axis, so

$\cos\frac{2\pi}{3} < 0$

$\therefore \cos\frac{2\pi}{3} = -\frac{1}{2}$

To find AB, use Pythagoras

$AB^2 = 1^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} \therefore \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Use the definition of tan x

$\tan\frac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$

The results for these and other special values are summarised below. You may be able to remember them all, but you should understand how they are derived, as shown in the above examples.

EXAM HINT

The table is easier to remember if you notice the pattern in the values of both sin and cos: $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

KEY POINT 9.9

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	0	30	45	60	90	120	135	150	180
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Exercise 9D

1. By marking the corresponding points on the unit circle, find the exact values of:

   (a) $\cos\left(\frac{3\pi}{4}\right)$ (b) $\cos\left(\frac{\pi}{2}\right)$ (c) $\sin\left(\frac{5\pi}{4}\right)$ (d) $\tan\left(\frac{3\pi}{4}\right)$

2. Find the exact value of:

   (a) $\sin\left(\frac{\pi}{6}\right)$ (b) $\sin\left(\frac{7\pi}{6}\right)$ (c) $\cos\left(\frac{4\pi}{3}\right)$ (d) $\tan\left(-\frac{\pi}{3}\right)$

3. Find the exact value of:

   (a) $\cos 45°$ (b) $\sin 135°$ (c) $\cos 225°$ (d) $\tan 225°$

4. Find the exact value of:

   (a) $\sin 210°$ (b) $\cos 210°$ (c) $\tan 210°$ (d) $\tan 330°$

5. Evaluate the following, simplifying as far as possible.

   (a) $1-\sin^2\left(\frac{\pi}{6}\right)$ (b) $\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)$ (c) $\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{6}\right)$

6. Show that:

   (a) $\sin 60° \cos 30° + \cos 60° \sin 30° = \sin 90°$

   (b) $(\sin 45°)^2 + (\cos 45°)^2 = 1$

   (c) $\cos^2\left(\frac{\pi}{6}\right)-\sin^2\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)$

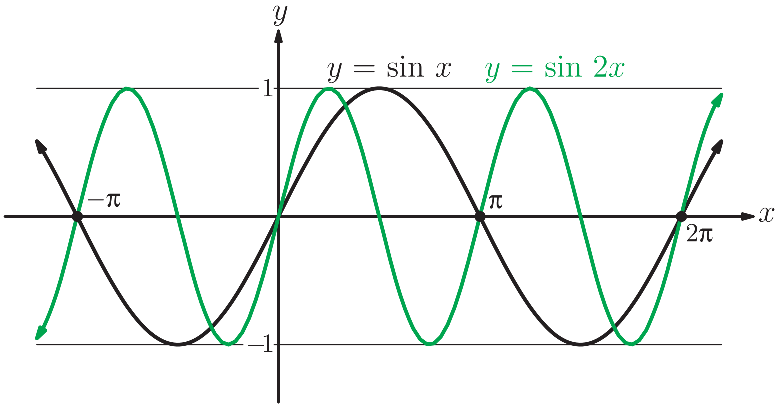
   (d) $\left(1+\tan\left(\frac{\pi}{3}\right)\right)^2 = 4+2\sqrt{3}$

9E Transformations of trigonometric graphs

See chapter 6 to revise transformations of graphs.

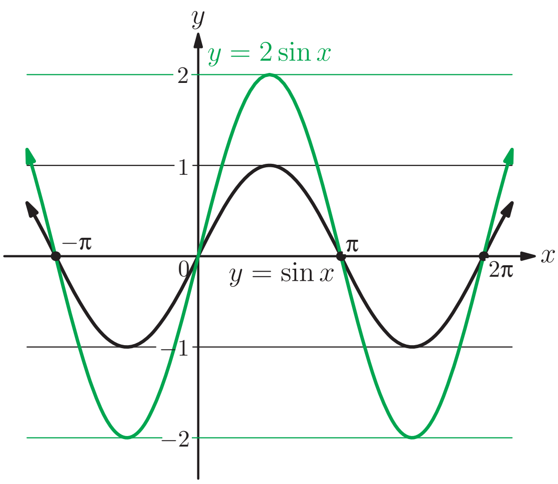
In this section we shall apply the ideas from chapter 6 to the trigonometric graphs we have met. It will allow us to model many real-life situations which show periodic behaviour in Section 9F and it will also be useful in solving equations in Section 9G.

Firstly, let us look at the relationship between $y=\sin x$ and $y=\sin 2x$. The equation $y=\sin 2x$ is of the form $y=f(2x)$, so we need to apply a horizontal stretch with scale factor $\frac{1}{2}$ to the graph of $y=\sin x$.



We can see that the amplitude of the function is still 1, but the period is halved to π.

We can compare this to the graph $y = 2\sin x$ which is of the form $y = 2f(x)$, so we need to apply a vertical stretch with scale factor 2 to the graph of $y = \sin x$.



EXAM HINT

sin 2x is not the same as 2 sin x, and $\frac{\sin 2x}{2}$ CANNOT be simplified to sin x.

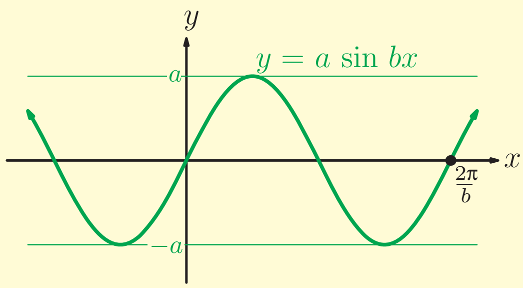
The resulting function has amplitude 2, while the period is unchanged.

The two types of transformation can be combined to change both the amplitude and the period of the function. The same transformations can also be applied to the graph of the cosine function.

KEY POINT 9.10

The function $y = a\sin bx$ has amplitude a and period $\frac{2\pi}{b}$.

The function $y = a\cos bx$ has amplitude a and period $\frac{2\pi}{b}$.



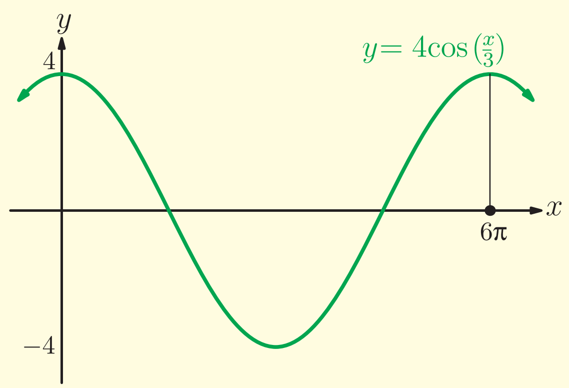
Worked example 9.10

(a) Sketch the graph of $y = 4\cos\left(\frac{x}{3}\right)$ for $0 \le x \le 6\pi$.

(b) Write down the amplitude and the period of the function.

Start with the graph of $y = \cos x$ and think what transformations to apply to it

(a) Vertical stretch with scale factor 4
Horizontal stretch with scale factor 3



EXAM HINT
'Write down' means that no working is required.

(b) amplitude = 4

period $= \dfrac{2\pi}{3^{-1}} = 6\pi$

As well as vertical and horizontal stretches, we can also apply translations to graphs. They will leave the period and the amplitude unchanged, but will change the positions of maximum and minimum points and the axis intercepts.

Worked example 9.11

(a) Sketch the graph of $y = \sin x + 2$ for $x \in [0, 2\pi]$.

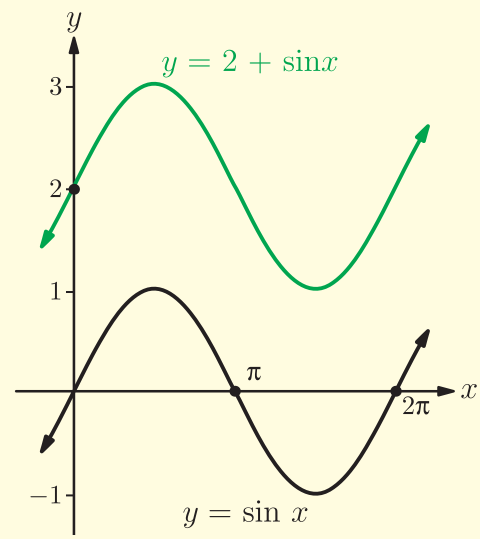
(b) Find the maximum and the minimum values of the function.

The equation is of the form $y = f(x) + 2$. What transformation does this give?

(a) Vertical translation by 2 units

continued . . .

Apply the transformation to the graph of $y = \sin x$



We know that the minimum and maximum values of sin x are 1 and −1. Add 2 to those values

(b) Minimum value: $-1+2=1$
Maximum value: $1+2=3$

In the next example we consider a horizontal translation.

Worked example 9.12

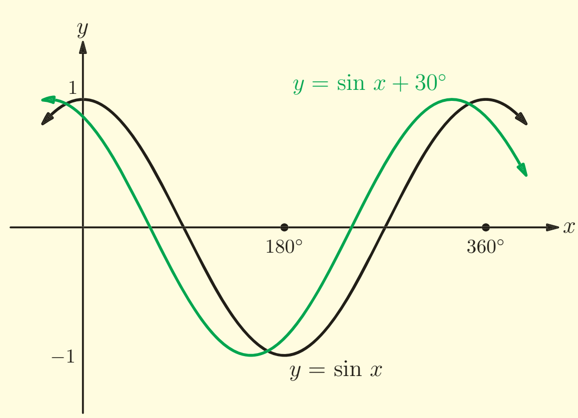
(a) Sketch the graph of $y = \cos(x + 30°)$ for $0 \le x \le 360°$.

(b) State the minimum and maximum values of the function, and the values of x for which they occur.

The equation is of the form $y = f(x + 30)$. What transformation does this give?

(a) Horizontal translation 30 degrees to the left

Apply the transformation to the graph of $y = \cos x$



continued . . .

The minimum value of cos x is -1, and it occurs for $x = 180°$, *etc*. The graph is shifted 30 units to the left

(b) Minimum value is -1.
It occurs for $x = 180 - 30 = 150°$

The maximum value of cos x is 1, and it occurs for $x = 0, 360°$..... Pick the value which is in the required interval after the translation to the left

Maximum value is 1.
It occurs for $x = 360 - 30 = 330°$

The result of combining all four transformations can be summarised as follows.

KEY POINT 9.11

The functions $y = a\sin b(x + c) + d$ and $y = a\cos b(x + c) + d$ have:

- amplitude a
- minimum value $d - a$ and maximum value $d + a$
- period $\frac{2\pi}{b}$.

Notice that the value of d is always half-way between the minimum and maximum values. In other words: $d = \frac{\min + \max}{2}$

The amplitude is half the difference between the minimum and maximum values: $a = \frac{\max - \min}{2}$

The value of c in Key point 9.11 determines the horizontal translation of the graph; therefore it affects the position of maximum and minimum points. The following example shows how to work them out.

Worked example 9.13

Find the values of x for which the function $y = \sin 3(x + 1)$ has its maximum value.

When does the sine function attain its maximum?

$\sin x$ has a maximum value when $x = \frac{\pi}{2}, \frac{5\pi}{2}$, *etc.*

The given function is of the form $f(3(x+1))$, which means that x has been replaced by $3(x + 1)$

$3(x+1) = \frac{\pi}{2}, \frac{5\pi}{2}$, *etc.*

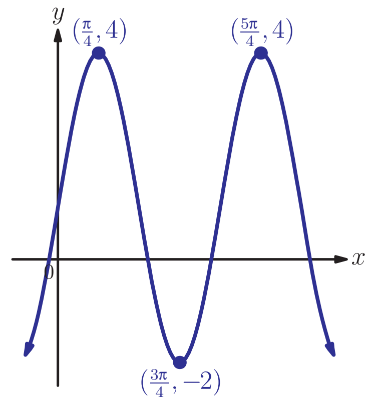
We can now solve for x

$x = \frac{\pi}{6} - 1, \frac{5\pi}{6} - 1$, *etc.*

We can use our knowledge of transformations of graphs to find an equation of a function given its graph.

Worked example 9.14

The graph shown has equation $y = a\sin(bx) + d$.



Find the values of a, b and d.

a is the amplitude, which is half the difference between the minimum and maximum values

$$\text{amplitude} = \frac{4-(-2)}{2} = 3$$

$$\therefore a = 3$$

b is related to the period, which is the distance between the two consecutive maximum points

The formula is $period = \frac{2\pi}{b}$

$$\text{period} = \frac{5\pi}{4} - \frac{\pi}{4} = \pi$$

$$\pi = \frac{2\pi}{b} \quad \therefore b = 2$$

d represents the vertical translation of the graph. It is the value half-way between the minimum and the maximum values

$$d = \frac{4+(-2)}{2}$$

$$\therefore d = 1$$

Exercise 9E

1. Sketch the following graphs, indicating any axes intercepts.

   (a) (i) $y = \sin 2x$ for $-180° \le x \le 180°$

   (ii) $y = \cos 3x$ for $0° \le x \le 360°$

   (b) (i) $y = \tan\left(x - \frac{\pi}{2}\right)$ for $0 \le x \le \pi$

   (ii) $y = \tan\left(x + \frac{\pi}{3}\right)$ for $0 \le x \le \pi$

   (c) (i) $y = 3\cos x - 2$ for $0° \le x \le 720°$

   (ii) $y = 2\sin x + 1$ for $-360° \le x \le 360°$

2. Sketch the following graphs, giving coordinates of maximum and minimum points.

   (a) (i) $y = \cos\left(x - \frac{\pi}{3}\right)$ for $0 \le x \le 2\pi$

   (ii) $y = \sin\left(x + \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$

   (b) (i) $y = 2\sin(x + 45°)$ for $-180° \le x \le 180°$

   (ii) $y = 3\cos(x - 60°)$ for $-180° \le x \le 180°$

   (c) (i) $y = -3\sin 2x$ for $-\pi \le x \le \pi$

   (ii) $y = 3 - 2\cos x$ for $0° \le x \le 360°$

3. Find the amplitude and the period of the following functions:

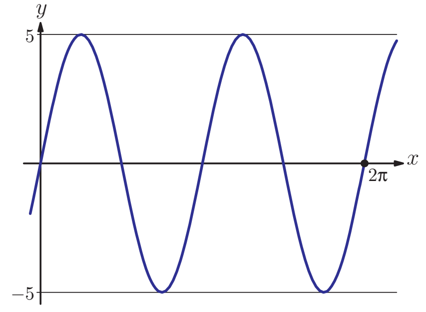
   (a) $f(x) = 3\sin 4x$, where x is in degrees

   (b) $f(x) = \tan 3x$, where x is in radians

   (c) $f(x) = \cos 3x$, where x is in degrees

   (d) $f(x) = 2\sin \pi x$, where x is in radians

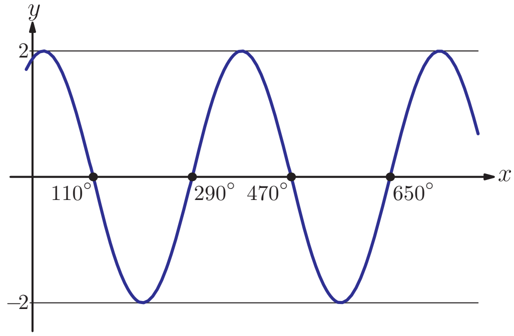
4. The graph has equation $y = p\sin(qx)$ for $0 \le x \le 2\pi$. Find the values of p and q.



[3 marks]

5. The graph shown below has equation $y = a\cos(x - b)$ for $0° \le x \le 720°$.

Find the values of a and b.



[3 marks]

6. (a) On the same set of axes sketch the graphs of $y = 1 + \sin 2x$ and $y = 2\cos x$ for $0 \le x \le 2\pi$.

(b) Hence state the number of solutions of the equation $1 + \sin 2x = 2\cos x$ for $0 \le x \le 2\pi$.

(c) Write down the number of solutions of the equation $1 + \sin 2x = 2\cos x$ for $-2\pi \le x \le 6\pi$. *[6 marks]*

7. (a) Sketch the graph of $y = 2\cos(x + 60°)$ for $x \in [0°, 360°]$.

(b) Find the coordinates of the maximum and minimum points on the graph.

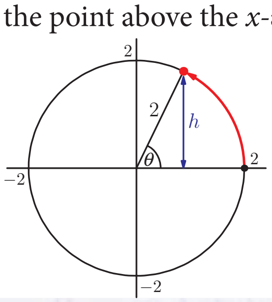
(c) Write down the coordinates the maximum and minimum points on the graph of

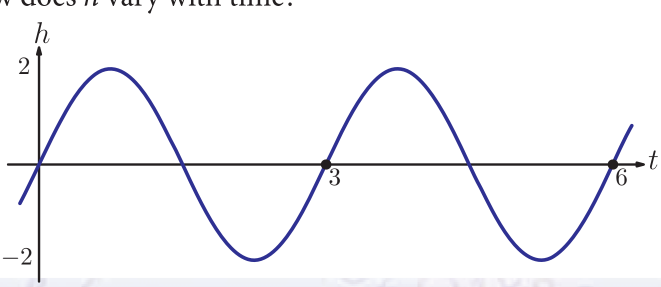
$y = 2\cos(x + 60°) - 1$ for $x \in [0°, 360°]$. *[6 marks]*

9F Modelling using trigonometric functions

In this section we will see how trigonometric functions can be used to model real-life situations which show periodic behaviour.

Imagine a point moving with constant speed around a circle of radius 2 cm, starting from the positive x-axis and taking 3 seconds to complete one full rotation. Let h be the height of the point above the x-axis. How does h vary with time?





We know that if the point is moving around the unit circle, the height above the x-axis is $\sin\theta$, where θ is the angle between the radius and the x-axis. As the circle has radius 2, the height is $2\sin\theta$.

We now need to find how θ depends on time. As the point starts on the positive x-axis, $\theta = 0$ when $t = 0$. After one complete rotation, $\theta = 2\pi$ and t = 3 (t is time measured in seconds). Because the point is moving with constant speed, we can use ratios to find that $\frac{\theta}{2\pi} = \frac{t}{3}$, so $\theta = \frac{2\pi}{3}t$.

Therefore the equation for the height in terms of time is $h = 2\sin\left(\frac{2\pi}{3}t\right)$. The second diagram above shows the graph of this function.

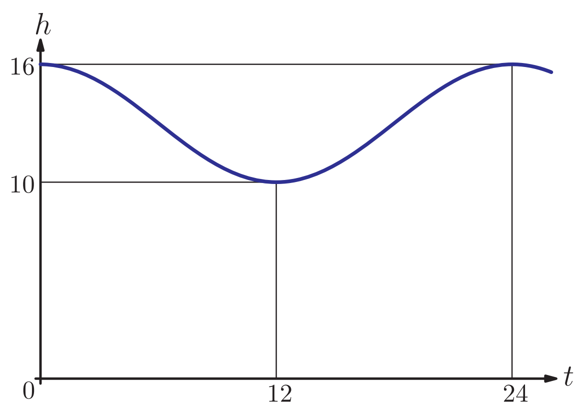
This is an example of **modelling** using trigonometric functions. Sine and cosine functions can be used to model **periodic motion**, such as motion around a circle, oscillation of a particle attached to the end of a spring, water waves, or heights of tides. In practice, we would collect experimental data to sketch a graph and then use our knowledge of trigonometric functions to find its equation. We can then use the equation to do further calculations.

Worked example 9.15

The height of water in the harbour is 16 m at high tide, and 12 hours later at low tide, it is 10 m. The graph below shows how the height of water changes with time over 24 hours.

(a) Find the equation for height (in metres) in terms of time (in hours) in the form $h = m + a\cos(bt)$.

(b) Find the first two times after the high tide when the height of water is 12 m.



From the previous section: m is half-way between minimum and maximum values, a is the amplitude, b is related to the period

(a) $m = \frac{16+10}{2} = 13$

continued . . .

The amplitude is half the distance between the minimum and maximum values

amplitude $= \dfrac{16-10}{2} = 3$

$\therefore a = 3$

The period is $\dfrac{2\pi}{b}$

period $= 24$

$24 = \dfrac{2\pi}{b} \quad \therefore b = \dfrac{\pi}{12}$

Write down the equation for height

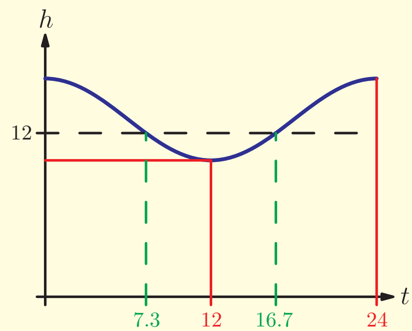
So $h = 13 + 3\cos\left(\dfrac{\pi}{12}t\right)$

Set $h = 12$

(b) $13 + 3\cos\left(\dfrac{\pi}{12}t\right) = 12$

The high tide is when $t = 0$, so we want the first two answers with $t > 0$

Solve for t using GDC

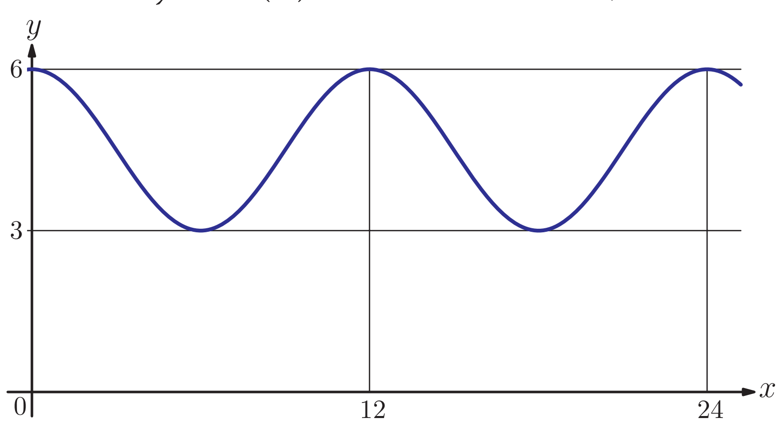


Answer the question

The height of the water will be 12 m 7.3 hours and 16.7 hours after the high tide.

Exercise 9F

1. The graph shows the height of water below the level of a walkway as a function of time. The equation of the graph is of the form $y = a\cos(bt) + m$. Find the values of a, b and m.

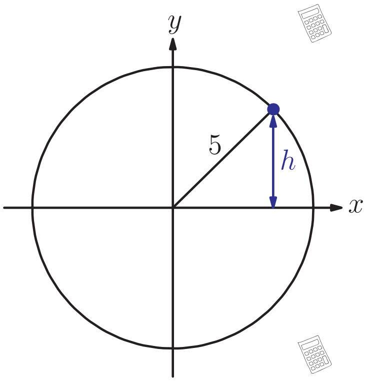


You will see how to solve equations like these without a calculator in Section 10B.

2. The depth of water in a harbour varies and is given by the equation $d = 16 + 7\sin\left(\frac{\pi}{12}t\right)$, where d is measured in metres and t in hours after midnight.

(a) Find the depth of the water at low and high tide.

(b) A boat can only enter the harbour when the depth of water is at least 19 m. Find the times between which the boat can enter the harbour. *[6 marks]*



3. A point moves around a vertical circle of radius 5 cm, as shown in the diagram. It takes 10 seconds to complete one revolution.

(a) The height of the point above the x-axis is given by $h = a\sin(kt)$, where t is time measured in seconds. Find the values of a and k.

(b) Find the times during the first revolution when the point is 3 cm below the x-axis. *[6 marks]*

4. A ball is attached to one end of an elastic string, with the other end held fixed above ground. When the ball is pulled down and released, it starts moving up and down, so that the height of the ball above the ground is given by the equation $h = 120 - 10\cos 400t$, where h is measured in cm and t is time in seconds.

(a) Find the least and greatest height of the ball above ground.

(b) Find the time required to complete one full oscillation.

(c) Find the first time after the ball is released when it reaches the greatest height. *[8 marks]*

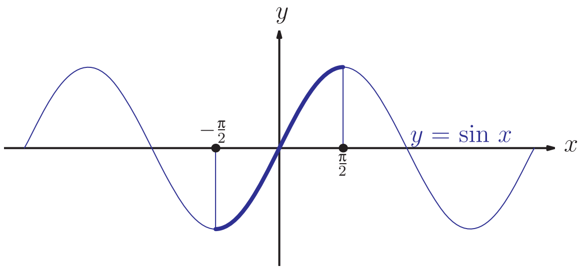
9G Inverse trigonometric functions

To solve equations it will be important that we can 'undo' trigonometric functions.

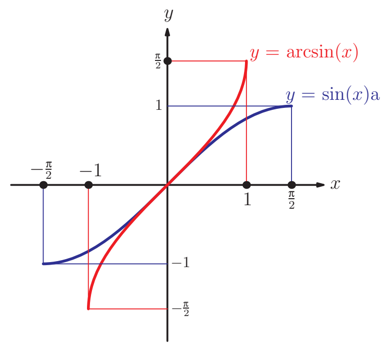
If you were told that the sine of a value is $\frac{1}{2}$ you now know that the original value might be $\frac{\pi}{6}$, but if you knew that the sine of a value was 0.8 the original value would not be so easy to find.

Inverse functions and their graphs were covered in Section 5E.

We need to find the inverse function of sine. Not every function has an inverse function; we need the original function to be one-to-one. Looking at the graph of the sine function, it is clear that it is not one-to-one. So how can we define inverse sine?



We need to consider the sine function on a restricted domain where the function is one-to-one. From the graph above we can see that a suitable domain is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. (There are other options but this is chosen by convention.) It is now possible to define the inverse function, called the **arcsine** and written $\arcsin x$. The graph of the inverse function is the reflection of the graph of the original function in the line $y = x$. From the graph alongside we can see the domain and range of the inverse function.



KEY POINT 9.12

The inverse function of $f(x) = \sin x$ is $f^{-1}(x) = \arcsin x$ or $\sin^{-1} x$. Its domain is $[-1, 1]$ and its range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAM HINT

Arcsine is also known as $\sin^{-1}$.

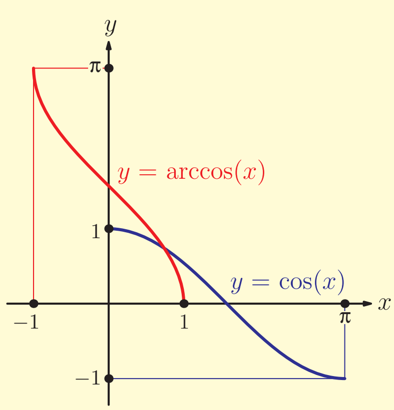
Calculators usually do not have a button labelled arcsin; use the $\sin^{-1}$ button instead.

We carry out similar analysis for cosine and tangent functions to identify the domains on which their inverse functions are defined. The results are as follows:

KEY POINT 9.13

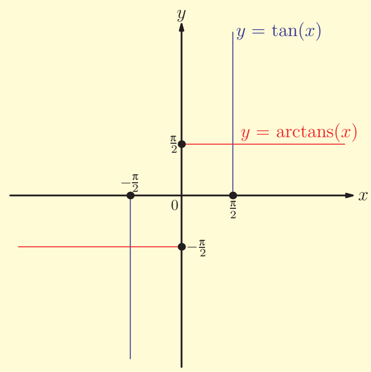
The inverse function of $f(x) = \cos x$ is $f^{-1}(x) = $ **arccos** x or $\cos^{-1} x$.

Its domain is $[-1, 1]$ and its range $[0, \pi]$.



KEY POINT 9.14

The inverse function of $f(x) = \tan x$ is $f^{-1}(x) = \textbf{arctan}\, x$ or $\tan^{-1} x$.
Its domain is $\mathbb{R}$ and its range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



You should remember what happens when we compose a function with its inverse: $f\left(f^{-1}(x)\right) = x$. Applying this to inverse trigonometric functions, we get:

$$\sin(\arcsin x) = x$$
$$\cos(\arccos x) = x$$
$$\tan(\arctan x) = x$$

We need to be more careful when composing the other way round: $\arcsin(\sin x) = x$ only when x is in the restricted domain we used to define the arcsin function (so $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

Inverse functions were introduced in chapter 5.

Worked example 9.16

Solve the equation $3\arcsin(3x) = \arcsin\left(\frac{1}{2}\right) + \arccos\left(\frac{1}{2}\right)$.

Evaluate the arcsine and arccosine terms on the RHS

$$3\arcsin(3x) = \arcsin\left(\frac{1}{2}\right) + \arccos\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{6} + \frac{\pi}{3}$$
$$= \frac{\pi}{2}$$

Isolate the arcsine

$$\arcsin(3x) = \frac{\pi}{6}$$

continued . . .

Taking sine of both sides undoes the arcsine

$$\sin(\arcsin(3x)) = \sin\left(\frac{\pi}{6}\right)$$

$$\Leftrightarrow \quad 3x = \frac{1}{2}$$

$$\Leftrightarrow \quad x = \frac{1}{6}$$

Exercise 9G

1. Use your calculator to evaluate in radians correct to three significant figures:

(a) (i) $\arccos 0.6$ (ii) $\arcsin(0.2)$

(b) (i) $\arctan(-3)$ (ii) $\arcsin(-0.8)$

2. Evaluate in radians without a calculator:

(a) (i) $\arcsin\left(\frac{1}{2}\right)$ (ii) $\arccos\left(\frac{\sqrt{3}}{2}\right)$

(b) (i) $\arctan(-\sqrt{3})$ (ii) $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

(c) (i) $\arcsin(-1)$ (ii) $\arctan(1)$

3. Evaluate the following, giving your answer in degrees correct to one decimal place:

(a) (i) $\arcsin(0.7)$ (ii) $\arcsin(0.3)$

(b) (i) $\arccos(-0.62)$ (ii) $\arccos(-0.75)$

(c) (i) $\arctan(6.4)$ (ii) $\arctan(-7.1)$

4. Find the value of:

(a) (i) $\sin(\arcsin 0.6)$ (ii) $\cos(\arccos(-0.3))$

(b) (i) $\tan(\arctan(-2))$ (ii) $\sin(\arcsin(-1))$

5. Find the exact value of:

(a) (i) $\cos\left(\arcsin\left(\frac{1}{2}\right)\right)$ (ii) $\sin\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$

(b) (i) $\sin\left(\arccos\left(-\frac{\sqrt{3}}{2}\right)\right)$ (ii) $\tan\left(\arccos\left(\frac{1}{2}\right)\right)$

6. Find the exact value of:

(a) (i) $\arcsin\left(\sin\left(\frac{\pi}{3}\right)\right)$ (ii) $\arccos\left(\cos\left(\frac{5\pi}{6}\right)\right)$

(b) (i) $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$ (ii) $\arccos(\cos(-3\pi))$

7. Solve the following equations:

(a) $\arcsin x = \frac{\pi}{3}$ (b) $\arccos 2x = \frac{5\pi}{6}$

(c) $\arctan(3x-1) = -\frac{\pi}{6}$

8. (a) Show by a counterexample that $\arctan x \neq \frac{\arcsin x}{\arccos x}$.

(b) Express $\arcsin x$ in terms of $\arccos x$.

(c) Solve the equation $2\arctan x = \arcsin x + \arccos x$. *[8 marks]*

9. A system of equations is given by:

$\sin x + \cos y = 0.6$

$\cos x - \sin y = 0.2$

(a) For each equation, express y in terms of x.

(b) Hence solve the system for

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. *[8 marks]*

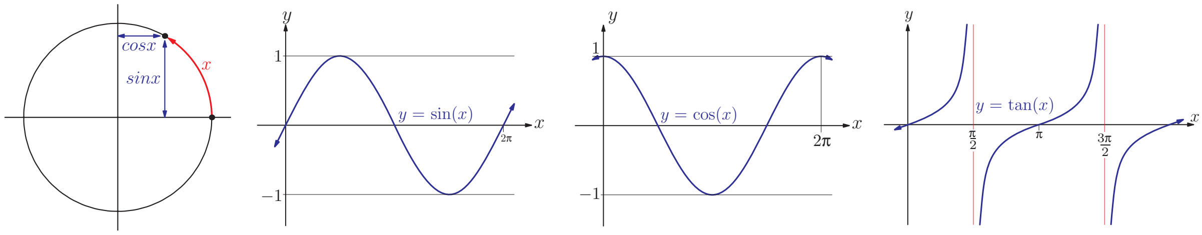
Summary

- The **unit circle** is a circle with centre O and radius 1.
- The **radian** measure is defined in terms of the distance travelled around the unit circle, so that a full turn $= 2\pi$ radians.
- To convert from degrees to radians, divide by 180 and multiply by π.
- To convert from radians to degrees, divide by π and multiply by 180.
- Using the unit circle and the real number α, the **sine** and **cosine** of this number is defined in terms of distance to the axes:

  $\sin \alpha$ is the distance of a point from the horizontal axis

  $\cos \alpha$ is the distance of the point from the vertical axis
- Useful properties of the sine and cosine function are summarised in Key point 9.3.
- The relationship between the sine and cosine function is summarised in Key point 9.4.
- The **tangent** function is another trigonometric function. It is defined as the ratio between the sine and cosine functions: $\tan x = \frac{\sin x}{\cos x}$

- The sine, cosine and tangent **trigonometric functions** can be defined for all real numbers.



- For some real numbers the three functions have exact values, which should be learnt. These are provided in Key point 9.9.

- The sine and cosine functions are periodic with period 2π and amplitude 1.

$$\text{period: } \sin x = \sin(x \pm 2\pi) = \sin(x \pm 4\pi) = \ldots$$

$$\cos x = \cos(x \pm 2\pi) = \cos(x \pm 4\pi) = \ldots$$

$$\text{amplituted: } -1 \le \sin x \le 1 \quad \text{and} \quad -1 \le \cos x \le 1$$

The cosine graph is a translation of the sine graph $\frac{\pi}{2}$ units to the left.

- The tangent function is periodic with period π.

$$\tan(x) = \tan(x \pm \pi) = \tan(x \pm 2\pi) = \ldots$$

- We can apply transformations to sine and cosine functions.

  $y = a\sin b(x+c)+d$ and $y = a\cos b(x+c)+d$ have:

  amplitude a and period $\frac{2\pi}{b}$

  minimum value $d-a$ and maximum value $d+a$.

- We introduced **inverse trigonometric functions**:

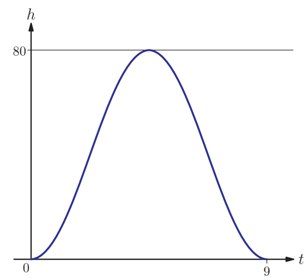
Inverse function	Domain	Range
arcsin x ($\sin^{-1} x$)	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
arccos x ($\cos^{-1} x$)	$[-1, 1]$	$[0, \pi]$
arctan x ($\cos^{-1} x$)	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Introductory problem revisited

The original Ferris Wheel was constructed in 1893 in Chicago. It was just over 80 m tall and could complete one full revolution in 9 minutes. During each revolution, how long did the passengers spend more than 5 m above ground?

A car on the Ferris Wheel moves in a circle. Its height above ground can therefore be modelled by sine or cosine function.

If the height of the wheel is 80 m then the radius of the circle is 40 m. A car starts on the ground, climbs to the maximum height of 80 m and returns to the ground. This takes 9 minutes. We can sketch the graph showing the height of the car above the ground as a function of time.



The period of the function is 9, so the equation will involve either $40\sin\left(\frac{2\pi}{9}t\right)$ or $40\cos\left(\frac{2\pi}{9}t\right)$. The amplitude is 40 and the graph has been translated up by 40 units. We can choose whether to use the sine or the cosine function so use the cosine function, as then there is no horizontal translation involved.

Combining all the transformations gives the equation for height in terms of time.

$$h = 40 - 40\cos\left(\frac{2\pi}{9}t\right)$$

To find how long the car spends more than 50 m above ground, we need to find the times when the height is exactly 50 m. This involves solving an equation.

$$40 - 40\cos\left(\frac{2\pi}{9}t\right) = 50$$

This can be solved on a calculator.

$$t_1 = 2.61 \text{ and } t_2 = 6.39$$

Therefore the time spent more than 50 m above ground is

$$6.39 - 2.61 \approx 3.8 \text{ minutes.}$$

Mixed examination practice 9

Short questions

EXAM HINT
Many examination questions combine ideas from this chapter with further trigonometry techniques you will meet in chapters 10–12.

1. The height of a wave at a distance x metres from a buoy is modelled by the function:

$f(x) = 1.4\sin(3x - 0.1) - 0.6$

(a) State the amplitude of the wave.

(b) Find the distance between consecutive peaks of the wave. *[4 marks]*

2. Sketch the graph of $y = \sin(2x) + 2\sin(6x)$ and hence find the exact period of the function. *[4 marks]*

3. A runner is jogging around a level circular track. His distance north of the centre of the track in metres is given by $60\cos 0.08t$ where t is measured in seconds.

(a) How long does is take the runner to complete one lap?

(b) What is the length of the track?

(c) At what speed is the runner jogging? *[7 marks]*

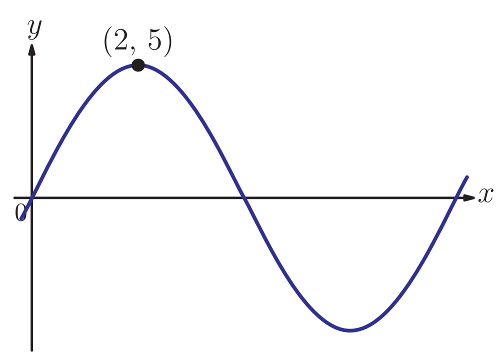
4. Let $f(x) = 3\sin 2\left(x - \frac{\pi}{3}\right)$.

(a) State the period of the function.

(b) Find the coordinates of the zeros of $f(x)$ for $x \in [0, 2\pi]$.

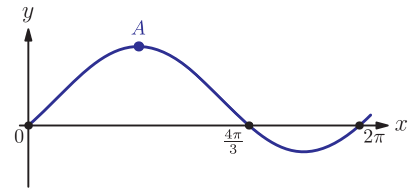
(c) Hence sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$, showing the coordinates of the maximum and minimum points. *[7 marks]*

5. The diagram shows the graph of the function $f(x) = a\sin(bx)$. Find the values of a and b. *[4 marks]*



Long questions

1. The graph shows the function $f(x) = \sin(x - k) + c$.



(a) (i) Write down the coordinates of A.
(ii) Hence find the values of k and c.

(b) Find all the zeros of the function in the interval $[-4\pi, 0]$.

(c) Consider the equation $f(x) = k$ with $-0.5 < k < 0$.
(i) Write down the number of solutions of this equation in the interval $[0, 9\pi]$.
(ii) Given that the smallest positive solution is α, write the next two solutions in terms of α. *[11 marks]*

2. (a) (i) Sketch the graph of $y = \tan x$ for $0 \le x \le 2\pi$.
(ii) On the same graph, sketch the line $y = \pi - x$.

(b) Consider the equation $x + \tan x = \pi$. Denote by x_0 the solution of this equation in the interval $]0, \frac{\pi}{2}[$.
(i) Find, in terms of x_0 and π, the remaining solutions of the equation in the interval $[0, 2\pi]$.
(ii) How many solutions does the equation $x + \tan x = \pi$ have for $x \in \mathbb{R}$?

(c) Given that $\cos A = c$ and $\sin A = s$,
(i) Write down the values of $\cos\left(\frac{\pi}{2} - A\right)$ and $\sin\left(\frac{\pi}{2} - A\right)$.
(ii) Hence show that $\tan\left(\frac{\pi}{2} - A\right) = \frac{1}{\tan A}$.
(iii) Given that $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$, find the possible values of $\tan A$.
(iv) Hence find the values of $x \in]0, \frac{\pi}{2}[$ for which $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$. *[16 marks]*

3. (a) Write down the minimum value of $\cos x$ and the smallest positive value of x (in radians) for which the minimum occurs.

(b) (i) Describe two transformations which transform the graph of $y = \cos x$ to the graph of $y = 2\cos\left(x + \frac{\pi}{6}\right)$.
(ii) Hence state the minimum value of $2\cos\left(x + \frac{\pi}{6}\right)$ and find the value of $x \in [0, 2\pi]$ for which the minimum occurs.

(c) The function f is defined for $x \in [0, 2\pi]$ by $f(x) = \dfrac{5}{2\cos\left(x + \frac{\pi}{6}\right) + 3}$.
(i) State, with a reason, whether f has any vertical asymptotes.
(ii) Find the range of f. *[13 marks]*

10 Trigonometric equations and identities

In this chapter you will learn:

- how to solve equations involving trigonometric functions
- about relationships between different trigonometric functions, called identities
- how to use identities to solve more complicated equations.

Introductory problem

Without a calculator solve the equation $2\cos x = 3\tan x$ for $0 \le x < 2\pi$.

When using trigonometric functions to model real life situations we often need to solve equations where the unknown is in the argument of a trigonometric function, for example, $12 - 4\sin\left(\frac{x}{\pi}\right) = 11$, or $\sin^2 x + \cos x = 0.5$. Because trigonometric functions are periodic, such equations can have more than one solution. In this chapter we will see how to find all solutions in a given interval. We will also see that we can use trigonometric identities, which tell us how different trigonometric functions are related to each other, to transform more complicated equations into simpler ones.

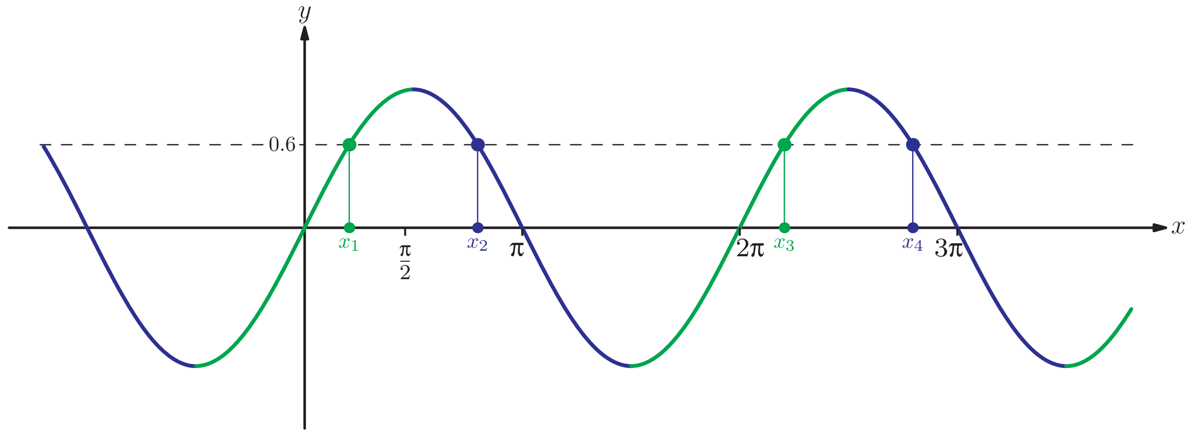
10A Introducing trigonometric equations

We start with the simplest type of trigonometric equation: Suppose we would like to find the x values which satisfy $\sin x = 0.6$.

By applying arcsine to both sides of the equation we get

$x = \arcsin 0.6 = 0.644$ from GDC (3SF).

The arcsine only gives us one solution. However, looking at the graph of $y = \sin x$ we can see that there are many points which satisfy this equation.



The solutions come in pairs – one in the green section of the graph and one in the blue section. The arcsine function will always give us a solution in the green section. To find the solution in the blue section we use fact that the graph has a line of symmetry at $x = \frac{\pi}{2}$. Therefore x_2 is as far below π as x_1 is above zero. In our example, this means that $x_2 = \pi - 0.644 = 2.50$.

Once we have this pair of solutions we can use the fact that the sine graph repeats with period 2π to find the other solutions: $x_3 = x_1 + 2\pi = 6.93$, $x_4 = 2.50 + 2\pi = 8.78$, and so on.

EXAM HINT

When doing subsequent calculations always use the ANS button or the stored value rather than the rounded answer.

See Calculator sheet 1 on the CD-ROM

KEY POINT 10.1

To find the possible values of x satisfying $\sin x = a$:

- use the calculator to find $x_1 = \arcsin a$
- the second solution is given by $x_2 = \pi - x_1$ (or $180 - x_1$ if working in degrees).

Other solutions are found by adding or subtracting 2π (or 360°) to any solution already found.

In the International Baccalaureate® you will only be asked to find solutions in a given interval.

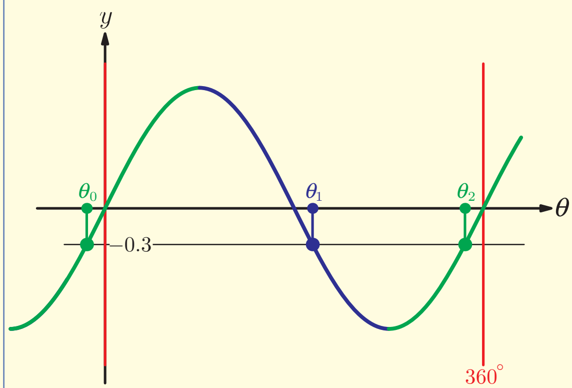
Worked example 10.1

Find the possible values of angle $\theta \in [0, 360°]$ for which $\sin\theta = -0.3$.

Check that the calculator is in degree mode

arcsin $(-0.3) = -17.5°$

Look at the graph to see how many solutions there are in the required interval



Note how many solutions

There are two solutions.

The second solution is given by $180° - \theta$

$180° - (-17.5°) = 197.5°$

The first solution is not in the required interval, so add $360°$

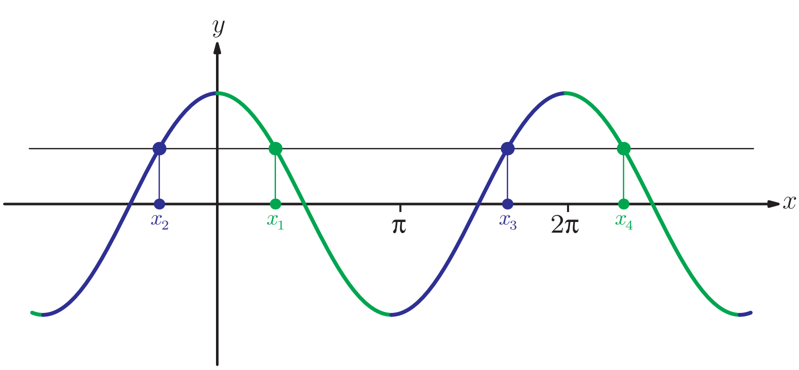
$\theta_1 = 197.5°$

$\theta_2 = 360° + (-17.5°) = 342.5°$

We can apply a similar analysis to the equation $\cos x = a$.

EXAM HINT

Always make sure that your calculator is in the correct mode, degree or radian, as indicated in the question.



The solution x_1 in the green region is given by $\arccos a$. We can use the symmetry of the cosine graph to find x_2; it is the negative of x_1. Once we have this pair of solutions we can use the fact that the cosine graph repeats with period 2π to find the other solutions.

EXAM HINT

It is often useful to know that the two positive solutions will be $\arccos a$ and $2\pi - \arccos a$.

KEY POINT 10.2

To find the possible values of x satisfying $\cos x = a$:

- use the calculator to find $x_1 = \arccos a$
- the second solution is given by $x_2 = -x_1$.

Other solutions are found by adding or subtracting 2π (or 360°) to any solution already found.

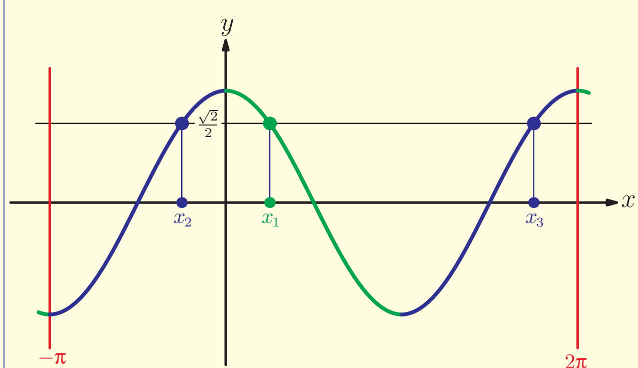
Worked example 10.2

Find the values of x between $-\pi$ and 2π for which $\cos x = \frac{\sqrt{2}}{2}$.

$\arccos \frac{\sqrt{2}}{2}$ is one that we should know

$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Sketch the graph to see how many solutions there are in the given interval



Note how many solutions

3 solutions

Use the symmetry of the graph to find the other solutions.

$x_1 = \frac{\pi}{4}$

$x_2 = -\frac{\pi}{4}$

$x_3 = 2\pi - \frac{\pi}{4}$

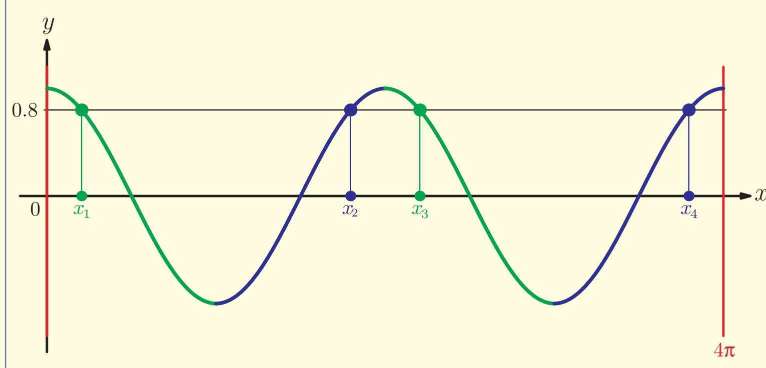
$= \frac{7\pi}{4}$

It can be difficult to know how many times to add or subtract 2π to make sure that we have found all the solutions in a given interval. Drawing a graph first can therefore help, as we can see how many solutions we are looking for and where they are.

Worked example 10.3

Find all the values of x between 0 and 4π for which $\cos x = 0.8$.

Sketch the graph



Note how many solutions

4 solutions

Arccosine on the calculator will give the first value of x

$x_1 = \arccos 0.8 = 0.644$ (3SF)

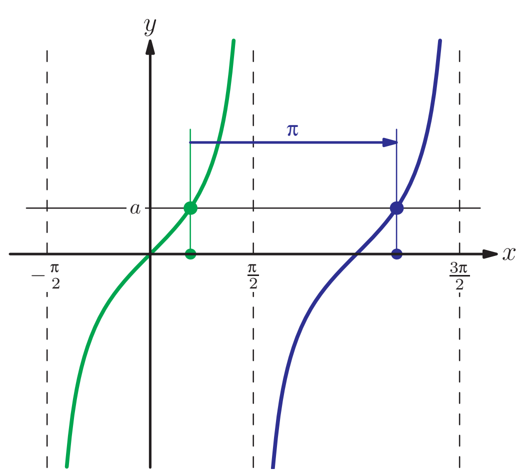
We can use the symmetry of the graph to find the other values

$x_2 = 2\pi - 0.644 = 5.64$ (3SF)

$x_3 = x_1 + 2\pi = 6.93$ (3SF)

$x_4 = x_2 + 2\pi = 11.9$ (3SF)

The procedure for solving equations of the type $\tan x = a$ is slightly different, because the tangent function has period π rather than 2π. It is best seen by looking at the graph of the tangent function.



KEY POINT 10.3

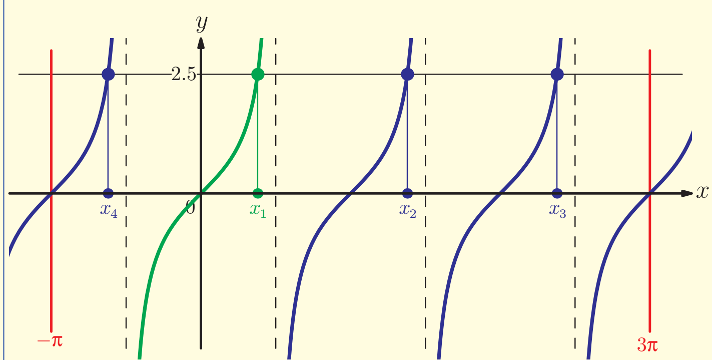
To find the possible values of x satisfying $\tan x = a$:

- use the calculator to find $x_1 = \arctan a$
- other solutions are found by adding or subtracting multiples of π.

Worked example 10.4

Find all real values of $x \in [-\pi, 3\pi]$ such that $\tan x = 2.5$.

Sketch the graph



Note the number of solutions

There are four solutions.

Use a calculator to find the arctan

$x_1 = \arctan 2.5 = 1.19$

The other solutions are found by adding or subtracting π

$x_2 = x_1 + \pi = 4.33$

$x_3 = x_2 + \pi = 7.47$

$x_4 = x_1 - \pi = -1.95$

Exercise 10A

1. Find the exact values of x between 0° and 360° for which:

(a) (i) $\sin x = \frac{1}{2}$ (ii) $\sin x = \frac{\sqrt{2}}{2}$

(b) (i) $\cos x = \frac{1}{2}$ (ii) $\cos x = \frac{\sqrt{3}}{2}$

(c) (i) $\sin x = -\frac{\sqrt{3}}{2}$ (ii) $\sin x = -\frac{1}{2}$

(d) (i) $\tan x = 1$ (ii) $\tan x = \sqrt{3}$

2. Find the exact values of x between 0 and 2π for which:

(a) (i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = \frac{\sqrt{2}}{2}$

(b) (i) $\cos x = -\frac{1}{2}$ (ii) $\cos x = -\frac{\sqrt{3}}{2}$

(c) (i) $\sin x = \frac{\sqrt{2}}{2}$ (ii) $\sin x = \frac{\sqrt{3}}{2}$

(d) (i) $\tan x = \frac{1}{\sqrt{3}}$ (ii) $\tan x = -1$

3. Solve these equations in the given interval, giving your answers to one decimal place.

(a) (i) $\sin x = 0.45$ for $x \in [0°, 360°]$

(ii) $\sin x = 0.7$ for $x \in [0°, 360°]$

(b) (i) $\cos x = -0.75$ for $-180° \le x \le 180°$

(ii) $\cos x = -0.2$ for $-180° \le x \le 180°$

(c) (i) $\tan\theta = \frac{1}{3}$ for $0° \le \theta \le 720°$

(ii) $\tan\theta = \frac{4}{3}$ for $0° \le \theta \le 720°$

(d) (i) $\sin t = -\frac{2}{3}$ for $t \in [-180°, 360°]$

(ii) $\sin t = -\frac{1}{4}$ for $t \in [-180°, 360°]$

4. Solve these equations in the given interval, giving your answers to three significant figures.

(a) (i) $\cos t = \frac{4}{5}$ for $t \in [0, 4\pi]$

(ii) $\cos t = \frac{2}{3}$ for $t \in [0, 4\pi]$

(b) (i) $\sin\theta = -0.8$ for $\theta \in [-2\pi, 2\pi]$

(i) $\sin\theta = -0.35$ for $\theta \in [-2\pi, 2\pi]$

(c) (i) $\tan\theta = -\frac{2}{3}$ for $-\pi \le \theta \le \pi$

(ii) $\tan\theta = -3$ for $-\pi \le \theta \le \pi$

(d) (i) $\cos\theta = 1$ for $\theta \in [0, 4\pi]$

(ii) $\cos\theta = 0$ for $\theta \in [0, 4\pi]$

5. Solve the following equations in the given interval, giving exact answers.

(a) (i) $\sin x = \frac{1}{2}$ for $0° \le x \le 360°$

(ii) $\sin x = \frac{\sqrt{2}}{2}$ for $0° \le x \le 360°$

(b) (i) $\cos x = -1$ for $-180° \le x \le 180°$

(ii) $\sin x = -1$ for $-180° \le x \le 180°$

(c) (i) $\tan x = \sqrt{3}$ for $0° < x < 360°$

(ii) $\tan x = 1$ for $0° < x < 360°$

(d) (i) $\cos x = -\frac{\sqrt{2}}{2}$ for $-360° \le x \le 360°$

(ii) $\cos x = -\frac{\sqrt{3}}{2}$ for $-360° \le x \le 360°$

6. Find the exact solutions of the following equations.

(a) (i) $\cos\theta = \frac{1}{2}$ for $-2\pi < \theta < 2\pi$

(ii) $\cos\theta = \frac{\sqrt{3}}{2}$ for $-2\pi < \theta < 2\pi$

(b) (i) $\sin\theta = -\frac{\sqrt{3}}{2}$ for $-\pi < \theta < 3\pi$

(ii) $\sin\theta = -\frac{\sqrt{2}}{2}$ for $-\pi < \theta < 3\pi$

(c) (i) $\tan\theta = -\frac{1}{\sqrt{3}}$ for $-\pi < \theta < \pi$

(ii) $\tan\theta = -1$ for $-\pi < \theta < \pi$

(d) (i) $\cos\theta = 0$ for $0 < \theta < 3\pi$

(ii) $\sin\theta = 0$ for $0 < \theta < 3\pi$

(e) $\sin\theta = \frac{\sqrt{2}}{2}$ for $-2\pi < \theta < 0$

7. Solve the following equations. Do not use the graphs on your calculator.

(a) (i) $2\sin\theta + 1 = 1.2$ for $0° < \theta < 360°$

(ii) $4\sin x + 3 = 2$ for $-90° < x < 270°$

(b) (i) $3\cos x - 1 = \frac{1}{3}$ for $0 < x < 2\pi$

(ii) $3\cos x - 1 = \frac{1}{3}$ for $0 < x < 2\pi$

(c) (i) $3\tan t - 1 = 4$ for $-\pi < t < \pi$

(ii) $5\tan t - 3 = 8$ for $0 < t < 2\pi$

8. Find the exact values of $x \in (-\pi, \pi)$ for which $2\sin x + 1 = 0$. *[5 marks]*

10B Harder trigonometric equations

In this section we shall look at two different problems which occur when solving trigonometric equations: equations which need to be rearranged first and equations in which the argument of the trigonometric function is more complicated.

It is not always obvious how to write an equation in the form trigonometric function = constant. There are three tactics which are often used.

See chapter 4 for a reminder on disguised quadratics and solving equations by factorising.

1. Look for disguised quadratics
2. Take everything over to one side and factorise
3. Use trigonometric identities

Squares of trigonometric functions come up very frequently and are given a special notation:

Using identities to solve trigonometric equations is covered in Section 10C.

KEY POINT 10.4

$(\cos x)^2$ is written $\cos^2 x$
$(\sin x)^2$ is written $\sin^2 x$
$(\tan x)^2$ is written $\tan^2 x$

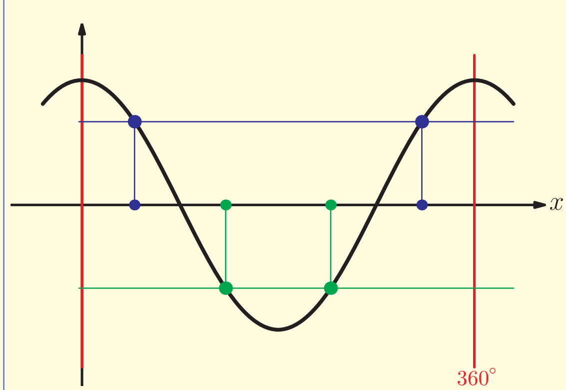
Worked example 10.5

Solve the equation $\cos^2\theta = \frac{4}{9}$ for $\theta \in [0°, 360°]$. Give answers correct to one decimal place.

First find possible values of $\cos\theta$. Remember ± when we take the square root

$\cos^2\theta = \frac{4}{9} \Rightarrow \cos\theta = \pm\frac{2}{3}$

Draw the graph to see how many solutions there are in the required interval



2 Solutions to each.

continued . . .

Solve each equation separately

When $\cos\theta = \frac{2}{3}$:

$$\arccos\left(\frac{2}{3}\right) = 48.2°$$

$$\theta = 48.2° \text{ or } 360° - 48.2° = 311.8°$$

When $\cos\theta = -\frac{2}{3}$:

$$\arccos\left(-\frac{2}{3}\right) = 131.8°$$

$$\theta = 131.8° \text{ or } 360° - 131.8° = 228.2°$$

List all the solutions.

$\theta_1 = 48.2°$
$\theta_2 = 131.8°$
$\theta_3 = 228.2°$
$\theta_4 = 311.8°$

In the next example we need to use factorising.

Worked example 10.6

Solve the equation $3\sin x\cos x = 2\sin x$ for $-\pi \le x \le \pi$.

We cannot find the inverse sine or inverse cosine directly. However, both sides have a factor of sin *x*, so we can make the equation equal to zero and factorise

$$3\sin x\cos x - 2\sin x = 0$$
$$\sin x(3\cos x - 2) = 0$$

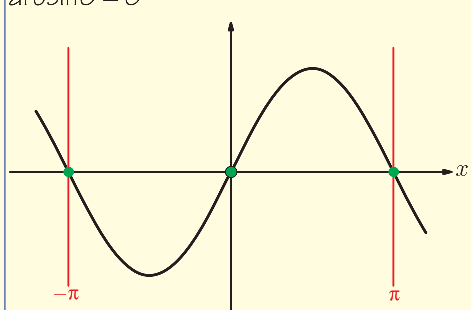
If a product is equal to 0, then one of the factors must be equal to 0

$\sin x = 0$ or $\cos x = \frac{2}{3}$

We can now solve each equation separately. Draw the graph for each equation to see how many solutions there are

When $\sin x = 0$:

$\arcsin 0 = 0$

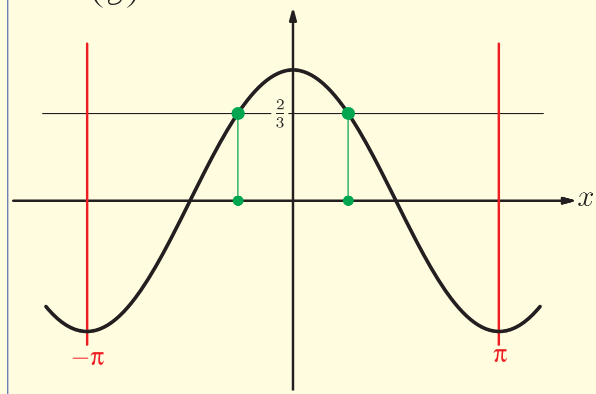


$$x = 0 \text{ or } \pi - 0 = \pi \text{ or } \pi - 2\pi = -\pi$$

continued . . .

When $\cos x = \frac{2}{3}$:

$\arccos\left(\frac{2}{3}\right) = 0.841$



$2\pi - 0.841 = 5.44$ is not in the interval

$5.44 - 2\pi = -0.841$ is in the interval

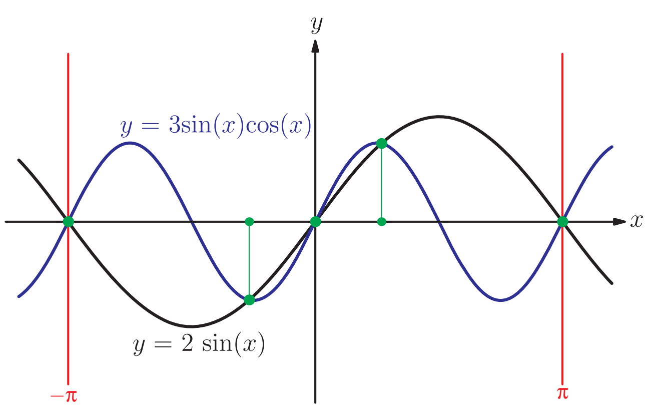
We have found 5 solutions

$x = -\pi, -0.841, 0, 0.841, \pi$

EXAM HINT

Do not be tempted to divide both sides of the original equation by sin x or you could lose some solutions, because sin x could be zero.

We can check these solutions using the graph on the calculator. Sketching graphs of $y = 3\sin x\cos x$ and $y = 2\sin x$ in the given interval shows that they intersect at five points.



Another common type of trigonometric equation is a disguised quadratic.

Disguised quadratics are often combined with the use of the Pythagorean identity – see Worked example 10.18.

Worked example 10.7

(a) Given that $3\sin^2 x - 5\sin x + 1 = 0$, find the possible values of sin x.

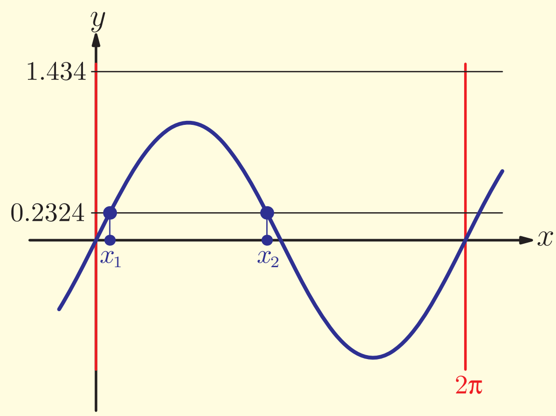
(b) Hence solve the equation $3\sin^2 x - 5\sin x + 1 = 0$ for $0 < x < 2\pi$.

Recognise that this is a quadratic equation in sin x. Since we cannot factorise it, use the quadratic formula

(a) $\sin x = \frac{5 \pm \sqrt{5^2 - 4\times3\times1}}{2\times3}$

$\sin x = 1.434$ or 0.2324

Draw the graph to see how many solutions there are



There are two solutions.

sin x is always between –1 and 1, so only one of the values is possible

$\sin x = 1.434 > 1$ is impossible.
Hence $\sin x = 0.2324$.

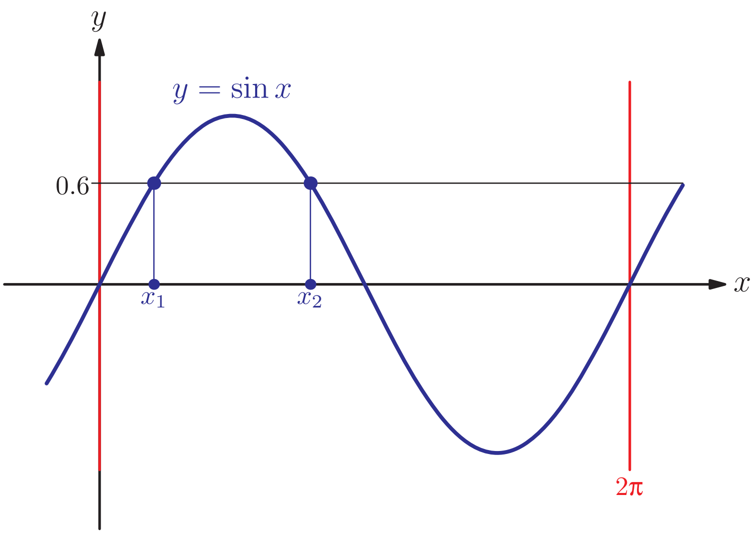
We can now solve the new equation as before

(b) $\arcsin(0.2324) = 0.235$
$x = 0.235$ or $\pi - 0.235 = 2.91$

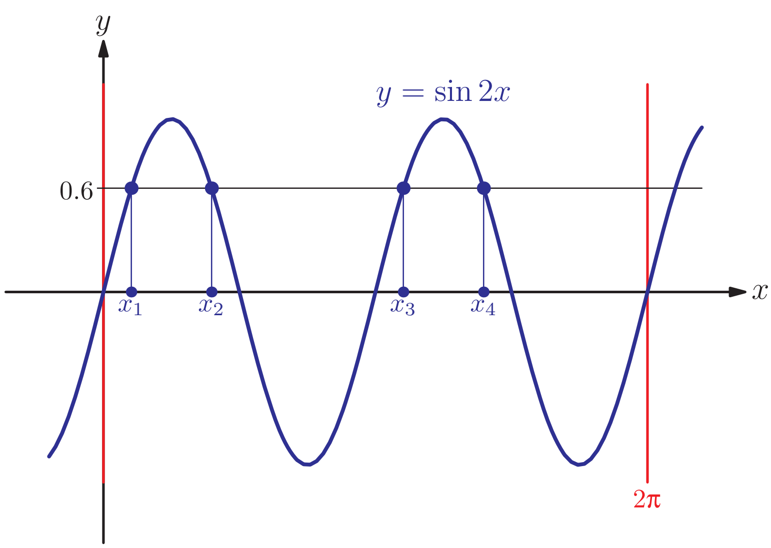
EXAM HINT

Questions like those in Worked example 10.7 force you to use algebraic methods to solve the equation. However, you can still use the graph on your calculator to check your solution. It is particularly useful to make sure that you have not missed out any solutions.

We will now look at equations where the argument of the trigonometric function is more complicated than just x. If we solve the equation $\sin x = 0.6$ for $0 \le x \le 2\pi$ we can see from the graph that there are two solutions:



If we solve the equation $\sin 2x = 0.6$ for $0 \le x \le 2\pi$ we can see from the graph that there are four solutions:



We need to extend the methods from Section 10A to deal with equations like this. A substitution is a useful approach.

Worked example 10.8

Find the zeros of the function $3\sin(2x)+1$ for $x \in [0, 2\pi]$.

Write down the equation to be solved

$3\sin(2x)+1=0$

Rearrange it into the form $\sin(A) = k$

$\sin(2x) = -\frac{1}{3}$

Make a substitution for the argument

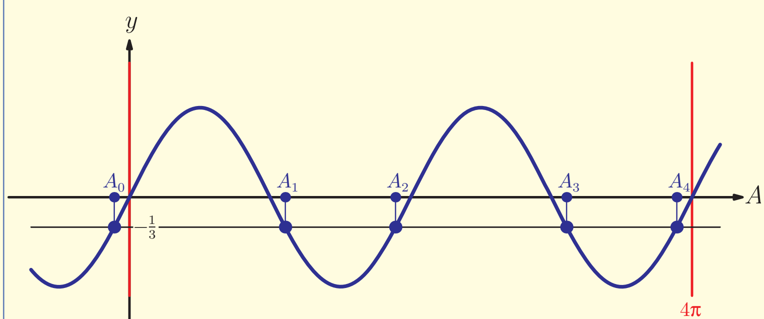
$A = 2x$

Rewrite the interval in terms of A

If $x \in [0, 2\pi]$ then $A \in [0, 4\pi]$.

Solve the equation for A

$\sin A = -\frac{1}{3}$



There are four solutions.

$A_0 = \arcsin\left(-\frac{1}{3}\right) = -0.3398$ (outside of interval)

$A_1 = \pi - A_0 = 3.481$
$A_2 = A_0 + 2\pi = 5.943$
$A_3 = A_1 + 2\pi = 9.764$
$A_4 = A_2 + 2\pi = 12.23$

Transform the solutions back into x

$x = \frac{A}{2}$

$= 1.74, 2.97, 4.88, 6.11$ (3SF)

This procedure may be summarised in the following four-step process:

KEY POINT 10.5

To solve trigonometric equations:

1. Make a substitution for the **argument** of the trigonometric function (such as $A = 2x$).
2. Change the interval for x into the interval for A.
3. Solve the equation in the usual way.
4. Transform the solutions back into the original variable.

The following example shows this method in a more complicated situation.

Worked example 10.9

Solve the equation $3\cos(2x+1) = 2$ for $x \in [-\pi, \pi]$.

Write the equation in the form $\cos(A) = k$

$\cos(2x+1) = \frac{2}{3}$

Make a substitution for the argument

$A = 2x + 1$

Rewrite the interval

If $-\pi \le x \le \pi$ then

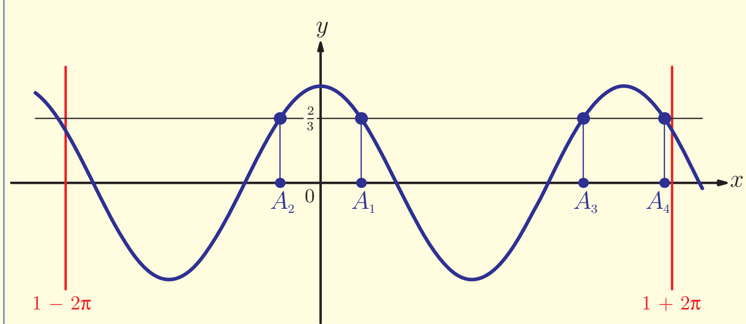
$-2\pi \le 2x \le 2\pi$

$-2\pi + 1 \le 2x + 1 \le 2\pi + 1$

So $A \in [-2\pi + 1,\ 2\pi + 1]$

Solve the equation for A

$\cos A = \frac{2}{3}$



There are four solutions

$A_1 = \arccos\left(\frac{2}{3}\right) = 0.841$

continued . . .

Transform the solutions back into x

$A_2 = -A_1 = -0.841$
$A_3 = A_2 + 2\pi = 5.44$
$A_4 = A_1 + 2\pi = 7.12$

$$x = \frac{A-1}{2}$$

$x = -0.0795, 2.22, 6.2035, -0.921$

We use the final example of this section to revisit the tangent function, working in degrees and finding exact solutions.

Worked example 10.10

Solve the equation $3\tan(\frac{1}{2}\theta - 30°) = \sqrt{3}$ for $0° \le \theta \le 720°$.

Rearrange the equation into the form tan $(A) = k$

$$\tan\left(\frac{1}{2}\theta - 30°\right) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Make a substitution for the argument

$$A = \frac{1}{2}\theta - 30°$$

Rewrite the interval

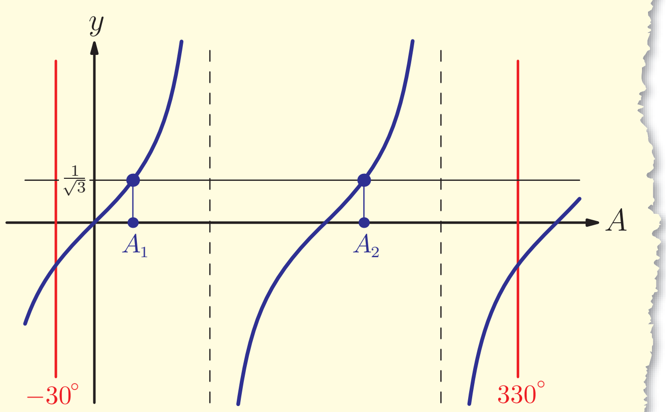
If $0° \le \theta \le 720°$ then

$$0° \le \frac{1}{2}\theta \le 360°$$

$$-30° \le \frac{1}{2}\theta - 30° \le 330°$$

Solve the equation for A

$$\tan A = \frac{1}{\sqrt{3}}$$



continued . . .

There are two solutions:

$A_1 = \arctan\left(\frac{1}{\sqrt{3}}\right) = 30°$

$A_2 = A_1 + 180° = 210°$

Transform the solutions back into θ.

$\theta = 2(A + 30°)$
$= 120°$ or $480°$

Exercise 10B

1. Solve, giving your answers to 3 significant figures:
   (a) (i) $\tan^2 x = 2$ for $-\pi \le x \le \pi$
   (ii) $\sin^2 x = 0.6$ for $-\pi \le x \le \pi$
   (b) (i) $9\cos^2\theta = 4$ for $0° < \theta < 360°$
   (ii) $3\tan^2\theta = 5$ for $0° < \theta < 360°$

2. Without using a graphing function on your calculator, find all solutions of these equations in the given interval. Use the graphs on your calculator to check your answers.
   (a) (i) $3\sin x - 2\sin x\cos x = 0$ for $0° \le x \le 360°$
   (ii) $4\cos x - \sin x\cos x = 0$ for $0° \le x \le 360°$
   (b) (i) $4\sin^2\theta = 3\sin\theta$ for $\theta \in [-\pi, \pi]$
   (ii) $3\cos^2\theta = -\cos\theta$ for $\theta \in [-\pi, \pi]$
   (c) (i) $\tan^2 t - 5\tan t + 5 = 0$ for $t \in]0, 2\pi[$
   (ii) $2\tan^2 t + \tan t - 1 = 0$ for $t \in]0, 2\pi[$
   (d) (i) $\sin\theta\tan\theta + \frac{1}{2}\tan\theta = 0$ for $\theta \in [0, 2\pi]$
   (ii) $2\cos\theta\tan\theta - 3\cos\theta = 0$ for $\theta \in [0, 2\pi]$
   (e) (i) $2\cos^2 x + 3\cos x = 2$ for $0° < x < 180°$
   (ii) $\cos^2 x - 2\cos x = 3$ for $0° < x < 180°$

3. Solve the following equations in the given interval, giving your answers to 3 significant figures. Do not use the graphing function on your GDC.
   (a) (i) $\cos 2x = \frac{1}{3}$ for $0° \le x \le 360°$
   (ii) $\cos 3x = \frac{2}{5}$ for $0° \le x \le 360°$

(b) (i) $\sin(3x-1) = -0.2$ for $0 \le x \le \pi$

(ii) $\sin(2x+1) = \frac{2}{3}$ for $0 \le x \le 2\pi$

(c) (i) $\tan(x-45°) = 2$ for $-180° \le x \le 180°$

(ii) $\tan(x+60°) = -3$ for $-180° \le x \le 180°$

4. Find the exact solutions in the given interval:

(a) (i) $\sin 2x = \frac{1}{2}$ for $0 \le x \le 2\pi$

(ii) $\sin 3x = -\frac{1}{2}$ for $0 \le x \le 2\pi$

(b) (i) $\cos 2x = -\frac{\sqrt{2}}{2}$ for $0° \le x \le 360°$

(ii) $\cos 3x = \frac{1}{2}$ for $-180° \le x \le 180°$

(c) (i) $\tan 4x = \sqrt{3}$ for $0 \le x \le \pi$

(ii) $\tan 2x = \frac{1}{\sqrt{3}}$ for $0 \le x \le \pi$

5. Find the exact solutions in the given interval:

(a) (i) $\cos(x+60°) = \frac{\sqrt{3}}{2}$ for $0° \le x \le 360°$

(ii) $\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ for $-\pi \le x \le \pi$

(b) (i) $\sin\left(x - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $-\pi \le x \le \pi$

(ii) $\sin(x-120°) = -\frac{\sqrt{2}}{2}$ for $0° \le x \le 360°$

(c) (i) $\tan\left(x + \frac{\pi}{2}\right) = 1$ for $0 < x < 2\pi$

(ii) $\tan\left(x - \frac{\pi}{4}\right) = -1$ for $0 < x < 2\pi$

6. Solve the equation $3\cos x = \tan x$ for $0 \le x \le 2\pi$. *[8 marks]*

7. (a) Given that $2\sin^2 x - 3\sin x = 2$, find the exact values of $\sin x$.

(b) Hence solve the equation $2\sin^2 x - 3\sin x = 2$ for $0° < x < 360°$. *[6 marks]*

8. Solve the equation $\sin x \tan x = \sin^2 x$ for $-\pi \le x \le \pi$. *[8 marks]*

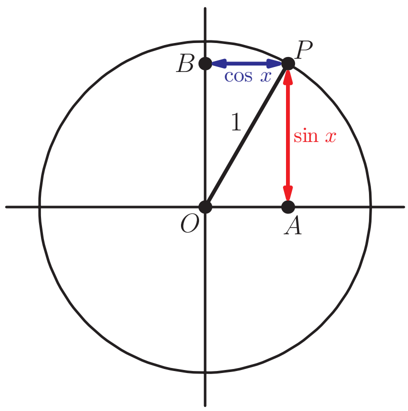
9. Solve the equation $\sin(x^2) = \frac{1}{2}$ for $-\sqrt{\pi} < x < \sqrt{\pi}$ *[5 marks]*

10C Trigonometric identities

We have already seen one example of an identity in chapter 9: $\frac{\sin x}{\cos x} = \tan x$. The two sides are equal for all values of x (except when $\cos = 0$, where $\tan x$ is undefined). There are many other identities involving trigonometric functions, and we will meet some of them in this section.

> **EXAM HINT**
>
> Many books write $\frac{\sin x}{\cos x} \equiv \tan x$ to emphasise that the expression is an identity (true for all values of x) rather than an equation (only true for some values of x which need to be found). However, the International Baccalaureate® syllabus and most exam questions use the equals sign in identities, so we will do the same in this book, unless there is a possibility of confusion.

Consider again the familiar unit circle diagram, with point P representing number x. According to the definitions of sine and cosine functions, $AP = \sin x$ and $BP = \cos x = OA$.



But triangle OAP is right angled, with hypotenuse 1. Using Pythagoras' Theorem, we get:

KEY POINT 10.6

Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

We have already seen examples where we used the values of $\sin x$ and $\cos x$ to find the value of $\tan x$. Using the Pythagorean identity, we only need to know the value of one of the functions to find the other two.

Worked example 10.11

Given that $\sin x = \frac{1}{3}$, find the possible values of cos x and tan x.

Use identity relating sin and cos

$$\sin^2 x + \cos^2 x = 1$$

$$\left(\frac{1}{3}\right)^2 + \cos^2 x = 1$$

$$\cos^2 x = 1 - \frac{1}{9} = \frac{8}{9}$$

Remember ± when you take square root

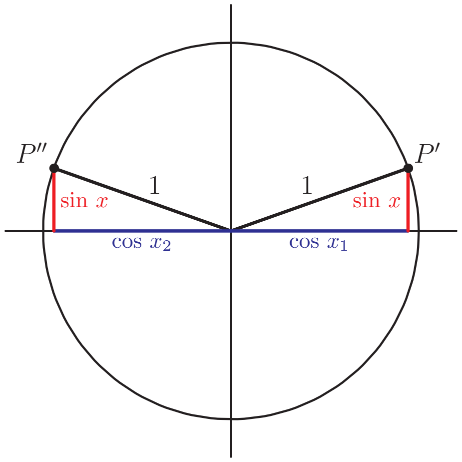
$$\therefore \cos x = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$$

Find tan given sin and cos

$$\tan x = \frac{\sin x}{\cos x}$$

$$\therefore \tan x = \frac{\frac{1}{3}}{\pm\frac{2\sqrt{2}}{3}} = \pm\frac{\sqrt{2}}{4}$$

Notice that for a given value of sin x, there are two possible values of cos x. This can be seen by looking at the unit circle: points P' and P'' are both distance, sin x above the horizontal axis, but have different values of cos x (equal in size but opposite in sign).



Also notice that we do not need to know what x is; in fact we know that there are infinitely many possible values for x.

We can specify one of the two possible values by restricting x to be in a particular quadrant.

Worked example 10.12

If $\tan x = 2$ and $\frac{\pi}{2} < x < \pi$ find the value of $\cos x$.

We need to link $\tan x$ and $\cos x$. We know $\frac{\sin x}{\cos x} = \tan x$ and can then use $\sin^2 x + \cos^2 x = 1$ to eliminate $\sin x$

$\sin x = \tan x \cos x$

We can substitute $\sin x$ from the first identity into the second

$\therefore \tan^2 x \cos^2 x + \cos^2 x = 1$

Now substitute the known value of $\tan x$

$2^2 \cos^2 x + \cos^2 x = 1$

$\Leftrightarrow 5\cos^2 x = 1$

$\Leftrightarrow \cos x = \pm\frac{\sqrt{5}}{5}$

x is in the second quadrant, so $\cos x$ is negative

$\cos x < 0$

$\therefore \cos x = -\frac{\sqrt{5}}{5}$

We can use known identities to derive new ones.

For example, multiplying both sides of the Pythagorean identity by 3 gives:

$$3\sin^2 x + 3\cos^2 x = 3$$

We can also rearrange an identity, just as we would with an equation. For example, we can rearrange the Pythagorean identity to get:

$$\cos^2 x = 1 - \sin^2 x \ (1)$$

In chapter 12, we shall meet a more direct way to relate $\tan x$ and $\cos x$.

We can also rearrange one identity and then substitute into another. For example, we can rearrange $\tan x = \frac{\sin x}{\cos x}$ to $\cos x = \frac{\sin x}{\tan x}$ and substitute it into the identity (1) above to get:

$$\frac{\sin^2 x}{\tan^2 x} = 1 - \sin^2 x$$

Finally, in each of these identities the variable x can be replaced by any other variable or expression, as long as each occurrence of x is replaced by the same thing.

This means that, for example:

$$\sin^2(2x+1)+\cos^2(2x+1)=1$$
$$\text{and } \frac{\sin 2x}{\cos 2x}=\tan 2x$$

By combining these ideas we can derive many more complicated identities, especially if we also use algebraic identities such as the difference of two squares.

Worked example 10.13

Starting from the identity $\sin^2 x+\cos^2 x=1$, derive the identity $\sin^4 x-\cos^4 x=2\sin^2 x-1$.

$\sin^2 x+\cos^2 x=1$

Look for some relationship between the two identities. In the second identity LHS is the difference of two squares: $a^4-b^4=(a^2-b^2)(a^2+b^2)$

$\sin^4 x-\cos^4 x$

This means that to get the LHS of the second identity, we need to multiply the first identity by $\sin^2 x-\cos^2 x$

$\Rightarrow (\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)=\sin^2 x-\cos^2 x$

$\Rightarrow \sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

Remove cos x from RHS, using $\cos^2 x=1-\sin^2 x$

$\Rightarrow \sin^4 x-\cos^4 x=\sin^2 x-(1-\sin^2 x)$

$\Rightarrow \sin^4 x-\cos^4 x=2\sin^2 x-1$

We know that the final equation is true for all x because we started from a known identity and only used known identities and algebraic manipulations. Therefore we have ended up with a new identity.

We have already looked at some methods for proving identities in Section 4H.

The question we want to ask now is, if we are given the final equation, how can we show that it is an identity? One way would be to try and recreate the steps we used to derive it from $\sin^2 x+\cos^2 x=1$. However, this seems to involve a lot of guessing! In the next three examples we look at various techniques for proving identities.

We begin with the same identity we just proved, but without the hint to use $\sin^2 x+\cos^2 x=1$ as a starting point.

Worked example 10.14

Prove the identity $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$.

The task is to show that the two sides are equal. One way to do this is to start from the left-hand side (LHS) and transform it until we get the right-hand side (RHS)

$LHS = \sin^4 x - \cos^4 x$

The LHS involves both sin x and cos x, while the RHS involves only sin x. So it seems like a good idea to replace $\cos^4 x$ on the LHS by using $\cos^2 x = 1 - \sin^2 x$

$$= \sin^4 x - \left(1 - \sin^2 x\right)^2$$
$$= \sin^4 x - \left(1 - 2\sin^2 x + \sin^4 x\right)$$
$$= -1 + 2\sin^2 x$$
$$= RHS$$

The two sides are equal, so we have proved the identity

$LHS = RHS$ ∴ proved

Sometimes it is not obvious how to get from one side to the other. In that case, we can transform each side separately and 'meet in the middle'.

Worked example 10.15

Prove the identity $\frac{1}{\cos\theta} - \cos\theta = \sin\theta\tan\theta$.

Write the whole LHS expression as a single fraction

$$LHS = \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$
$$= \frac{1-\cos^2\theta}{\cos\theta}$$

The RHS involves $\sin\theta$, so use $\sin^2\theta + \cos^2\theta$ in the numerator

$$= \frac{\sin^2\theta}{\cos\theta}$$

Look at the RHS. We know an identity for $\tan\theta$

$$RHS = \sin\theta\frac{\sin\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta}{\cos\theta}$$

The two sides are equal, so we have proved the identity

$LHS = RHS$ ∴ proved

Worked example 10.15 illustrates the importance of being able to simplify expressions with fractions. In a more complicated example like this, it is useful to write what you are doing, so you can check your working more easily.

Worked example 10.16

Prove the identity $\sin x\cos x = \dfrac{\tan x}{1+\tan^2 x}$.

There is no obvious way to simplify the LHS, but the RHS involves $\tan x$, so rewrite in terms of $\sin x$ and $\cos x$

$$RHS = \frac{\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} \qquad \left(\text{using } \tan x = \frac{\sin x}{\cos x}\right)$$

To simplify the 'big' fraction, multiply top and bottom by the common denominator, which is $\cos^2 x$

$$= \frac{\frac{\sin x}{\cos x}\times\cos^2 x}{\cos^2 x+\sin^2 x} \qquad (\text{multiplying by } \cos^2 x)$$

$$= \frac{\sin x\cos x}{\cos^2 x+\sin^2 x}$$

Now use $\sin^2 x + \cos^2 x = 1$

$$= \sin x\cos x \qquad (\text{using } \sin^2 x+\cos^2 x = 1)$$

This is equal to the LHS

$$= LHS \quad \therefore \text{proved}$$

Exercise 10C

1. Find the exact values of $\cos x$ and $\tan x$ given that:

   (a) $\sin x = \frac{1}{3}$ and $0° < x < 90°$

   (b) $\sin x = \frac{4}{5}$ and $0° < x < 90°$

2. Find the exact values of $\sin\theta$ and $\tan\theta$ given that:

   (a) $\cos\theta = -\frac{1}{3}$ and $180° < \theta < 270°$

   (b) $\sin\theta = -\frac{3}{4}$ and $180° < \theta < 270°$

3. (a) Find the exact value of $\cos x$ if:

   (i) $\sin x = \frac{1}{5}$ and $90° < x < 180°$

   (ii) $\sin x = -\frac{1}{2}$ and $270° < x < 360°$

   (b) Find the exact value of $\tan x$ if:

   (i) $\cos x = \frac{3}{5}$ and $-\frac{\pi}{2} < x < 0$

(ii) $\cos x = -1$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$

4. (a) Find the possible values of $\cos x$ if $\tan x = \frac{2}{3}$.

(b) Find the possible values of $\sin x$ if $\tan x = -\frac{1}{2}$.

5. Find the exact value of:

(a) $3\sin^2 x + 3\cos^2 x$

(b) $\sin^2 5x + \cos^2 5x$

(c) $-2\cos^2 2x - 2\sin^2 2x$

(d) $2\tan^2 2x - \frac{2}{\cos^2 2x}$

(e) $\frac{1}{\sin^2 x} - \frac{1}{\tan^2 x}$

(f) $\frac{3}{2\sin^2 4x} - \frac{3}{2\tan^2 4x}$

6. (a) (i) Express $3\sin^2 x + 4\cos^2 x$ in terms of $\sin x$ only.

(ii) Express $\cos^2 x - \sin^2 x$ in terms of $\cos x$ only.

7. If $t = \tan x$, express the following in terms of t:

(a) $\cos^2 x$

(b) $\sin^2 x$

(c) $\cos^2 x - \sin^2 x$

(d) $\frac{2}{\sin^2 x} + 1$

8. Prove the following identities:

(a) (i) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

(ii) $(2\sin x - \cos x)^2 + (\sin x + 2\cos x)^2 = 5$

(b) (i) $\sin\theta\tan\theta + \cos\theta = \frac{1}{\cos\theta}$

(ii) $\frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \sin\theta$

9. Express $3 - 2\tan^2 x$ in terms of $\cos x$ only. *[3 marks]*

10. Express $\frac{1+\tan^2 x}{\cos^2 x}$ in terms of $\sin x$, simplifying your answer. *[4 marks]*

11. Prove the following identities:

(a) $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$

(b) $\frac{1}{\cos\theta} + \tan\theta = \frac{\cos\theta}{1-\sin\theta}$ *[8 marks]*

10D Using identities to solve equations

Now we can use trigonometric identities to solve more complicated equations.

Usually we either start by replacing $\tan x$ by $\frac{\sin x}{\cos x}$, or using the Pythagorean identity. The latter can only be used if the equation contains squares and usually results in a quadratic equation.

Worked example 10.17

Solve the equation $4\sin x = \tan x$ in the interval $0 \le x \le 2\pi$.

Use the identity for tan	$4\sin x = \frac{\sin x}{\cos x}$
Eliminate fractions, so multiply both sides by $\cos x$	$\Rightarrow \quad 4\sin x\cos x = \sin x$
Both sides contain $\sin x$, so make the equation equal to zero and factorise	$\Rightarrow \; 4\sin x\cos x - \sin x = 0$ $\Rightarrow \; \sin x(4\cos x - 1) = 0$
One of the factors must be equal to zero	$\Rightarrow \; \sin x = 0 \text{ or } \cos x = \frac{1}{4}$
Solve each equation separately	When $\sin x = 0$: $x = 0,\ \pi,\ 2\pi$ When $\cos x = \frac{1}{4}$: $x = \arccos\left(\frac{1}{4}\right) = 1.32\,(3SF)$ or $2\pi - 1.32 = 4.97\,(3SF)$

The next example shows how using the Pythagorean identity can result in a quadratic equation. Note that you could solve this equation using a graph on your calculator, but the question can force you to use an algebraic method, for example by asking you to find possible values of $\cos\theta$ first.

Worked example 10.18

Find all values of θ in the interval $[-180°,\ 180°]$ which satisfy the equation $2\sin^2\theta + 3\cos\theta = 1$.

The sine term is squared, so replace $\sin^2\theta$ by $1-\cos^2\theta$	$2(1-\cos^2\theta) + 3\cos\theta = 1$ $\Rightarrow 2 - 2\cos^2\theta + 3\cos\theta = 1$
This is a quadratic equation in $\cos\theta$, so write it in the standard form and then solve it	$\Rightarrow 2\cos^2\theta - 3\cos\theta - 1 = 0$ $\Rightarrow \cos\theta = 1.78 \text{ or } -0.281$

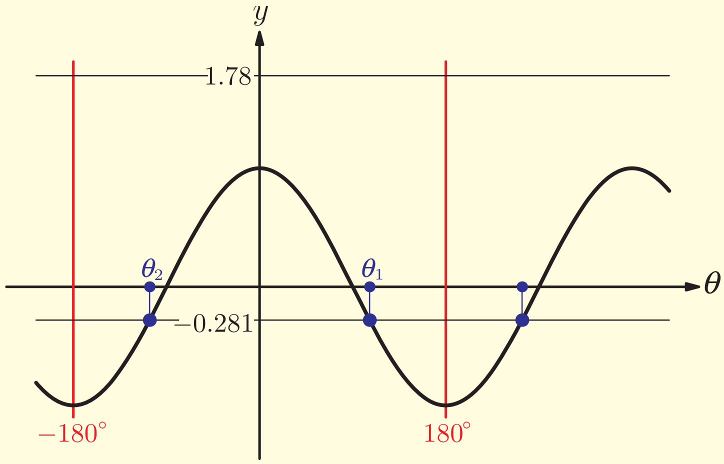
continued . . .

cos θ is between −1 and 1

$\cos\theta = 1.78$ is impossible

Solve the remaining equation

$\cos\theta = -0.281$



$\theta_1 = \arccos(-0.281) = 106°$

$\theta_2 = 254° - 360° = 106°$

Exercise 10D

1. By using the identity $\tan x = \dfrac{\sin x}{\cos x}$, solve the following equations:

(a) (i) $3\sin x = 2\cos x$ for $0° \le x \le 180°$

(ii) $3\sin x = 5\cos x$ for $0° \le x \le 180°$

(b) (i) $\cos x = 3\sin x$ for $0 \le x \le \dfrac{\pi}{2}$

(ii) $3\cos x = -\sin x$ for $0 \le x \le \pi$

(c) (i) $3\sin x + 5\cos x = 0$ for $0 \le x \le 2\pi$

(ii) $4\cos x + 3\sin x = 0$ for $0 \le x \le 2\pi$

(d) (i) $7\cos x - 3\sin x = 0$ for $-180° \le x \le 180°$

(ii) $\sin x - 5\cos x = 0$ for $-180° \le x \le 180°$

2. Solve the following equations in the given interval, giving exact answers.

(a) (i) $\sin 3\theta = \cos 3\theta$ for $0 < \theta < \dfrac{\pi}{2}$

(ii) $\sin 2t = \sqrt{3}\cos 2t$ for $t \in [0, \pi]$

(b) (i) $\sin 2x + \sqrt{3}\cos 2x = 0$ for $0 \le x \le 2\pi$

(ii) $\sin 3a + \cos 3a = 0$ for $a \in \left[0, \frac{\pi}{2}\right]$

3. Use trigonometric identities to solve these equations:

(a) $\sin x + \frac{\sin^2 x}{\cos x} = 0$ for $0° \le x \le 360°$

(b) $3\sin^2 x = 2\sin x \cos x$ for $x \in [-\pi, \pi]$

(c) $\frac{\cos\theta}{\sin\theta} - 2 = 0$ for $\theta \in [-90°, 90°]$

(d) $3\cos^2\theta + 4\sin\theta\cos\theta = 0$ for $0 \le \theta \le 2\pi$

4. Use the identity $\sin^2 x + \cos^2 x = 1$ to solve the following equations in the interval $[0°, 360°]$:

(a) (i) $7\sin^2 x + 3\cos^2 x = 5$ (ii) $\sin^2 x + 4\cos^2 x = 2$

(b) (i) $3\sin^2 x - \cos^2 x = 1$ (ii) $\cos^2 x - \sin^2 x = 1$

5. Find the exact values of $\theta \in [-180°, 180°]$ for which $\sqrt{3}\tan\theta = 2\sin\theta$.

6. Use an algebraic method to solve the equation $5\sin^2\theta = 4\cos^2\theta$ for $-180° \le \theta \le 180°$. *[4 marks]*

7. Solve the equation $2\cos^2 t - \sin t - 1 = 0$ for $0 \le t \le 2\pi$. *[4 marks]*

8. Solve the equation $4\cos^2 x - 5\sin x - 5 = 0$ for $x \in [-\pi, \pi]$. *[4 marks]*

9. Given that $\cos^2 t + 5\cos t = 2\sin^2 t$, find the exact values of $\cos t$. *[4 marks]*

10. (a) Given that $6\sin^2 x + \cos x = 4$, find the exact values of $\cos x$.

(b) Hence solve the equation $6\sin^2 x + \cos x = 4$ for $0° \le x \le 360°$. *[6 marks]*

11. (a) Show that the equation $2\sin^2 x - 3\sin x\cos x + \cos^2 x = 0$ can be written in the form $2\tan^2 x - 3\tan x + 1 = 0$.

(b) Hence solve the equation

$2\sin^2 x - 3\sin x\cos x + \cos^2 x = 0$,

giving all solutions in the interval $-\pi < x < \pi$. *[6 marks]*

Summary

- To solve trigonometric equations, you should follow the following procedure:
  - First rearrange into the form $\sin A = k$, $\cos A = k$ or $\tan A = k$ and then make a substitution for the **argument** of the trigonometric function e.g. A = 2*x*. (It is not necessary to substitute A for *x* in simple trigonometric equations.)
  - Change the interval for x to the interval for A and draw a graph to see how many solutions there are.
  - Find solutions in the interval e.g. $[0,\ 2\pi]$:
    - $\sin A = k$: $A_1 = \arcsin k$, $A_2 = \pi - A_1$
    - $\cos A = A$: $A_1 = \arccos k$, $A_2 = -A_1$ (i.e. the reflection of A_1)
    - $\tan A = k$: $A_1 = \arctan k$, $A_2 = A_1 \pm \pi$
  - Other solutions are found by adding or subtracting multiples of 2π (π for tan).
  - Use the values of A to find the value of x i.e. transform the solutions back into the original variable.
- Trigonometric functions are related through identities:

$$\text{e. g. } \tan x = \frac{\sin x}{\cos x} \text{ and } \sin^2 x + \cos^2 x = 1$$

- These identities can be used to solve more complicated equations.

Introductory problem revisited

Without a calculator solve the equation $2 \cos x = 3 \tan x$, for $0 \le x < 2\pi$.

Using the identity $\tan x = \dfrac{\sin x}{\cos x}$ the equation becomes: $2\cos x = 3\dfrac{\sin x}{\cos x}$.

Multiplying both sides by $\cos x$: $2\cos^2 x = 3\sin x$

We can now use an identity to rewrite the entire equation in terms of $\sin x$:

$2(1-\sin^2 x) = 3\sin x$

However, this is a disguised quadratic:

$$\begin{aligned} 0 &= 2\sin^2 x + 3\sin x - 2 \\ &= (2\sin x - 1)(\sin x + 2) \end{aligned}$$

Therefore either $\sin x = \dfrac{1}{2}$ or $\sin x = -2$.

The latter is impossible, so we have: $x = \arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$

Or

$$x = \pi - \arcsin\frac{1}{2} = \frac{5\pi}{6}$$

Mixed examination practice 10

Short questions

1. Solve the equation $\tan x = -0.62$ for $x \in (-90°, 270°)$. *[4 marks]*

2. Prove the identity $\frac{2}{\cos^2 x} - \tan^2 x = 2 + \tan^2 x$. *[5 marks]*

3. Solve the equation $5\sin^2\theta = 4\cos^2\theta$ for $-\pi \le \theta \le \pi$. *[5 marks]*

4. Prove the identity $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = \frac{2}{\sin^2 x}$. *[5 marks]*

5. Solve the equation $\cos\theta - 2\sin^2\theta + 2 = 0$ for $\theta \in [0°, 360°]$ *[6 marks]*

6. Use an algebraic method to solve the equation $6\sin^2 x + \cos x = 4$ for $0° \le x \le 360°$. *[6 marks]*

7. Find the exact values of $x \in [-\pi, \pi]$ satisfying the equation
$$2\cos\left(2x + \frac{\pi}{3}\right) = \sqrt{2}.$$ *[6 marks]*

8. (a) Given that $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{16}{3}$, find the possible values of $\sin x$.

   (b) Hence find the exact solutions of the equation $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{16}{3}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. *[6 marks]*

Long questions

1. The shape of a small bridge can be modelled by the equation $y = 1.8\sin\left(\frac{x}{3}\right)$, where y is the height of the bridge above water, and x is the distance from one river bank, both measured in metres.

   (a) Find the width of the river.

   (b) A barge has height 1.2 metres above the water level. Find the maximum possible width of the barge so it can pass under the bridge.

   (c) Another barge has width 3.5 m. What is the maximum possible height of the barge so it can pass under the bridge? *[10 marks]*

2. (a) Sketch the graph of the function $C(x) = \cos x + \frac{1}{2}\cos 2x$ for $-2\pi \le x \le 2\pi$.

   (b) Prove that the function $C(x)$ is periodic and state its period.

   (c) For what values of x, $-2\pi \le x \le 2\pi$, is $C(x)$ a maximum?

(d) Let $x = x_0$ be the smallest positive value of x for which $C(x) = 0$. Find an approximate value of x_0 which is correct to two significant figures.

(e) (i) Prove that $C(x) = C(-x)$ for all x.

(ii) Let $x = x_1$ be that value of x, $\pi < x < 2\pi$, for which $C(x) = 0$. Find the value of x_1 in terms of x_0. *[16 marks]*

3. (a) Find the value of k for which the equation $4x^2 - kx + 1 = 0$ has a repeated root.

(b) Show that the equation $4\sin^2 x = 5 - k\cos x$ can be written as $4\cos^2\theta - k\cos\theta + 1 = 0$.

(c) Let $f_k(\theta) = 4\cos^2\theta - k\cos\theta + 1$.

(i) State the number of values of $\cos\theta$ which satisfy the equation $f_4(\theta) = 0$.

(ii) Find all the values of $\theta \in [-2\pi, 2\pi]$ which satisfy the equation $f_4(\theta) = 0$.

(iii) Find the value of k for which $x = 1$ is a solution of the equation $4x^2 - kx + 1 = 0$.

(iv) For this value of k, find the number of solutions of the equation $f_k(\theta) = 0$ for $\theta \in [-2\pi, 2\pi]$. *[14 marks]*

11 Geometry of triangles and circles

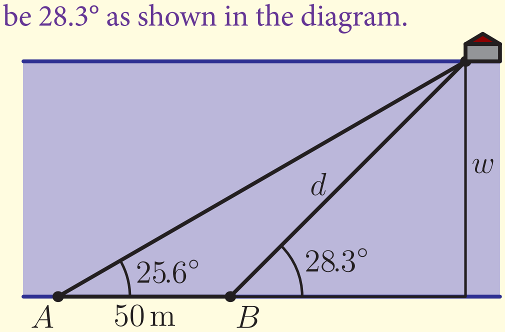
In this chapter you will learn:

- a reminder of the use of trigonometry in right angled triangles
- how to use the sine rule to find sides and angles of any triangle
- how to use the cosine rule to find sides and angles of any triangle
- an alternative formula for the area of a triangle
- to use techniques to solve problems in two and three dimensions
- how to calculate the length of an arc of a circle
- how to calculate the area of a sector of a circle
- to apply trigonometry to solving problems involving circles and triangles.

The first steps in developing trigonometry were made by Babylonian astronomers as early as the 2nd millennium BCE. It is thought that Egyptians used trigonometric calculations when building the pyramids. These were further developed by Greek and Indian mathematicians. Some of the most significant contributions to trigonometry were made by Islamic mathematicians in the second half of the 1st millennium BCE.

Introductory problem

Two observers are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. They stand at A and B, 50 m apart, on the opposite side of the river to a tower. The person at A measures the angle between the line (AB) and the line from A to the tower as 25.6°. The observer at B similarly measures the corresponding angle to be 28.3° as shown in the diagram.



Can they use this information to calculate the width of the river?

The problem above involves finding lengths and angles in triangles. Such problems can be solved using trigonometric functions. In fact, trigonometry was first used to solve similar problems in land measurement, building and astronomy. The word *trigonometry* means ‘measuring triangles’.

In this chapter we will use what we already know about trigonometric functions and also develop some new results to enable us to calculate lengths and angles in triangles.

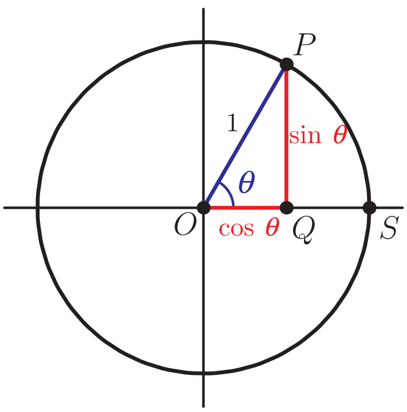
11A Right-angled triangles

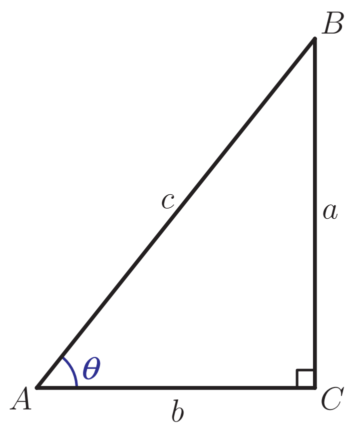
In previous studies, you may have already seen definitions of sine, cosine and tangent functions in terms of the sides of a right-angled triangle. In this section we will briefly discuss how they are related to the definitions we introduced in the previous chapter.

Remember the definition of sine and cosine functions using the unit circle. Let $0° < \theta < 90°$, and let P be the point on the unit circle such that SOP $= \theta$. Then $PQ = \sin\,\theta$ and $OQ = \cos\,\theta$.

Trigonometric functions were defined in Section 9B.

Consider now a right-angled triangle ABC with right angle at C and $B\hat{A}C = \theta$.





Triangles OPQ and ABC have the same angles, so they are similar. Therefore:

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = \frac{c}{1}$$

From this we find that the ratios of sides in a right-angled triangle are trigonometric functions of angle θ.

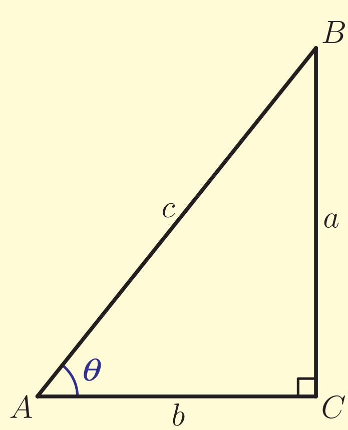
KEY POINT 11.1

In a right-angled triangle:

$$\frac{a}{c} = \sin\theta$$

$$\frac{b}{c} = \cos\theta$$

$$\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

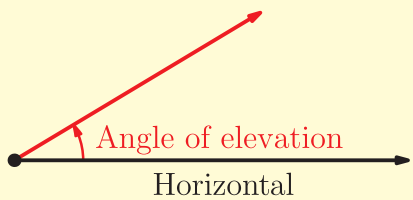


See Prior learning Section U on the CD-ROM for practice questions on right-angled triangles.

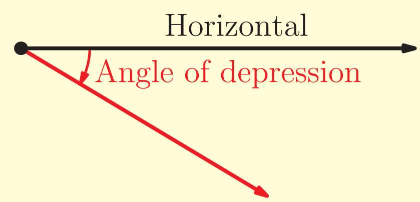
These equations only apply to right-angled triangles, so the angles are always acute. In the remainder of this chapter we shall deal with triangles which can also have obtuse angles. In those cases we need to use our definitions of trigonometric functions from the unit circle.

Two terms which you will see frequently in the context of trigonometry are **angle of elevation** and **angle of depression**.

KEY POINT 11.2



The angle of elevation is the angle above the horizontal.

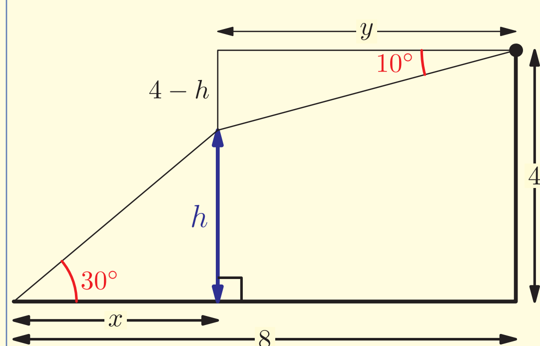


The angle of depression is the angle below the horizontal.

Worked example 11.1

Daniel and Theo are trying to find the height of a birds' nest in their garden. From Theo's bedroom window, which is 4 m above the ground, the angle of depression of the nest is 10°. From the end of the flat garden, 8 m away from the house, the angle of elevation is 30°. Find the height of the nest.

Sketch a diagram



EXAM HINT

If a diagram is not given it is always a good use of time to sketch one.

Apply trigonometry to the right-angled triangles

$$y = \frac{4-h}{\tan 10°}$$

$$x = \frac{h}{\tan 30°}$$

→

continued . . .

From the diagram we can find an equation linking x and y

$x + y = 8$

$\therefore \frac{4-h}{\tan 10°} + \frac{h}{\tan 30°} = 8$

Evaluate tan 10 and tan 30 on the calculator

$\Rightarrow \frac{4-h}{0.176} + \frac{h}{0.577} = 8$

$\Rightarrow 22.69 - 5.67h + 1.73h = 8$

$\Rightarrow 14.69 = 3.94h$

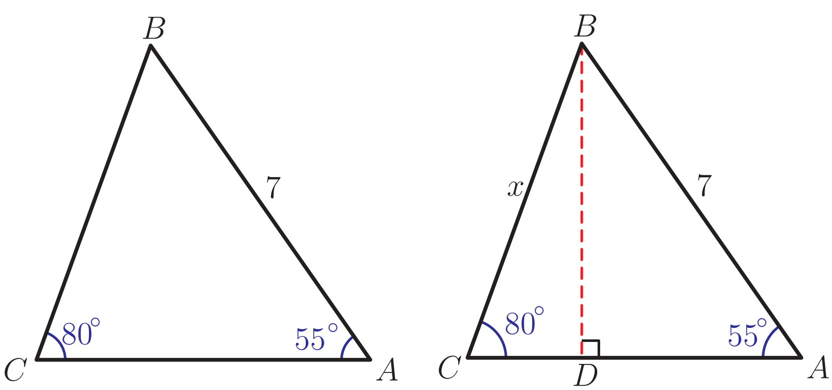
$\Rightarrow h = 3.73\text{m}\ (3\text{SF})$

EXAM HINT

In a question with several steps like this, it is important not to round intermediate values. Use the answer button on your calculator and round at the end. However, you do not have to write down all the digits on your calculator display.

11B The sine rule

Can we use trigonometry to calculate sides and angles in triangles that do not have a right angle? The answer is yes, and one way to do this is called the **sine rule**. In the triangle shown on the left below, $AB = 7$, $B\hat{A}C = 55°$ and $A\hat{C}B = 80°$. Can we find the length of BC?



There are no right angles in the diagram, but we can create some by drawing the line [BD] perpendicular to [AC] as shown in the second diagram.

We now have two right-angled triangles. In triangle ABD,

$$\frac{BD}{7} = \sin 55°, \text{ so } BD = 7\sin 55°.$$

In triangle BCD, $\frac{BD}{x} = \sin 80°$, so $BD = x\sin 80°$.

Comparing the two expressions for BD, we get:

$$x\sin 80° = 7\sin 55° \qquad (*)$$

And rearranging gives:

$$x = \frac{7\sin 55°}{\sin 80°} = 5.82$$

Notice that we did not actually need to calculate the length of BD but can go straight to equation marked (*). This equation is more commonly written as:

$$\frac{x}{\sin 55°} = \frac{7}{\sin 80°}$$

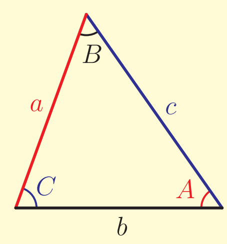
This last equation is an example of the sine rule. Notice that the length of each side is divided by the sine of the angle *opposite* that side. We can repeat the same process to obtain a general equation.

You may think that the sum of angles in a triangle being 180° is an absolute fact, but it is only the case on a flat plane. If you draw a triangle on a sphere, the sum of the angles will be greater than 180°. The study of these spherical triangles is vital to navigation and astronomy. There are even analogues of the cosine and sine rules for these triangles.

KEY POINT 11.3

The sine rule

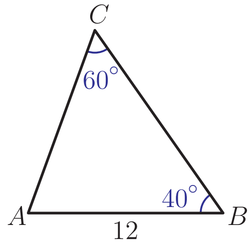
$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$



To use the sine rule you need to have both an angle and its opposite side. When using the sine rule you will normally use only two of the three ratios. To decide which ones, you need to look at what information is given in the question.

Worked example 11.2

Find the length of side AC.



continued . . .

We are given the angles opposite sides AB and AC, so use the sine rule with those two sides

$$\frac{12}{\sin 60°} = \frac{AC}{\sin 40°}$$

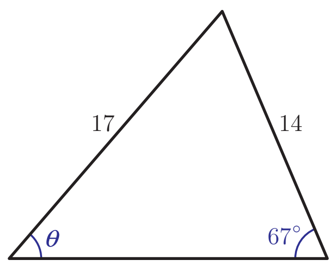
$$\Rightarrow AC = \frac{12\sin 40°}{\sin 60°}$$

$$= 8.91 \text{ (3SF)}$$

We can also use the sine rule to find angles.

Worked example 11.3

Find the size of the angle marked θ.



Use the sine rule with the two given sides, as we have one of the opposite angles and want to find the other one

$$\frac{17}{\sin 67°} = \frac{14}{\sin\theta}$$

$$\Rightarrow \sin\theta = \frac{14\sin 67°}{17} = 0.758$$

$$\therefore \theta = \arcsin 0.758 = 49.3°$$

You should remember from your work on trigonometric equations that there is more than one value of θ with $\sin\theta = 0.758$. So does that mean that the last question has more than one possible answer?

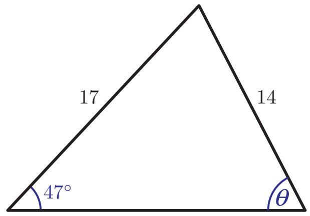
See Section 10A for solving trigonometric equations.

Another solution of the equation $\sin\theta = 0.758$ is $180 - 49.3 = 130.7°$. However, as one of the other angles is 67°, this value is impossible, because all three angles of the triangle must add up to 180°, and $130.7 + 67 = 190.7 > 180$. All other possible values of θ are outside the interval [0°, 180°], so cannot be angles of a triangle. In this example, there is only one possible value of angle θ.

The next example shows that this is not always the case.

Worked example 11.4

Find the size of the angle marked θ, giving your answer to the nearest degree.



Use the sine rule with the two given sides

$$\frac{17}{\sin\theta} = \frac{14}{\sin 47°}$$

$$\Rightarrow \sin\theta = \frac{17\sin 47°}{14} = 0.888$$

Find the two possible values of θ

$\arcsin 0.888 = 62.6°$

$\Rightarrow \theta = 62.6°$ or $180 - 62.6 = 117.4°$

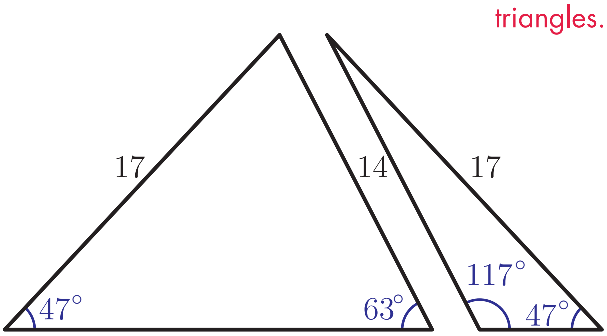
Check whether both solutions are possible: do the two angles add up to less than 180°?

$62.6 + 47 = 109.6 < 180$

$117.4 + 47 = 164.4 < 180$

Both solutions are possible. The diagram below shows the two possible triangles.

$\therefore \theta = 63°$ or $117°$



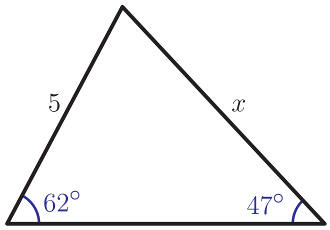
EXAM HINT

In an examination, a question will often alert you to look for two possible answers. However, if it doesn't, you should check whether the second solution is possible by finding the sum of the known angles.

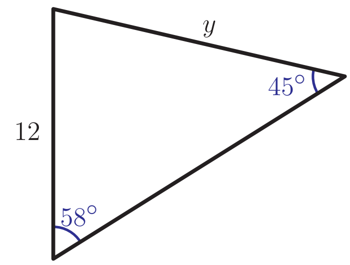
Exercise 11B

1. Find the lengths of sides marked with letters.

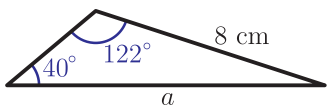
(a) (i)



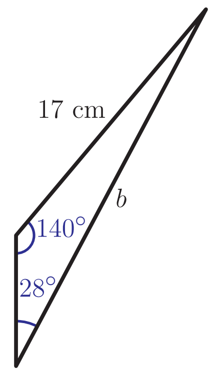
(ii)



(b) (i)

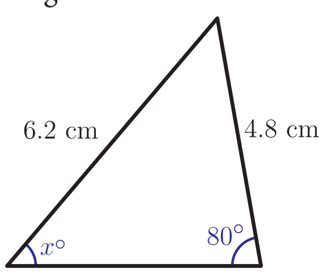


(ii)

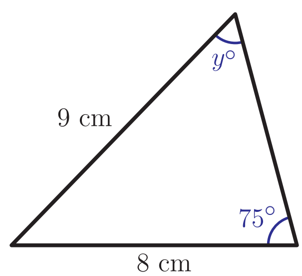


2. Find the angles marked with letters, checking whether there is more than one solution.

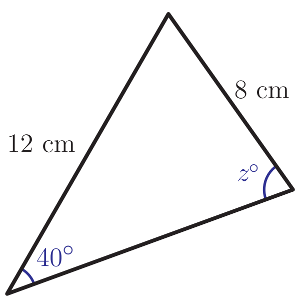
(a) (i)



(ii)



(b) (i)



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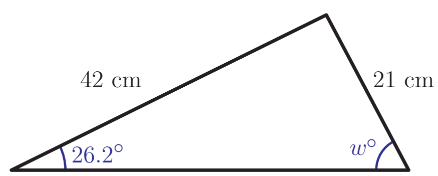
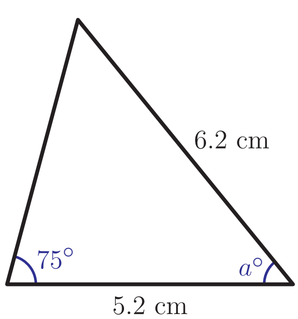
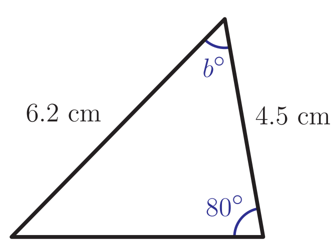


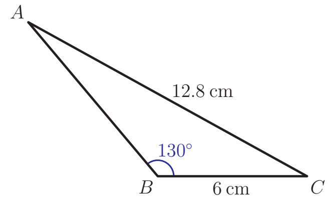
(c) (i)



(ii)

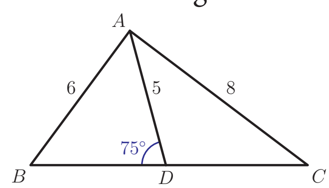


3. Find all the unknown sides and angles of triangle ABC.



4. In triangle ABC, $AB = 6$ cm, $BC = 8$ cm, $A\hat{C}B = 35°$. Show that there are two possible triangles with these measurements and find the remaining side and angles for each. *[4 marks]*

5. In the triangle shown in the diagram, $AB = 6$, $AC = 8$, $AD = 5$ and $A\hat{D}B = 75°$. Find the length of the side BC.



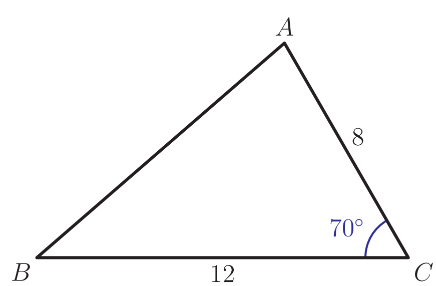
[5 marks]

6. Show that it is impossible to draw a triangle ABC with $AB = 12$ cm, $AC = 8$ cm and $A\hat{B}C = 47°$. *[5 marks]*

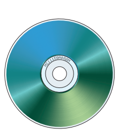
11C The cosine rule

Not all information about triangles can be found using the sine rule. In particular, if we have two sides and the angle between them or all three sides we can find further information using a new rule called the **cosine rule**.

Can we find the length of the side AB in the triangle shown?



The sine rule for this triangle gives $\frac{AB}{\sin 70°} = \frac{8}{\sin \hat{B}} = \frac{12}{\sin \hat{C}}$. But we do not know either of the angles B or C, so it is impossible to find AB from this equation. We need a different strategy.

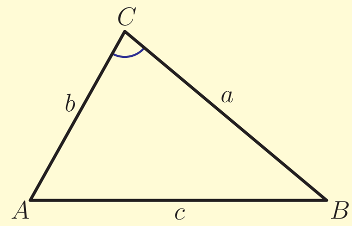


As before, by creating two right-angled triangles we can derive a formula for the length of AB (See Fill-in proof 8 'Cosine rule' for proof on the CD-ROM).

KEY POINT 11.4

The Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos\hat{c}$



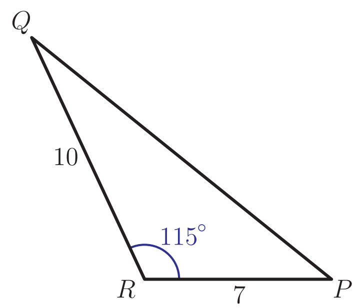
EXAM HINT

This is the form of the cosine rule given in the Formula booklet. However, you can change the names of the variables to anything you like as long as the angle corresponds to the side given on the left-hand side of the equation.

The cosine rule can be used to find the third side of the triangle when we know the other two sides and the angle between them.

Worked example 11.5

Find the length of the side PQ.



The question involves two sides and an angle, so use the cosine rule. The known angle is opposite PQ

$$PQ^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos 115°$$

$$\Rightarrow PQ^2 = 208.2$$

$$\Rightarrow PQ = \sqrt{208.2} = 14.4$$

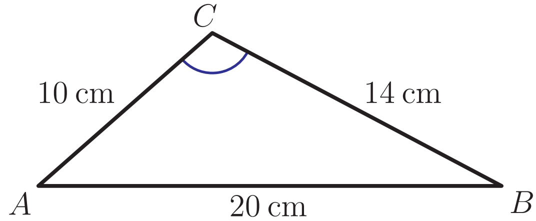
We can also use the cosine rule to find an angle if we know all three sides of a triangle. To help with this there is a rearrangement of the cosine rule:

KEY POINT 11.5

The cosine rule: $\cos\hat{C} = \dfrac{a^2 + b^2 - c^2}{2ab}$

Worked example 11.6

Find the size of the angle $A\hat{C}B$ correct to the nearest degree.



The question involves three sides and an angle, so use the cosine rule. Side AB is opposite the angle we want

$$\cos\hat{C} = \frac{196+100-400}{2\times14\times10} = -\frac{104}{280} = -0.371$$

Use inverse cosine to find the angle

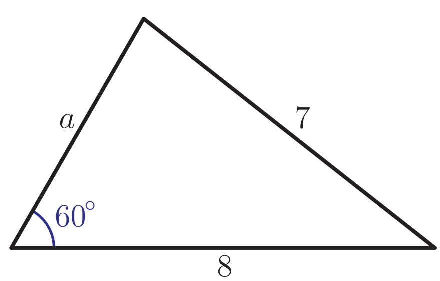
$$\hat{C} = \arccos(-0.371) = 112°$$

Notice that in the last two examples the angle is obtuse and its cosine is negative. Notice also that when using the cosine rule to find an angle, there is no second solution. This is because the second possibility is (360° – first solution) and so it will always be greater than 180°.

It is possible to use the cosine rule even when the given angle is not opposite the required side, as illustrated in the next example. This example also reminds you that there are some exact values of trigonometric functions you should know.

Worked example 11.7

Find the possible lengths of the side marked a.



continued . . .

The question involves three sides and an angle, so use the cosine rule. The known angle is opposite the side marked 7

$7^2 = a^2 + 8^2 - 2a \times 8\cos 60°$

Use the fact that $\cos 60° = \frac{1}{2}$

$\Rightarrow 49 = a^2 + 64 - 8a$

We recognise that this is a quadratic equation

$\Rightarrow a^2 - 8a + 15 = 0$

$\Rightarrow (a-3)(a-5) = 0$

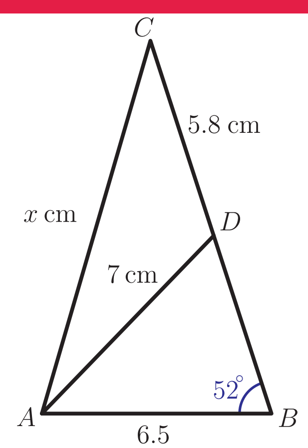
$\Rightarrow a = 3 \text{ or } 5$

It is also possible to answer this question using the sine rule twice, first to find the angle opposite the side marked 8, and then to find side a. Try to see if you can get the same answers.

The next example illustrates how to select which of the two rules to use. For both sine and cosine rule, we need to know three measurements in a triangle to find a fourth one.

Worked example 11.8

In the triangle shown in the diagram, $AB = 6.5$ cm, $AD = 7$ cm, $CD = 5.8$ cm, $A\hat{B}C = 52°$ and $AC = x$. Find the value of x correct to one decimal place.



The only triangle in which we know three measurements is ABD. We know two sides and a non-included angle (i.e. the angle does not lie between two given sides), so we can use the sine rule to find $A\hat{D}B$

Sine rule in triangle ABD; let $A\hat{D}B = \theta$:

$\frac{6.5}{\sin\theta} = \frac{7}{\sin 52^o}$

$\Rightarrow \sin\theta = \frac{6.5\sin 52^o}{7} = 0.7317$

$\arcsin 0.7317 = 47°$

continued . . .

Are there two possible solutions?

$180 - 47 = 133,\ 133 + 52 > 180$

There is only one solution.

$\therefore \theta = 47^o$ so, $A\hat{D}B = 47°$

In triangle ADC, we know two sides and want to find the third. We can find $A\hat{D}C$ and then use the cosine rule.

$A\hat{D}C = 180 - 47 = 133^o$

Cosine rule in triangle ADC:

$x^2 = 7^2 + 5.8^2 - 2 \times 7 \times 5.8 \cos 133^o$

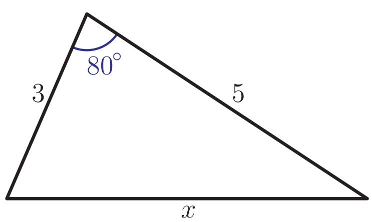
$\Rightarrow x^2 = 137.99$

$\Rightarrow x = \sqrt{137.99} = 11.7\text{cm}$

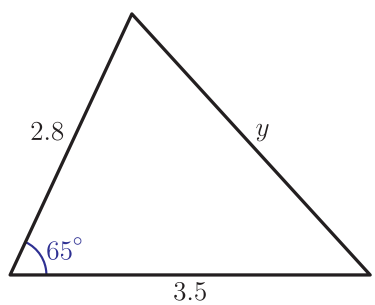
Exercise 11C

1. Find the lengths of the sides marked with letters.

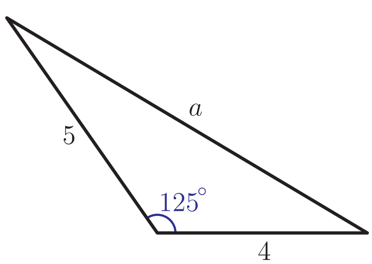
(a) (i)



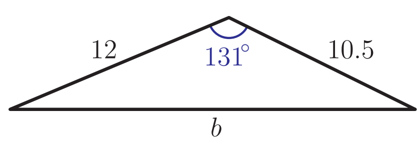
(ii)



(b) (i)

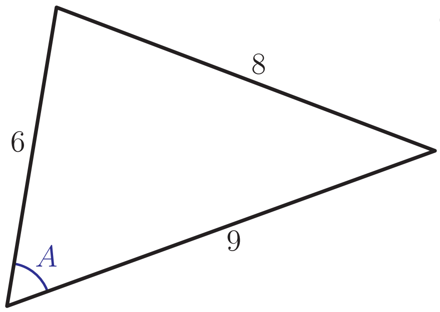


(ii)

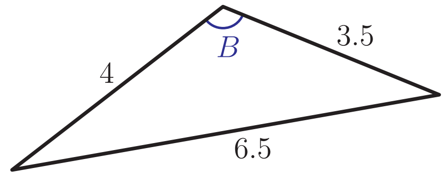


2. Find the angles marked with letters.

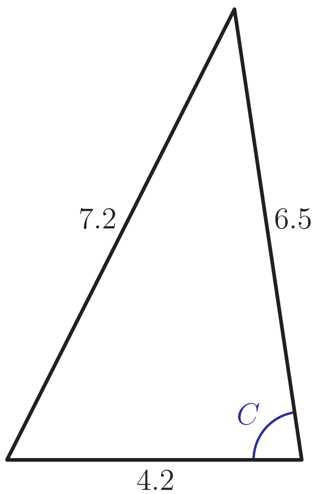
(a) (i)



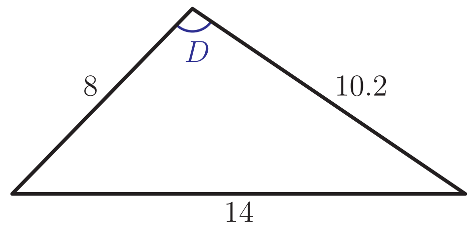
(ii)



(b) (i)



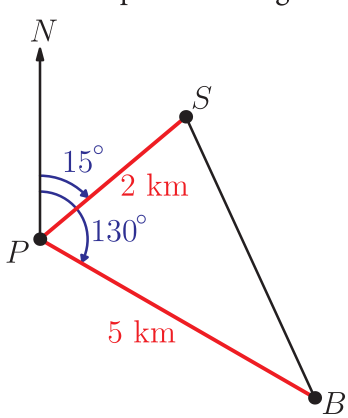
(ii)



3. (i) Triangle PQR has sides $PQ = 8$ cm, $QR = 12$ cm, $RP = 7$ cm. Find the size of the largest angle.

(ii) Triangle ABC has sides $AB = 4.5$ cm, $BC = 6.2$ cm, $CA = 3.7$ cm. Find the size of the smallest angle.

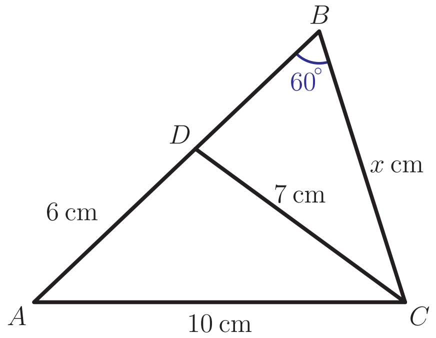
4. Ship S is 2 km from the port at an angle of 15° from North and boat B is 5 km from the port at an angle of 130° from North.



Find the distance between the ship and the boat. *[6 marks]*

Land surveying and astronomy were two areas which motivated the development of trigonometry. Supplementary sheet 4 'Coordinate systems and graphs' on the CD-ROM shows you some examples of how the sine and cosine rule can be applied in these areas.

5. Find the value of x in the diagram below.



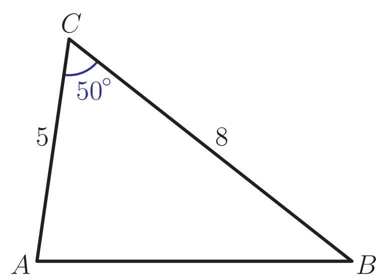
[6 marks]

6. In triangle ABC, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $B\hat{A}C = 60°$. Find the value of x. *[6 marks]*

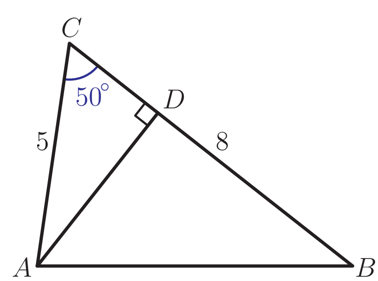
7. In triangle KLM, $KL = 4$, $LM = 7$ and $L\hat{K}M = 45°$. Find the exact length of KM. *[6 marks]*

11D Area of a triangle

Now that we know how to calculate sides and angles in a triangle, we can ask how to calculate the area. We know that the formula for the area of a triangle is $\frac{1}{2}$ × base × perpendicular height and we can use this to find the area of the triangle shown in the diagram:



We need to find a height of the triangle in order to calculate the area. For example, we can draw the line (AD) perpendicular to (BC), as in the second diagram:



Then $AD = 5\sin 50°$, so the area of the triangle is:

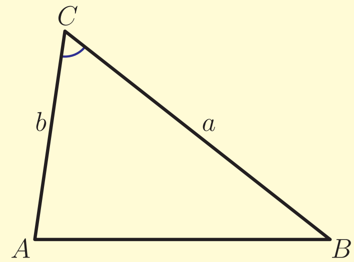
$$\frac{1}{2}BC \times AD = \frac{1}{2} \times 8 \times 5 \sin 50° = 15.2 \text{ cm}^2$$

This method can be applied to any triangle so the general formula for the area is:

KEY POINT 11.6

The area of the triangle is given by:

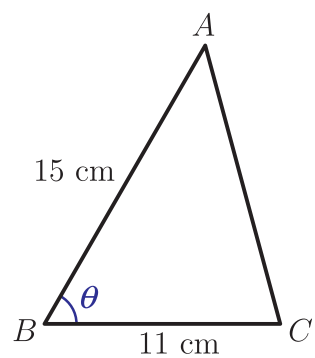
$$\text{area} = \frac{1}{2}ab\sin\hat{C}$$



There is also a formula for the area of a triangle if you know all three of its sides. It is called *Heron's formula*.

Worked example 11.9

The area of the triangle shown in the diagram is 52 cm². Find the two possible values of $A\hat{B}C$, correct to one decimal place.



Use the formula for the area of a triangle with known angle $\hat{B}$: area $= \frac{1}{2}ac\sin\hat{B}$

$$\frac{1}{2}(11 \times 15)\sin\theta = 52$$

$$\Rightarrow \sin\theta = \frac{2 \times 52}{11 \times 15} = 0.6303$$

$$\Rightarrow \arcsin 0.6303 = 39.07°$$

$$\therefore \theta = 39.1° \text{ or } 180 - 39.1 = 140.9°$$

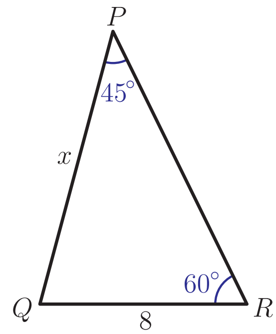
The next example combines the area of the triangle with the sine rule and working with exact values.

Worked example 11.10

For triangle PQR shown in the diagram:

(a) Calculate the exact value of x.

(b) Find the area of the triangle.



The question involves two angles and two sides, so we can use the sine rule

(a) $\dfrac{8}{\sin 45°} = \dfrac{x}{\sin 60°}$

We need the exact value of x, so use the exact values for $\sin 45°$ and $\sin 60°$

$\Rightarrow \dfrac{8}{\sqrt{2}/2} = \dfrac{x}{\sqrt{3}/2}$

$\Rightarrow \dfrac{16}{\sqrt{2}} = \dfrac{2x}{\sqrt{3}}$

$\Rightarrow x = \dfrac{16\sqrt{3}}{2\sqrt{2}}$

$\Rightarrow x = 4\sqrt{6}$

To use the formula for the area of the triangle, we need $P\hat{Q}R$

(b) $P\hat{Q}R = 180 - 60 - 45 = 75°$

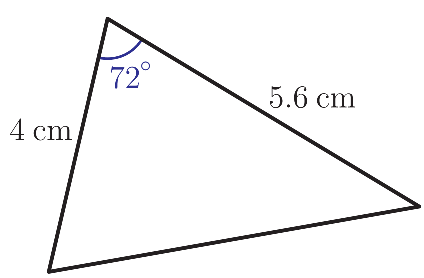
$\therefore \text{area} = \dfrac{1}{2}\left(8 \times 4\sqrt{6}\right)\sin 75°$

$= 37.9 \text{ (3SF)}$

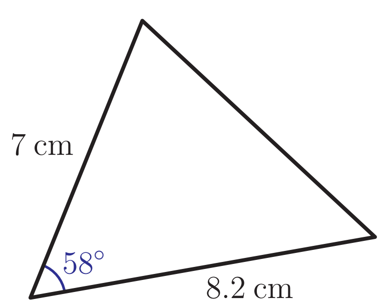
Exercise 11D

1. Calculate the areas of these triangles.

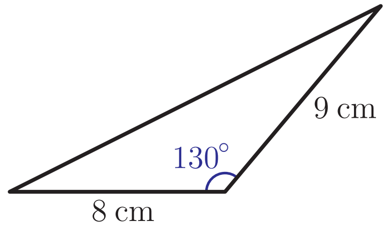
(a) (i)



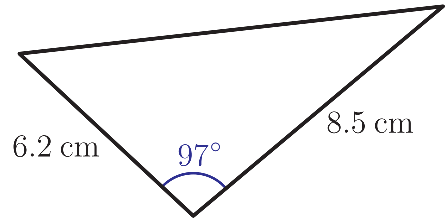
(ii)



(b) (i)

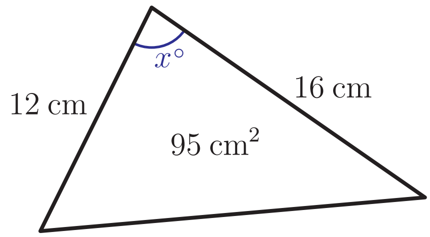


(ii)

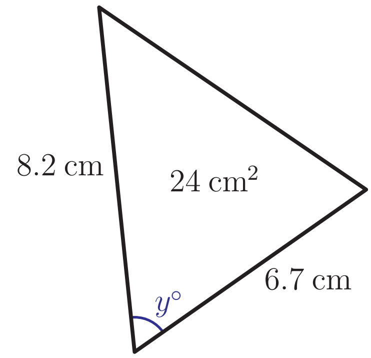


2. Each triangle has the area shown. Find two possible values of each marked angle.

(a)



(b)



3. In triangle LMN, $LM = 12$ cm, $MN = 7$ cm and $L\hat{M}N = 135°$.
Find the length of the side LN and the area of the triangle.

[6 marks]

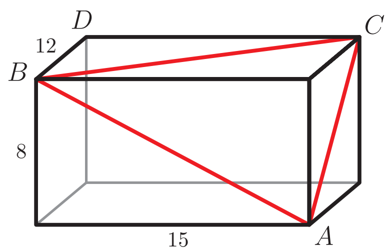
4. In triangle PQR, $PQ = 8$ cm, $RQ = 7$ cm and $R\hat{P}Q = 60°$.
Find the exact difference in areas between the two possible triangles. *[6 marks]*

11E Trigonometry in three dimensions

In many applications we work with three-dimensional objects. Examples in this section show you how to apply trigonometry in three dimensions. The general strategy is to identify a suitable triangle and then apply one of the rules from the previous sections.

Worked example 11.11

A cuboid has sides of length 8 cm, 12 cm and 15 cm. Diagonals of three of the faces are drawn as shown.



continued . . .

(a) Find the lengths of AB, BC and CA.

(b) Find the size of the angle $A\hat{C}B$.

(c) Calculate the area of the triangle ABC.

(d) Find the length AD.

AB is the diagonal of the front face, so it is a hypotenuse of a right-angled triangle with sides 8 and 12

(a) $AB^2 = 15^2 + 8^2 = 289$

$AB = \sqrt{298} = 17 \text{ cm}$

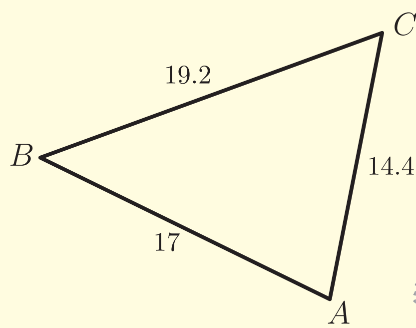
Find *BC* and *CA* in a similar way

$BC = \sqrt{12^2 + 15^2} = 19.2 \text{ cm}$

$CA = \sqrt{12^2 + 8^2} = 14.4 \text{ cm}$

We now look at triangle *ABC*. Draw the triangle if it helps

(b)



We know all three sides and want to find the angle, so we use the cosine rule

$$\cos \hat{C} = \frac{14.4^2 + 19.2^2 - 17^2}{2 \times 14.4 \times 19.2} = 0.519$$

$C = \arccos 0.519 = 58.7°$

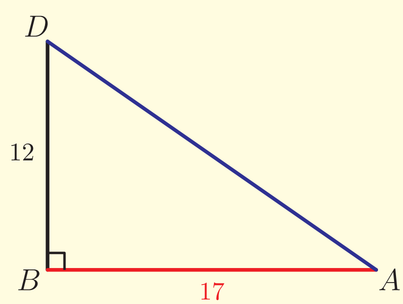
Use the formula for the area, with the angle we have just found

(c) $\text{area} = \frac{1}{2}(14.4 \times 19.2)\sin 58.7°$

$= 118 \text{ cm}^2$

ABD is a right-angled triangle

(d)



$AD^2 = 12^2 + 17^2$

$AD = \sqrt{433} = 20.8 \text{ cm}$

Part (d) illustrates a very useful fact about the diagonal of a cuboid.

KEY POINT 11.7

The diagonal of a $p \times q \times r$ cuboid has length $\sqrt{p^2 + q^2 + r^2}$.

In chapter 13 you will meet vectors which can also be used to solve three-dimensional problems.

The key to solving three-dimensional problems is spotting right angles. This is not always easy, as diagrams are drawn in perspective, but there are some common configurations to look for, such cross-sections and bisectors of isosceles triangles.

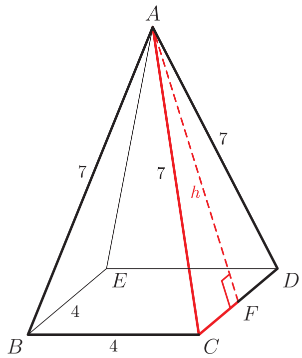
Worked example 11.12

The base of a pyramid is a square of side 4 cm. The other four faces are isosceles triangles with sides 7 cm. The height of one of the side faces is labelled h.

(a) Find the exact value of h.

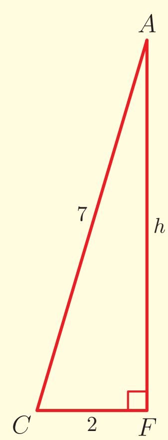
(b) Find the exact height of the pyramid.

(c) Calculate the volume of the pyramid, correct to 3 significant figures.



Triangle *AFC* is right angled. Draw it separately and label known sides

(a)



Use Pythagoras to find *h*

$h^2 = 7^2 - 2^2$

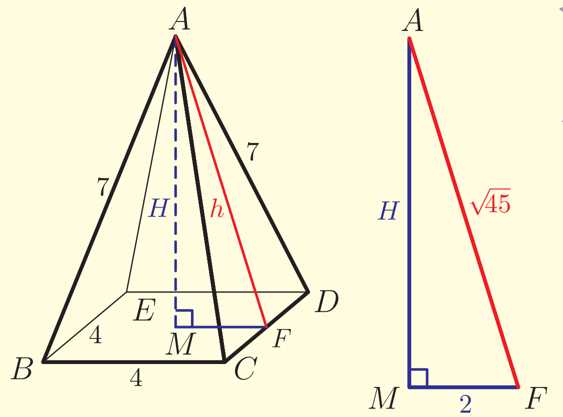
$h = \sqrt{45}$ cm

continued . . .

Add the height to the diagram. It is perpendicular to the base, so it makes a right-angle with MF. M is in the centre of the base, so $MF = 2$ cm

Look at triangle AMF

(b)



Use Pythagoras to find H

$$H^2 = \left(\sqrt{45}\right)^2 - 2^2$$

$$H = \sqrt{41}\text{ cm}$$

From the Formula booklet, the formula for the volume of a pyramid is $\frac{1}{3}$(area of the base × height)

(c) $V = \frac{1}{3}(4^2 \times \sqrt{41})$

$V = 34.1\text{ cm}^3$

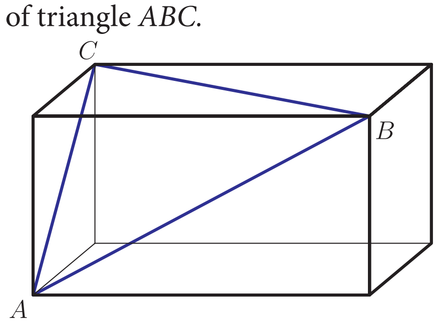
See Prior learning Section V on the CD-ROM for volumes of three-dimensional shapes.

Exercise 11E

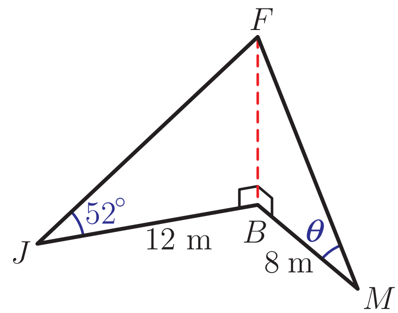
1. Find the length of the diagonal of the following cuboids:

   (a) 3 cm × 5 cm × 10 cm (b) 4 cm × 4 cm × 8 cm

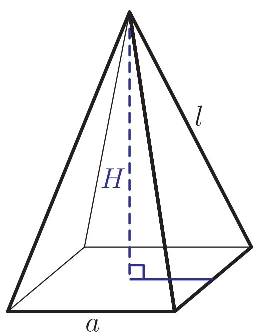
2. A cuboid has sides 12.5 cm, 10 cm and 7.3 cm. It is intersected by a plane passing through vertices A, B and C. Find the angles and the area of triangle ABC. *[8 marks]*



3. Johann stands 12 m from the base of a flagpole and sees the top at the angle of elevation of 52°. Marit stands 8 m from the flagpole. At what angle of elevation does she see the top? *[6 marks]*



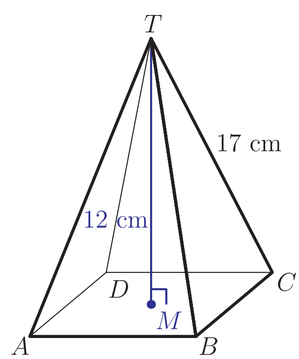
4. A square-based pyramid has a base of side $a = 8$ cm and a height $H = 12$ cm. Find the length of the sloping side, l. *[6 marks]*



5. The base of a pyramid $TABCD$ is a square. The height of the pyramid $TM = 12$ cm and the length of a sloping edge $TC = 17$ cm.

(a) Calculate the length of MC.

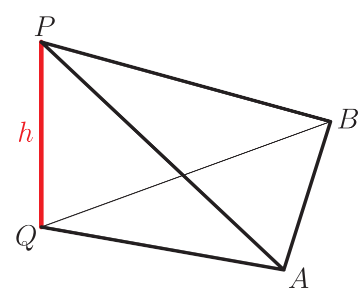
(b) Find the length of a side of the base. *[6 marks]*



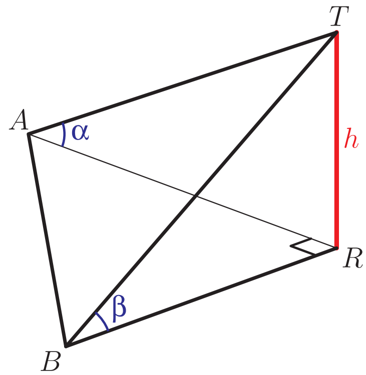
6. The diagram shows a vertical tree PQ and two observers at A and B, standing on horizontal ground. $AQ = 25$ m, $Q\hat{A}P = 37°$, $Q\hat{B}P = 42°$ and $A\hat{Q}B = 75°$.

(a) Find the height of the tree, h.

(b) Find the distance between the two observers. *[8 marks]*



7. Annabelle and Berta are trying to measure the height, h, of a vertical tree RT. They stand on horizontal ground, a distance d apart so that ARB is a right-angle. From where Annabelle is standing, the angle of elevation of the top of the tree is α, and from where Berta stands the angle of elevation of the top of the tree is β.

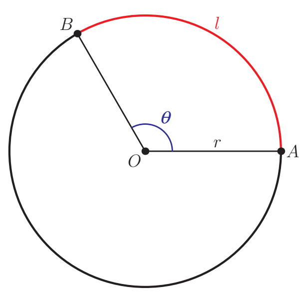


(a) Find expressions for RA and RB in terms of h, α and β, and hence show that:

$$h^2\left(\frac{1}{\tan^2\alpha}+\frac{1}{\tan^2\beta}\right)=d^2$$

(b) Given that $d = 26\,\text{m}$, $\alpha = 45°$ and $\beta = 30°$, find the height of the tree. *[8 marks]*

11F Length of an arc



The diagram shows a circle with centre O and radius r, and points A and B on its circumference. The part of the circumference between points A and B is called an **arc** of a circle. You can see that there are in fact two such parts; the shorter one is called the **minor arc**, and the longer one the **major arc**. We say that the minor arc AB **subtends** angle θ at the centre of the circle.

You know that a measure of angle θ in radians is equal to the ratio of the length of the arc AB, l, to the circumference of the circle; in other words, $\theta = \frac{l}{r}$.

Radian measure was introduced in Section 9A.

This gives us a very simple formula to calculate the length of an arc of a circle if we know the angle it subtends at the centre.

KEY POINT 11.8

Length of an arc of a circle

$l = \theta r$

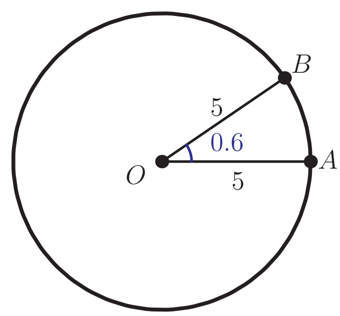
where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 11.13

Arc AB of a circle with radius 5 cm subtends an angle of 0.6 at the centre, as shown on the diagram.

(a) Find the length of the minor arc AB.

(b) Find the length of the major arc AB.



We know the formula for the length of an arc

(a) $l = r\theta$

$l = 5 \times 0.6$

$= 3\text{ cm}$

The angle subtended by the major arc is equal to a full turn minus the smaller angle. A full turn is 2π radians

(b) $\theta_1 = 2\pi - 0.6$

$= 5.683$

$l = r\theta_1 = 5 \times 5.683$

$= 28.4\text{ cm (3SF)}$

We could also have done the second part differently, by finding the length of the whole circumference and then taking away the minor arc: circle circumference is $2\pi r = 2\pi \times 5 = 31.42$, so the length of the major arc is $31.42 - 3 = 28.4\text{ cm (3 SF)}$.

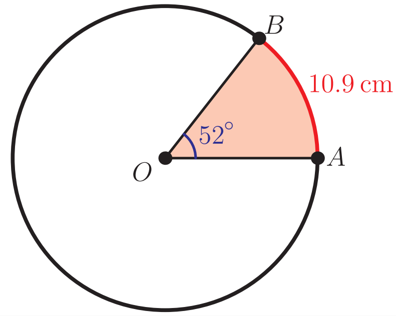
If the angle is given in degrees, we need to convert to radians before using the formula for the arc length.

Worked example 11.14

Two points, A and B, lie on the circumference of a circle of radius r cm. The arc AB has length 10.9 cm and subtends an angle of 52° at the centre of the circle.

(a) Find the value of r.

(b) Calculate the perimeter of the shaded region.



We know the arc length and the angle, so we can find the radius

(a) $l = r\theta \Rightarrow r = \frac{l}{\theta}$

To use the formula $l = r\theta$, the angle must be in radians

$\theta = 52 \times \frac{\pi}{180} = 0.908$

$\therefore r = \frac{10.9}{0.908} = 12.0\text{ cm}$

The perimeter is made up of two radii and the arc

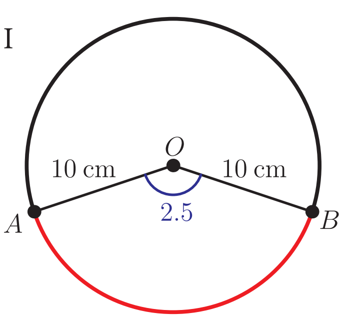
(b) $p = 2r + l$

$= 2 \times 12.0 + 10.9$

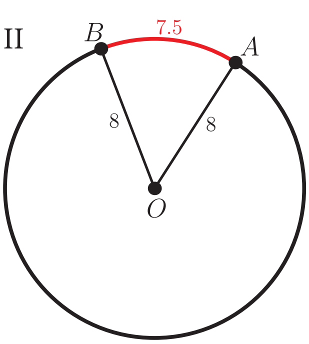
$= 34.9\text{ cm}$

Exercise 11F

I



II



1. Calculate the length of the minor arc subtending an angle of θ radians at the centre of a circle of radius r cm.

(a) $\theta = 1.2,\quad r = 6.5$ (b) $\theta = 0.4,\quad r = 4.5$

2. Points A and B lie on the circumference of a circle with centre O and radius r cm.

Angle $A\hat{O}B$ is θ radians. Calculate the length of the major arc AB.

(a) $r = 15,\quad \theta = 0.8$ (b) $r = 1.4,\quad \theta = 1.4$

3. Calculate the length of the minor arc AB in the upper diagram (I). *[4 marks]*

4. In the lower diagram (II), the radius of the circle is 8 cm and the length of the minor arc AB is 7.5 cm. Calculate the size of the angle $A\hat{O}B$:

(a) in radians

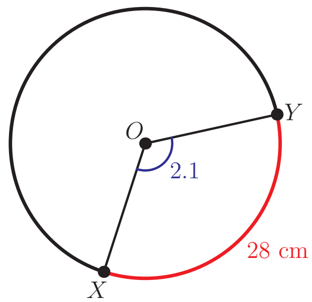
(b) in degrees. *[5 marks]*

5. Points M and N lie on the circumference of a circle with centre C and radius 4 cm. The length of the *major* arc MN is 15 cm. Calculate the size of the *smaller* angle $M\hat{C}N$. *[4 marks]*

6. Points P and Q lie on the circumference of a circle with centre O. The length of the minor arc PQ is 12 cm and $P\hat{O}Q = 1.6$. Find the radius of the circle. *[4 marks]*

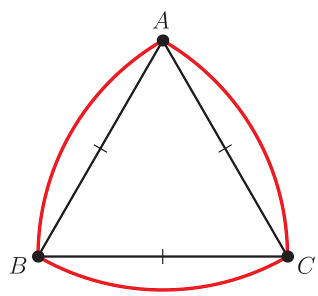
7. In the diagram, the length of the major arc XY is 28 cm. Find the radius of the circle.

[4 marks]



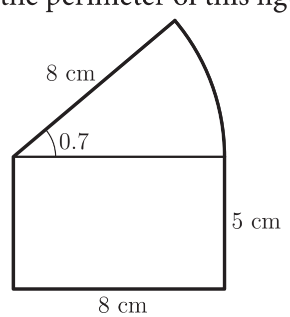
8. The figure below shows an equilateral triangle ABC with side $a = 5$ cm, and three arcs of circles with centres at the vertices of the triangle. Calculate the perimeter of the figure.

[5 marks]



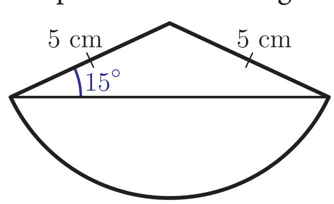
9. Calculate the perimeter of this figure:

[6 marks]



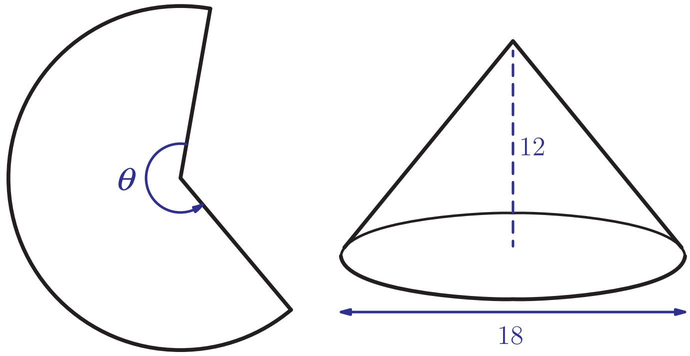
10. Find the exact perimeter of the figure shown:

[6 marks]



11. A sector of a circle has perimeter $p = 12$ cm and angle at the centre $\theta = 0.4$. Find the radius of the circle. *[5 marks]*

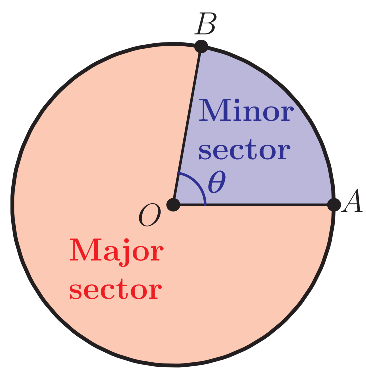
12. A cone is made by rolling a piece of paper as shown in the diagram.



If the cone is to have height 12 cm and base diameter 18 cm, find the size of the angle marked θ. *[6 marks]*

11G Area of a sector

A sector is a part of a circle bounded by two radii. As with arcs, we distinguish between a **minor sector** and a **major sector**.



The ratio of the area of the sector to the area of the whole circle is the same as the ratio of angle θ to a full turn. If A is the area of the sector, and angle θ is measured in radians, this means that $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$. Rearranging this equation gives the formula for the area of the sector.

KEY POINT 11.9

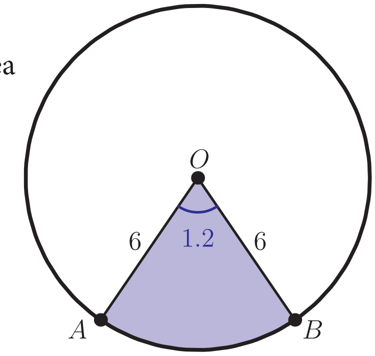
The area of a sector of a circle

$A = \frac{1}{2}r^2\theta$,

where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 11.15

The diagram shows a circle with centre O and radius 6 cm, and two points on its circumference such that $A\hat{O}B = 1.2$. Find the area of the minor sector AOB.



Formula for the area of a sector

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2} \times 6^2 \times 1.2 = 21.6 \text{ cm}^2$$

We may also have to use the formulae for area and arc length in reverse.

Worked example 11.16

A sector of a circle has perimeter $p = 12$ cm and angle at the centre $\theta = 50°$.

Find the area of the sector.

If we are going to use the formula for the area of a sector, $A = \frac{1}{2}r^2\theta$, we first need to find r

We are given the perimeter, and we know that it consists of arc length $l = r\theta$ and two radii

$$p = r\theta + 2r$$

The angle needs to be in radians

$$\theta = 50 \times \frac{\pi}{180} = 0.873$$

Substitute all the values into the formula for perimeter to find r

$$12 = 0.873r + 2r = 2.873r$$

$$r = \frac{12}{2.873} = 4.177$$

Substitute θ and r into the formula for sector area

$$A = \frac{1}{2}(4.177)^2(0.873)$$

$$A = 7.62 \text{ cm}^2 \text{ (3SF)}$$

Exercise 11G

1. Points M and N lie on the circumference of a circle, centre O and radius r cm, with $M\hat{O}N = \alpha$. Calculate the area of the minor sector MON.

   (a) $r = 5, \alpha = 1.3$ (b) $r = 0.4, \alpha = 0.9$

2. Points A and B lie on the circumference of a circle with centre C and radius r cm. The size of the angle ACB is θ radians. Calculate the area of the major sector $A\hat{C}B$.

   (a) $r = 13, \quad \theta = 0.8$ (b) $r = 1.4, \quad \theta = 1.4$

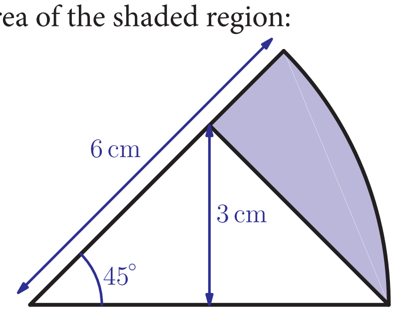
3. A circle has centre O and radius 10 cm. Points A and B lie on the circumference of the circle so that the area of the minor sector AOB is 40 cm^2. Calculate the size of the smaller angle $A\hat{O}B$. *[5 marks]*

4. Points P and Q lie on the circumference of a circle with radius 21 cm. The area of the *major* sector POQ is 744 cm^2. Find the size of the *smaller* angle $P\hat{O}Q$ in *degrees*. *[5 marks]*

5. A sector of a circle with angle 1.2 radians has area 54 cm^2. Find the radius of the circle. *[4 marks]*

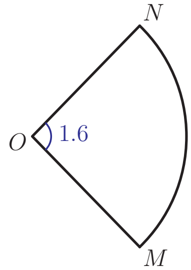
6. A sector of a circle with angle 162° has area 180 cm^2. Find the radius of the circle. *[4 marks]*

7. Find the area of the shaded region:



[6 marks]

8. The perimeter of the sector shown in the diagram is 28 cm. Find its area.



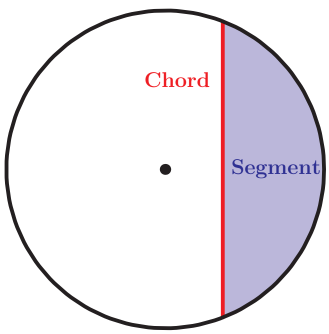
[5 marks]

9. A sector of a circle has perimeter 7 cm and area 3 cm^2. Find the possible values of the radius of the circle. *[6 marks]*

10. Points P and Q lie on the circumference of the circle with centre O and radius 5 cm. The difference between the areas of the major sector POQ and the minor sector POQ is 15 cm^2. Find the size of the angle $P\hat{O}Q$. *[5 marks]*

11H Triangles and circles

You already know how to calculate lengths of arcs and areas of sectors of circles. In this section we will look at two other important parts of circles: **chords** and **segments.**

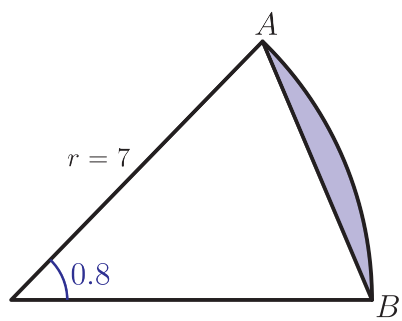


Worked example 11.17

The diagram shows a sector of a circle of radius 7 cm and the angle at the centre 0.8 radians. Find:

(a) the perimeter of the shaded region

(b) the area of the shaded region



The perimeter is made up of the arc AB and the chord $[AB]$

(a) $p = \text{arc} + \text{chord}$

The formula for the length of the arc is $l = r\theta$

$l = 7 \times 0.8 = 5.6$ cm

The chord $[AB]$ is the third side of the triangle ABC. As we know the other two sides and the angle between them, we can use the cosine rule. Remember that the angle is in radians

Cosine rule:

$AB^2 = 7^2 + 7^2 - 2 \times 7 \times 7 \cos 0.8$

$AB^2 = 29.7$

$AB = \sqrt{29.7} = 5.45$ cm

continued . . .

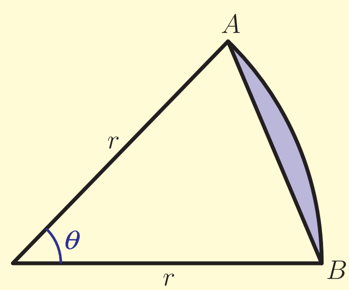
We can now find the perimeter	$\therefore p = 5.6 + 5.45 = 11.1\text{ cm}$
We know how to calculate the area of a sector. If we subtract the area of triangle ABC, we are left with the area of the segment	(b) $\text{area} = \text{sector} - \text{triangle}$
The formula for the area of a sector is $\frac{1}{2}r^2\theta$	$\text{sector} = \frac{1}{2}(7^2 \times 0.8) = 19.6\text{ cm}^2$
The formula for the area of a triangle is $\frac{1}{2}ab\sin C$	$\text{triangle} = \frac{1}{2}(7 \times 7)\sin 0.8 = 17.58\text{ cm}^2$
We can now find the area of the segment	$\therefore \text{area} = 19.6 - 17.6 = 2.02\text{ cm}^2$

We can follow the same method to derive general formulae for the length of a chord and area of a segment.

EXAM HINT

These formulae are not given in the Formula booklet, so you need to know how to derive them.

KEY POINT 11.10



The length of a chord of a circle is given by:

$$AB^2 = 2r^2(1 - \cos\theta)$$

and **the area of the shaded segment** is

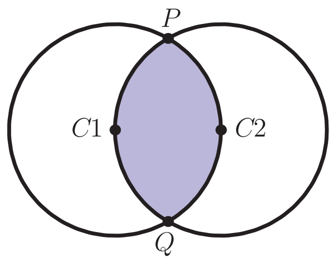
$$\frac{1}{2}r^2(\theta - \sin\theta),$$

where angle θ is measured in radians.

The next example shows how we can solve more complex geometry problems by splitting up the figure into basic shapes such as triangles and sectors.

Worked example 11.18

The diagram shows two equal circles of radius 12 such that the centre of one circle is on the circumference of the other.

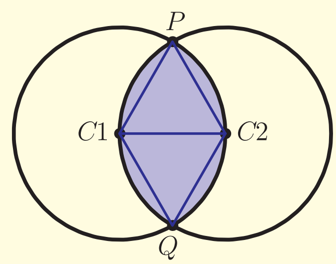


(a) Find the exact size of angle $P\hat{C}_1Q$ in radians.

(b) Calculate the exact area of the shaded region.

The only thing we know is the radius of the circle, so draw all the lengths which are equal to the radius

(a)



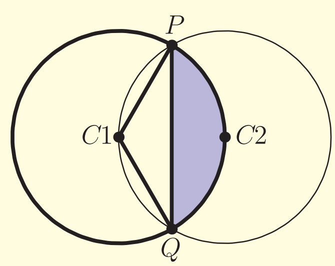
The lengths C_1P, C_2P and C_1C_2 are all equal to the radius of the circle. Therefore triangle PC_1C_2 is equilateral

$P\hat{C}_1C_2 = \frac{\pi}{3}$

$\therefore P\hat{C}_1Q = \frac{2\pi}{3}$

The shaded area is made up of two segments, each with angle at the centre $\frac{2\pi}{3}$

(b)



We can find area of one segment using the formula. Remember to use the exact value of $\sin\left(\frac{2\pi}{3}\right)$

$$\text{area of one segment} = \frac{1}{2}(12^2)\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= 72\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 48\pi - 36\sqrt{3}$$

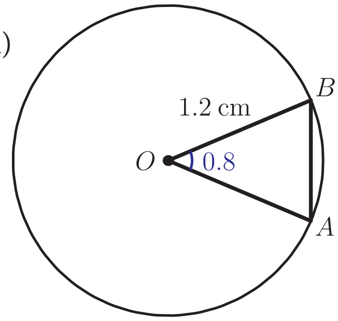
The shaded area consists of two segments

$$\therefore \quad \text{shaded area} = 96\pi - 72\sqrt{3}$$

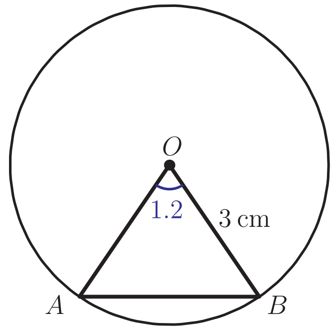
Exercise 11H

1. Find the length of the chord AB:

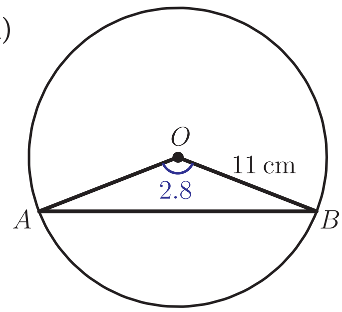
(a) (i)



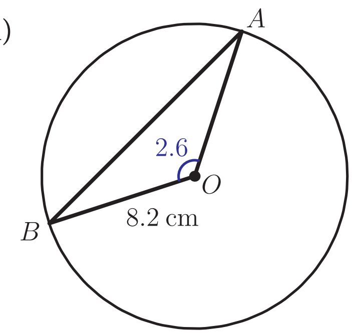
(ii)



(b) (i)



(ii)



2. Find the perimeters of the minor segments from Question 1.

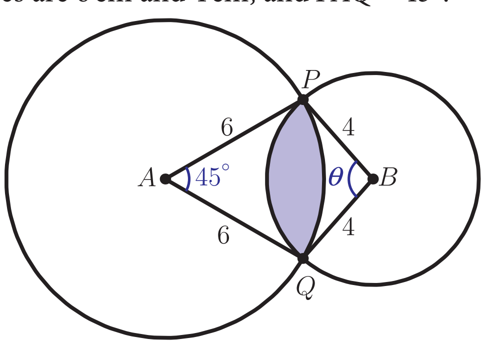
3. Find the areas of minor segments from Question 1.

4. A circle has centre O and radius 5 cm. Chord PQ subtends angle θ at the centre of the circle.

   (a) Write down an expression for the area of the minor segment.

   (b) Given that the area of the minor segment is 15 cm^2, find the value of θ. *[6 marks]*

5. Two circles, with centres A and B, intersect at P and Q. The radii of the circles are 6 cm and 4 cm, and $P\hat{A}Q = 45°$.



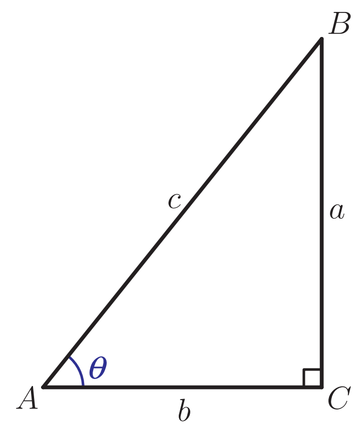
   (a) Show that $PQ = 6\sqrt{2-\sqrt{2}}$.

   (b) Find the size of $P\hat{B}Q$.

   (c) Find the area of the shaded region. *[9 marks]*

Summary

- In a right-angled triangle: $\frac{a}{c} = \sin\theta, \frac{b}{c} = \cos\theta, \frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

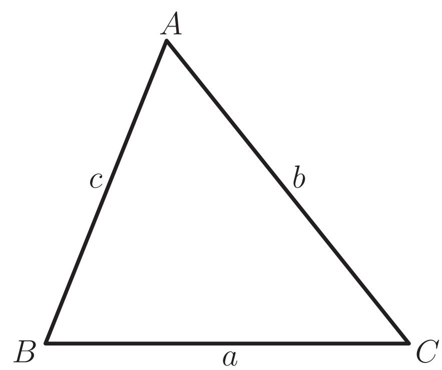


- The **angle of elevation** is the angle above the horizontal.
- The **angle of depression** is the angle below the horizontal.
- To find a side when two angles and a side are given, or an angle when two sides and a non-included angle are given, we can use the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- To find a side when two sides and an angle are given, or an angle when all three sides are given, we can use the cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



- An alternative formula for the area of a triangle:

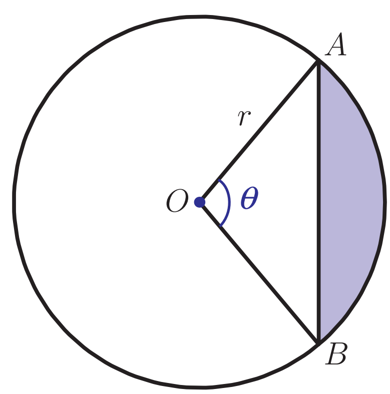
$$\text{area} = \frac{1}{2}ab\sin C$$

- To solve problems in three dimensions, find a suitable triangle and apply one of the above rules. Look out for right angles.
- The diagonal of a $p \times q \times r$ cuboid has length $\sqrt{p^2 + q^2 + r^2}$.
- In a circle of radius r and an angle θ in radians subtended at the centre:
  - Arc length $l = r\theta$
  - Area of sector $A = \frac{1}{2}r^2\theta$
- You need to know how to derive the formulae for the length of a chord and the area of a segment:

$$AB^2 = 2r^2(1 - \cos\theta)$$

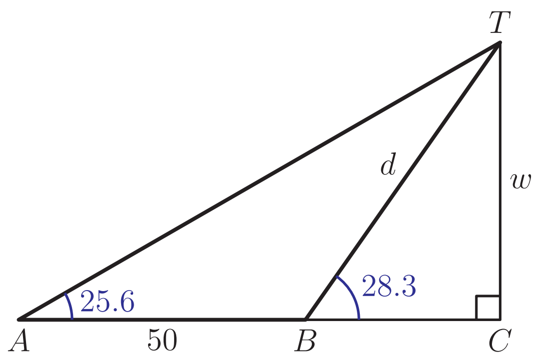
$$\text{area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

where θ is measured in radians.



Introductory problem revisited

Two observers are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. They stand at A and B, 50 m apart, on the opposite side of the river to a tower. The person at A measures that the angle between the line (AB) and the line from A to the tower as 25.6°.
The observer at B similarly measures the corresponding angle to be 28.3° as shown in the diagram.
Can they use this information to calculate the width of the river?



We want to find the width w. There are two right-angled triangles, ACT and BCT. We do not know the lengths of any of their sides, but in triangle ABT we know two angles and one side, so we can calculate the remaining sides. In particular, knowing the length of BT would allow us to find w from triangle BCT.

In order to use the sine rule in triangle ABT, we need to know the size of $A\hat{T}B$. Since $A\hat{B}T = 180° - 28.3° = 151.7°$, we can find that $A\hat{T}B = 2.7°$.

We can now use the sine rule:

$$\frac{d}{\sin 25.6°} = \frac{50}{\sin 2.7°}$$

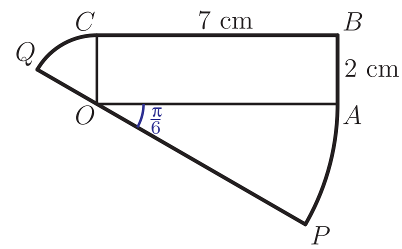
$$d = \frac{50 \sin 25.6°}{\sin 2.7°} = 458.6\,\text{m}$$

Finally, we can use the right-angled triangle BCT to find the width of the river:

$$w = d \sin 28.3° = 217\,\text{m}\,(3\text{SF})$$

Mixed examination practice 11

Short questions



1. In the diagram, $OABC$ is a rectangle with sides 7 cm and 2 cm. PQ is a straight line. AP and CQ are circular arcs, and $A\hat{O}P = \frac{\pi}{6}$.

(a) Write down the size of $C\hat{O}Q$.

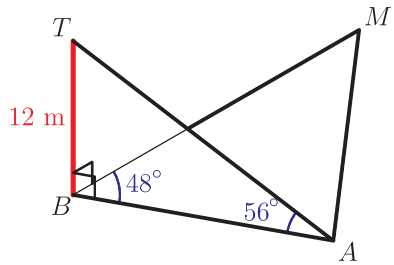
(b) Find the area of the whole shape.

(c) Find the perimeter of the whole shape. *[9 marks]*

2. A sector has perimeter 36 cm and radius 10 cm. Find its area. *[6 marks]*

3. In triangle ABC, $AB = 6.2$ cm, $CA = 8.7$ cm and $A\hat{C}B = 37.5°$. Find the two possible values of $A\hat{B}C$. *[6 marks]*

4. A vertical tree of height 12 m stands on horizontal ground. The bottom of the tree is at the point B. Observer A, standing on the ground, sees the top of the tree, T, at an angle of elevation of 56°.

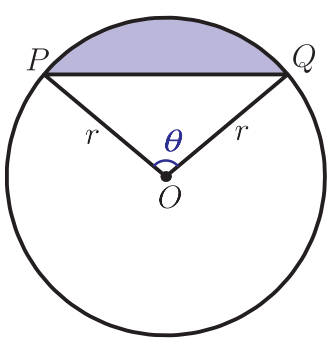


(a) Find the distance of A from the bottom of the tree.

Another observer, M, stands the same distance away from the tree, and $A\hat{B}M = 48°$.

(b) Find the distance AM. *[6 marks]*

5. The diagram shows a circle with centre O and radius $r = 7$ cm. The chord PQ subtends angle $\theta = 1.4$ radians at the centre of the circle.



Find:

(a) the area of the shaded region

(b) the perimeter of the shaded region. *[6 marks]*

6. In triangle ABC, $AB = 2\sqrt{3}$, $AC = 10$ and $B\hat{A}C = 150°$. Find the exact length of BC. *[6 marks]*

7. The perimeter of the sector shown in the diagram is 34 cm and its area is 52 cm^2. Find the radius of the sector. *[6 marks]*

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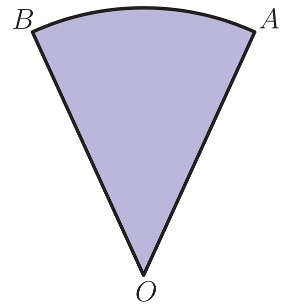
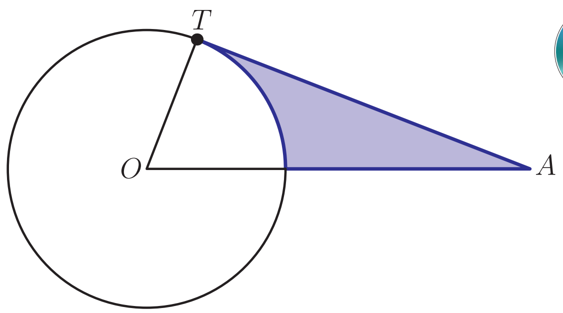


8. In the diagram, O is the centre of the circle and AT is the tangent to the circle at T.



Properties of circles and basic trigonometry are covered in the Prior learning Section W of the CD-ROM.

If $OA = 12$ cm, and the circle has a radius of 6 cm, find the area of the shaded region. *[4 marks]*

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9. The diagram shows a triangle and two arcs of circles.

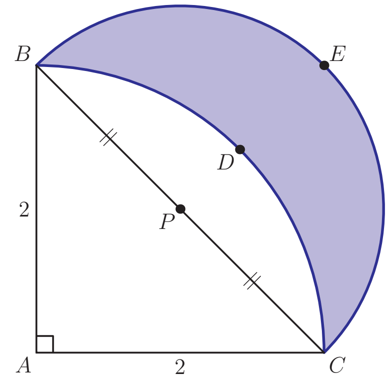
The triangle ABC is a right-angled isosceles triangle, with $AB = AC = 2$. The point P is the midpoint of BC.

The arc BDC is part of a circle with centre A.

The arc BEC is part of a circle with centre P.

(a) Calculate the area of the segment $BDCP$.

(b) Calculate the area of the shaded region $BECD$.



[6 marks]

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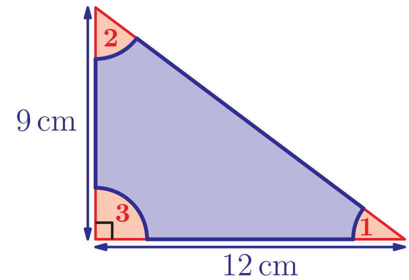
10. A right-angled triangle has sides 12 cm and 9 cm. At each vertex, a sector of radius 2 cm is cut out, as shown in the diagram. The angle at sector 1 is θ.



(a) Write down an expression for the area of sector 2 in terms of θ.

(b) Find the remaining area. *[6 marks]*

11. In the obtuse-angled triangle KLM, $LM = 6.1$ cm, $KM = 4.2$ cm and $K\hat{L}M = 42°$.

Find the area of the triangle. *[6 marks]*

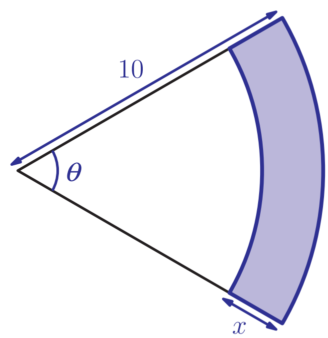
12. In triangle ABC, $AB = 10$ cm, $BC = 8$ cm and $CA = 7$ cm.

(a) Find the exact value of $\cos(A\hat{B}C)$.

(b) Find the exact value of $\sin(A\hat{B}C)$. *[8 marks]*

(c) Find the exact value of the area of the triangle.

13. The diagram shows two circular sectors with angle θ at the centre. The radius of the larger sector is 10 cm, the radius of the smaller sector is x cm smaller.



(a) Show that the area of the shaded region is given by $\frac{x(20-x)\theta}{2}$.

(b) If $\theta = 1.2$ find the value of x such that the area of the shaded region is equal to 54.6 cm^2. *[8 marks]*

Long questions

1. In triangle ABC, $AB = 5$, $AC = x$ and $\hat{A} = \theta$. M is the midpoint of the side AC.

(a) Use the cosine rule to find an expression for MB^2 in terms of x and θ.

(b) Given that $BC = MB$, show that cos, $= \frac{3x}{20}$.

(c) If $x = 5$, find the value of the angle θ such that $MB = BC$. *[9 marks]*

2. Two circles have equal radius r and intersect at points S and T. The centres of the circles are A and B, and $A\hat{S}B = 90°$.

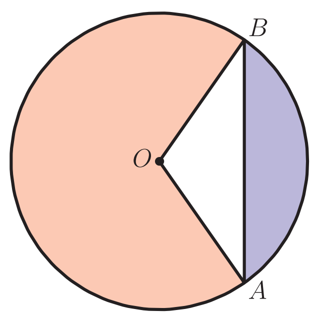
(a) Explain why $S\hat{A}T$ is also $90°$.

(b) Find the length AB in terms of r.

(c) Find the area of the sector AST.

(d) Find the area of the overlap of the two circles. *[10 marks]*

3. The diagram shows a circle with centre O and radius r. Chord AB subtends an angle at the centre of size θ radians. The minor segment and the major sector are shaded.



(a) Show that the area of the minor segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

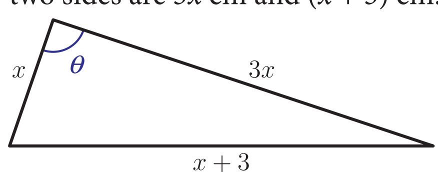
(b) Find the area of the major sector.

(c) Given that the ratio of the areas of the blue: pink regions is 1 : 2, show that:

$$\sin, = \frac{3}{2}, - \pi$$

(d) Find the value of θ. *[10 marks]*

4. The area of the triangle shown is 2.21 cm^2. The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x + 3)$ cm.



(a) Using the formula for the area of the triangle, write down an expression for $\sin\theta$ in terms of x.

(b) Using the cosine rule, write down and simplify an expression for $\cos\theta$ in terms of x.

(c) (i) Using your answers to parts (a) and (b), show that:

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)$$

(ii) Hence find the possible values of x and the corresponding values of θ, in radians, using your answer to part (b) above. *[10 marks]*

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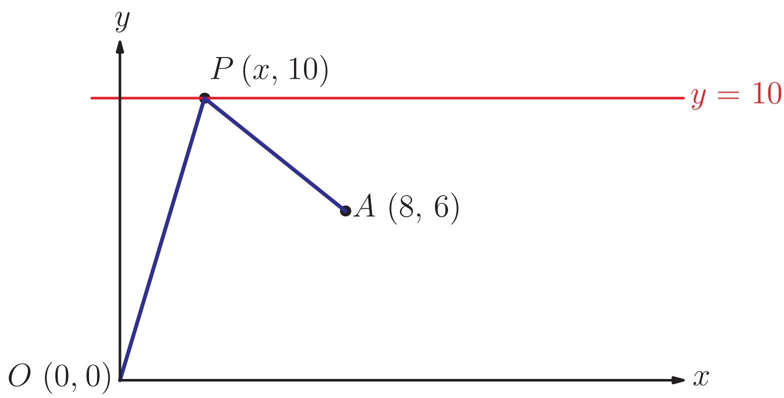
5. In triangle ABC, $AB = 10$, $BC = 5$, $CA = x$ and $B\hat{A}C = \theta°$.

(a) Show that $x^2 - 20x\cos\theta + 75 = 0$.

(b) Find the range of values of $\cos\theta$ for which the above equation has real roots.

(c) Hence find the set of values of θ for which it is possible to construct triangle ABC with the given measurements. *[8 marks]*

6. In the diagram below, the points O (0, 0) and A (8, 6) are fixed. The angle $O\hat{P}A$ varies as the point P $(x, 10)$ moves along the horizontal line $y = 10$.



(a) (i) Show that $AP = \sqrt{x^2 - 16x + 80}$.

(ii) Write down a similar expression for OP in terms of x.

(b) Hence, show that:

$$\cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}$$

(c) Find, in degrees, the angle $O\hat{P}A$ when $x = 8$.

(d) Find the positive value of x such that $O\hat{P}A = 60°$.

Let the function f be defined by

$$f(x) = \cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}, \quad 0 \le x \le 15.$$

(e) Consider the equation $f(x) = 1$.

(i) Explain, in terms of the position of the points O, A and P, why this equation has a solution.

(ii) Find the exact solution to the equation.

[17 marks]

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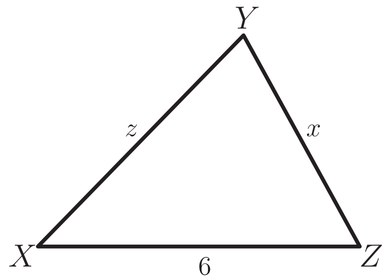
7. (a) Let $y = -16x^2 + 160x - 256$. Given that y has a maximum value, find:

(i) the value of x giving the maximum value of y

(ii) this maximum value of y.

The triangle XYZ has $XZ = 6$, $YZ = x$, $XY = z$ as shown. The perimeter of triangle XYZ is 16.



(b) (i) Express z in terms of x.

(ii) Using the cosine rule, express z^2 in terms of x and $\cos Z$.

(iii) Hence, show that $\cos Z = \dfrac{5x - 16}{3x}$.

Let the area of triangle XYZ be A.

(c) Show that $A^2 = 9x^2 \sin^2 Z$.

(d) Hence, show that $A^2 = -16x^2 + 160x - 256$.

(e) (i) Hence, write down the maximum area for triangle XYZ.

(ii) What type of triangle is the triangle with maximum area?

[15 marks]

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8. Two circular cogs are connected by a chain as shown in diagram 1. The radii of the cogs are 3 cm and 8 cm and the distance between their centres is 25 cm.

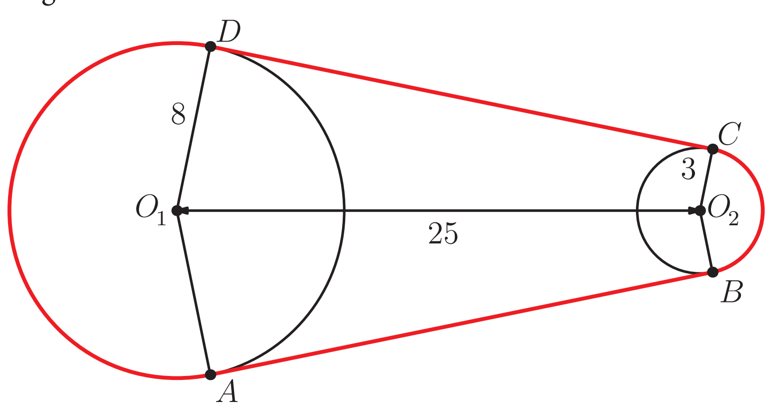


diagram 1

Diagram 2 shows the quadrilateral $O_1\hat{A}BO_2$. Line O_2P is drawn parallel to AB.

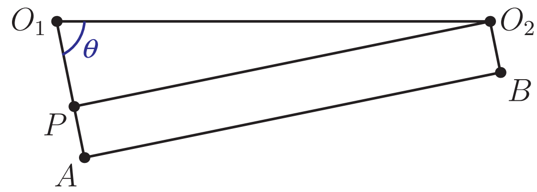


diagram 2

(a) Write down the size of $O_1\hat{A}B$ in radians, giving a reason for your answer.

(b) Explain why $PO_2 = AB$.

(c) Hence find the length AB.

(d) Find the size of the angle marked θ, giving your answer in radians correct to 4 significant figures.

(e) Calculate the length of the chain $ABCD$. *[12 marks]*

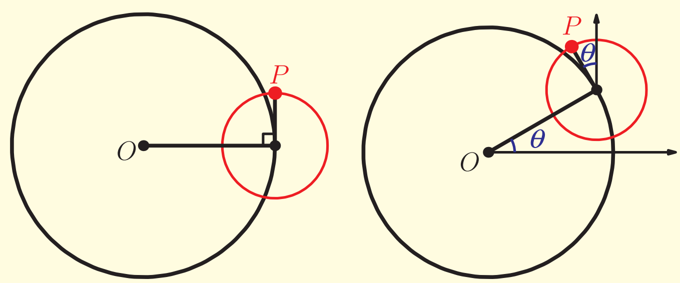
12 Further trigonometry

In this chapter you will learn:

- how to find trigonometric functions of sums and differences of two angles (for example $\sin(A+B)$)
- a useful method for working with sums of trigonometric functions (for example $\sin A + \cos A$)
- about some new trigonometric functions.

Introductory problem

A fairground ride consists of a vertical wheel with radius 10 m with smaller wheels of radius 2 m attached to it. The seats are arranged around the edges of the smaller wheels. The large wheel rotates at constant speed with period of 30 seconds, and the smaller wheels rotate with the same period around their own centres. The wheels start in the position shown in the diagram, and we consider the passenger in the seat labelled P.



The wheels are connected in such a way that when the large wheel has rotated through angle θ, the small wheel has also rotated through angle θ relative to its starting position, as shown in the second diagram.

The height, h, above ground is measured in metres (so that at the starting position, $\theta = 0$ and $h = 12$). Find the maximum height of the passenger above ground and the time he takes to return to the starting position.

See Section 9F for modelling using trigonometric functions.

We know that motion around the circle can be described in terms of sine and cosine functions. The height of the passenger is the sum of two such functions with the same periods but different amplitudes. Is this another sine function? If so, can we find its amplitude and period?

In this chapter we will learn how to simplify expressions involving sums and products of sine and cosine functions.

12A Double angle identities

This section looks at trigonometric functions when the argument of the function doubles.

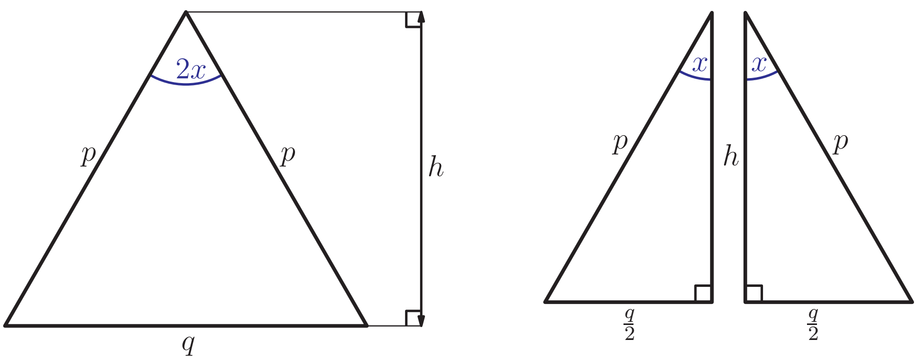
Working in radians, use your calculator to find:

1. $\sin 1.2$ and $\sin 2.4$
2. $\cos 1.2$ and $\cos 2.4$

It seems that there is no connection between the values of trigonometric functions of an angle and those for twice that angle. Are there any rules that relate $\sin 2x$ and $\cos 2x$ to $\sin x$ and $\cos x$?

To try and find such rules, a sensible starting point would seem be the familiar right-angled triangle containing the angle x. We are interested in the angle $2x$ which we can make by simply adjoining an identical right-angled triangle as shown.

We used this idea of joining two right-angled triangles to derive sine and cosine rules in chapter 11.



Let us consider the whole isosceles triangle, and find its area using two familiar formulae.

Using the formula Area $= \frac{1}{2}ab\sin\hat{C}$ we get:

$$\text{Area} = \frac{1}{2}p^2\sin(2x) \qquad (1)$$

We can also calculate the area by using the length of the base and the height of the triangle and the formula Area $= \frac{1}{2}bh$.

Working in one of the right-angled triangles we have:
$h = p\cos x$

and $\frac{q}{2} = p\sin x \Rightarrow q = 2p\sin x$

So a second expression for the area of the triangle is:

$$\text{Area} = \frac{1}{2}(2p\sin x)(p\cos x) = p^2\sin x\cos x \qquad (2)$$

Comparing the two expressions (1) and (2) for the area, we get:

$$\frac{1}{2}p^2\sin(2x) = p^2\sin x\cos x$$

EXAM HINT

You will often see sin, cos and tan of a multiple of x written without brackets: sin2x, cos5x, etc.

Rearranging this equation gives the **double angle identity** for sine.

KEY POINT 12.1

$$\sin(2x) = 2\sin x \cos x$$

Although x was assumed to be an acute angle in the derivation of Key point 12.1, the identity actually applies to all values of x.

Working from the same triangle and using the cosine rule we can derive another double angle identity for cosine:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

See Fill in proofs 9–11 on the CD-ROM.

Substituting $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ gives us two further ways of expressing this double angle identity.

KEY POINT 12.2

$$\cos 2x = \begin{cases} 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \\ \cos^2 x - \sin^2 x \end{cases}$$

A useful application of Key point 12.2 is finding the exact values of half-angles.

The really useful applications of double angle identities are in proving more complex identities and in solving equations.

Worked example 12.1

Using the exact value of $\cos 30°$, show that $\sin 15° = \sqrt{\dfrac{2-\sqrt{3}}{4}}$.

We know that $\cos 30° = \dfrac{\sqrt{3}}{2}$ but we need a way of relating this to $\sin 15°$

Since $30 = 2 \times 15$, the obvious choice seems to be the cosine double angle identity

Using $\cos(2x) = 1 - 2\sin^2 x$

continued . . .

$$\cos(2\times 15°) = 1 - 2\sin^2 15°$$

$$\Rightarrow \quad \cos 30° = 1 - 2\sin^2 15°$$

$$\text{As } \cos 30° = \frac{\sqrt{3}}{2}$$

$$\frac{\sqrt{3}}{2} = 1 - 2\sin^2 15°$$

$$\Rightarrow \quad \sqrt{3} = 2 - 4\sin^2 15°$$

$$\Rightarrow \quad 4\sin^2 15° = 2 - \sqrt{3}$$

$$\Rightarrow \quad \sin^2 15° = \frac{2-\sqrt{3}}{4}$$

We have to choose between the positive and negative square root here. $\sin 15° > 0$ so we take the positive square root

$$\therefore \sin 15° = \sqrt{\frac{2-\sqrt{3}}{4}} \quad (\text{as } \sin 15° > 0)$$

Although they are called the double angle identities, these identities can also be used with higher multiples.

In chapter 19 we shall use double angle identities to integrate some trigonometric functions.

Worked example 12.2

Find an expression for $\cos 4x$ in terms of:

(a) $\cos 2x$ (b) $\cos x$

$4x = 2\times(2x)$, so use one of the cosine double angle identities. Since we want an expression involving only cos it has to be $\cos(2a) = 2\cos^2 a - 1$ with $a = 2x$

(a) $\cos 2(2x) = 2\cos^2(2x) - 1$

$\Rightarrow \cos(4x) = 2\cos^2(2x) - 1$

Use the answer from the previous part

(b) From part (a)

$\cos(4x) = 2\cos^2(2x) - 1$

Replace $\cos(2x)$ in the answer to part (a) with an expression involving only $\cos x$

$= 2(2\cos^2 x - 1)^2 - 1$

EXAM HINT

In the exam, any equivalent form of the answer would be acceptable. Note that, unless you are asked to do so, there is no need to go any further than we did for part (b), for example by expanding the brackets.

Recognising the form of double angle formulae can be useful in solving trigonometric equations.

Worked example 12.3

Solve the equation $6\sin x\cos x = 1$ for $-\pi < x < \pi$.

We see the LHS is similar to 2sinx cosx (which equals sin 2x), so isolate this and use the double angle identity

$6\sin x\cos x = 1$

$\Rightarrow 2\sin x\cos x = \frac{1}{3}$

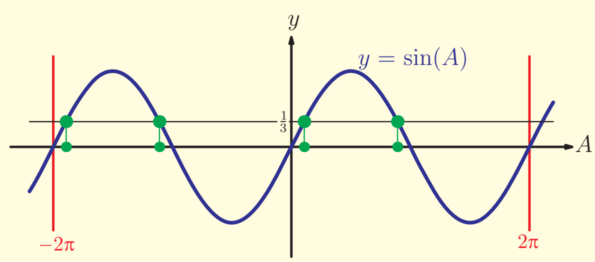
$\Rightarrow \sin(2x) = \frac{1}{3}$

Follow the standard procedure and look at the graph to find that there are 4 solutions in the given domain

Substitute $A = 2x$

Since $-\pi < x < \pi$

$-2\pi < 2x < 2\pi$



$\arcsin\left(\frac{1}{3}\right) = 0.3398$

$\therefore A = 0.3398, 2.802, -5.943, -3.481$

$\therefore x = 0.170, 1.40, -2.97, -1.74$ (3SF)

If an equation contains both a $\cos 2\theta$ and a $\cos\theta$, we can use identities to turn it into an equation involving only $\cos\theta$.

Worked example 12.4

Solve the equation $\cos 2x = \cos x$ for $0° \le x \le 360°$.

$\cos(2x) = \cos x$

As before, write the equation in terms of only one trig function (Note that $\cos(2x)$ and $\cos x$ are not the same function!)

We can write the LHS in terms of cosine using $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x - 1 = \cos x$

Rewrite as a quadratic in $\cos x$ and factorise

$\Rightarrow \quad 2\cos^2 x - \cos x - 1 = 0$

$\Rightarrow \quad (2\cos x + 1)(\cos x - 1) = 0$

$\Rightarrow \quad \cos x = -\frac{1}{2}$ or $\cos x = 1$

Solve each equation separately

When $\cos x = -\frac{1}{2}$, $x = 120°$ or $360° - 120° = 240°$

When $\cos x = 1$, $x = 0°$ or $360°$

List all the solutions

$x = 0, 120°, 240°, 360°$

There is also a double angle identity for the tangent function.

KEY POINT 12.3

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

You should be able to prove this.

Worked example 12.5

Prove that $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

We know the formulae for $\sin 2x$ and $\cos 2x$, so write $\tan 2x$ in terms of $\sin(2x)$ and $\cos(2x)$

$$\tan(2x) = \frac{\sin 2x}{\cos 2x}$$

Convert from double to single angles using the sine and cosine identities

Use $\cos(2x) = \cos^2 x - \sin^2 x$

$$= \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$$

Divide through by $\cos^2 x$ to leave $1 - \tan^2 x$ in the denominator

$$= \frac{2\left(\dfrac{\sin x}{\cos x}\right)}{1-\dfrac{\sin^2 x}{\cos^2 x}}$$

$$\therefore \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

In Worked example 12.5 the most useful version of the $\cos(2x)$ identity may not have been obvious. The good news is that here, and in many other cases, even if you choose the 'wrong' one you can still complete the calculation; it may just take a little longer.

Exercise 12A

1. (a) (i) Given that $\cos\theta = -\dfrac{1}{4}$, find the exact value of $\cos 2\theta$.

(ii) Given that $\sin A = -\dfrac{2}{3}$, find the exact value of $\cos 2A$.

(b) (i) Given that $\sin x = \dfrac{1}{3}$ and $0 < x < \dfrac{\pi}{2}$, find the exact value of $\cos x$.

(ii) Given that $\sin x = \dfrac{3}{5}$ and $0 < x < \dfrac{\pi}{2}$, find the exact value of $\cos x$.

(c) (i) Given that $\sin x = \dfrac{1}{3}$ and $0 < x < \dfrac{\pi}{2}$, find the exact value of $\sin 2x$.

(ii) Given that $\sin x = \dfrac{3}{5}$ and $0 < x < \dfrac{\pi}{2}$, find the exact value of $\sin 2x$.

2. Find the exact value of:

(a) $\sin^2 22.5°$ (b) $\cos^2 75°$ (c) $\cos^2\left(\dfrac{\pi}{12}\right)$

3. Find the exact value of $\tan 22.5°$.

4. Simplify using a double angle identity:

(a) $2\cos^2(3A)-1$ (b) $4\sin 5x\cos 5x$

(c) $3-6\sin^2\left(\frac{b}{2}\right)$ (d) $5\sin\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)$

5. Use an algebraic method to solve these equations:

(a) $\sin 2x = 3\sin x$ for $x\in[0, 2\pi]$

(b) $\cos 2x - \sin^2 x = -2$ for $0° \le x \le 180°$

(c) $5\sin 2x = 3\cos x$ for $-\pi < x < \pi$

(d) $\tan 2x - \tan x = 0$ for $0° \le x \le 360°$

6. Prove these identities:

(a) $(\sin x + \cos x)^2 = 1 + \sin 2x$

(b) $\cos^4\theta - \sin^4\theta = \cos 2\theta$

(c) $\tan 2A - \tan A = \frac{\tan A}{\cos 2A}$

(d) $\tan\alpha - \frac{1}{\tan\alpha} = -\frac{2}{\tan 2\alpha}$

7. Find all the values of $\theta\in[-\pi,\pi]$ which satisfy the equation $\cos^2\theta + \cos 2\theta = 0$. *[6 marks]*

8. Show that $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$. *[5 marks]*

9. (a) Given that $\tan\alpha\tan 2\alpha = 6$, find the possible values of $\tan\alpha$.

(b) Solve the equation $\tan\alpha\tan 2\alpha = 1$ for $\alpha\in(0,\pi)$, giving your answer in terms of π. *[6 marks]*

10. Express $\cos 4\theta$ in terms of:

(a) $\cos\theta$ (b) $\sin\theta$ *[6 marks]*

In chapter 15, you will learn an alternative method for expressing $\cos n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

11. (a) Show:

(i) $\cos^2\left(\frac{1}{2}x\right) = \frac{1}{2}(1+\cos x)$

(ii) $\sin^2\left(\frac{1}{2}x\right) = \frac{1}{2}(1-\cos x)$

(b) Express $\tan^2\left(\frac{1}{2}x\right)$ in terms of $\cos x$. *[8 marks]*

12. Given that $a\sin 4x = b\sin 2x$ and $0 < x < \frac{\pi}{2}$, express $\sin^2 x$ in terms of a and b. *[6 marks]*

12B Compound angle identities

In Section 12A we derived identities for trigonometric functions of double angles. We now look at identities for expressions like $\sin(x + y)$. These are often called the **compound angle identities**.

Using a generalisation of the methods for the sine and cosine double angle identities from Section 12A we can show that:

See Fill-in proofs 10 'Sine compound angle formula' and 11 'Cosine compound angle formula' on the CD-ROM.

KEY POINT 12.4

Compound angle identities

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

EXAM HINT

In the Formula booklet these identities are summarised as

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

Notice the signs in the cosine identities: In the identity for the sum we use the *minus* sign, and in the identity for the difference we use the *plus* sign.

Note that the compound angle formulae can be used to derive the double angle formulae by setting $x = y$.

As with the double angle identities, one of the simplest applications is to calculate exact values of trigonometric functions.

Worked example 12.6

Find the exact values of:

(a) $\sin 75°$ (b) $\cos 15°$

We know the exact values of sin for a few angles: $0°, 30°, 45°, 60°, 90°$

Use $30° + 45° = 75°$

(a) $\sin 75° = \sin(30° + 45°)$

Use the sine addition identity

$= \sin 30° \cos 45° + \cos 30° \sin 45°$

$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

$15° = 45° - 30°$ so use cosine subtraction identity

(b) $\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

We can now use the sine and cosine compound angle identities to derive new ones. In particular:

KEY POINT 12.5

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Again, you should be able to prove this.

Worked example 12.7

Prove that $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

Express tangent in terms of sine and cosine.

$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$

continued . . .

To remove compound angles, use the identities

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

We want to express this in terms of tan. Looking at the top of the fraction, if we divide by cos x we will get tan x in the first term, and if we divide by cos y we will get tan y in the second term. So we divide top and bottom by cos x cos y

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x}\frac{\sin y}{\cos y}}$$

(dividing top and bottom by $\cos x \cos y$)

$$\therefore \tan(x+y) = \frac{\tan x + \tan}{1 - \tan x \tan}$$

The compound angle identities can be used to derive 'triple-angle identities'.

Worked example 12.8

(a) Show that $\sin(3A) = 3\sin A - 4\sin^3 A$.

(b) Solve the equation $\sin 3x = \sin x$ for $x \in [0, 2\pi]$.

Split the angle and use the sine addition identity

(a) $\sin(3A) = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

Use the double angle identities to get an expression involving only single angles

Since $\sin(2A) = 2\sin A \cos A$
and $\cos(2A) = 1 - \sin^2 A$:
$\sin(3A) = (2\sin A \cos A)\cos A + (1 - 2\sin^2 A)\sin A$
$= 2\sin A \cos^2 A + \sin A - 2\sin^3 A$

Replace $\cos^2 x$ with $1 - \sin^2 x$

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$
(since $\cos^2 x = 1 - \sin^2 x$)
$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$
$\therefore \sin(3A) = 3\sin A - 4\sin^3 A$

Use the equation from (a)

(b) $\sin(3x) = \sin x$
$\Rightarrow 3\sin x - 4\sin^3 x = \sin x$

continued . . .

This is a cubic in $\sin x$ so rearrange to make the RHS = 0 and factorise

$$\Rightarrow \quad 2\sin x - 4\sin^3 x = 0$$
$$\Rightarrow \quad 2\sin x(1-2\sin^2 x) = 0$$
$$\Rightarrow \quad \sin x = 0 \text{ or } \sin^2 x = \frac{1}{2}$$

Solve each equation separately

When $\sin x = 0$: $x = 0, \pi, 2\pi$

When $\sin^2 x = \frac{1}{2}$: $\sin x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The compound and double angle formulae allow us to simplify some nasty expressions involving functions and inverse functions.

Worked example 12.9

If $x \in [-1,1]$ find an expression for:

(a) $\cos(2\arccos x)$ (b) $\sin(2\arccos x)$ in terms of x.

$\cos(\arccos x) = x$, so use the cosine double angle identity on $\cos(2\arccos x)$

(a) Using $\cos(2\theta) = 2\cos^2\theta - 1$ we have:

$$\cos(2\arccos x) = 2\cos^2(\arccos x) - 1$$
$$= 2(\cos(\arccos x))^2 - 1$$
$$\therefore \cos(2\arccos x) = 2x^2 - 1$$

This time use the sine double angle identity

(b) Using $\sin(2\theta) = 2\sin\theta\cos\theta$ we have:

$$\sin(2\arccos x) = 2\sin(\arccos x)\cos(\arccos x) \quad (1)$$

We can't do anything with $\sin(\arccos x)$ but we can work with $\cos(\arccos x)$ so use the Pythagorean identity to change sine to cosine

Using $\sin^2\theta = 1 - \cos^2\theta$:

$$\sin^2(\arccos x) = 1 - \cos^2(\arccos x) = 1 - x^2$$

The square root could have two possible answers

$\arccos x \in [0, \pi]$, so $\sin(\arccos x) > 0$

$$\therefore \sin(\arccos x) = \sqrt{1-x^2}$$

Put everything together

Substituting into (1)

$$\sin(2\arccos x) = 2x\sqrt{1-x^2}$$

Exercise 12B

1. Express the following in the form $A\sin x + B\cos x$, giving exact values of A and B.

(a) $\sin\left(x+\frac{\pi}{3}\right)$ (b) $\sin\left(x-\frac{\pi}{4}\right)$

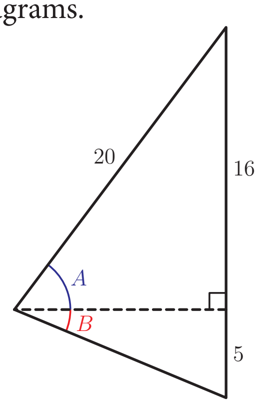
(c) $\cos\left(x+\frac{3\pi}{4}\right)$ (d) $\cos\left(x-\frac{3\pi}{2}\right)$

2. Find the exact value of:

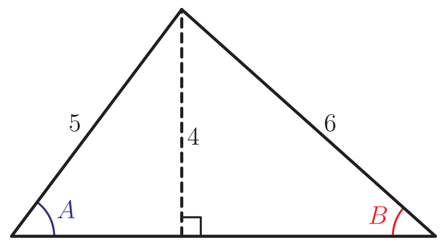
(a) $\cos 75°$ (b) $\sin\left(\frac{7\pi}{12}\right)$ (c) $\tan 105°$

3. Find the exact value of $\cos(A-B)$ for angles shown in the diagrams.

(a)



(b)



4. (a) Show that $\sin\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{3}\right)=\sin x$.

(b) Simplify $\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)$. *[6 marks]*

5. (a) Express $\tan\left(\theta-\frac{\pi}{4}\right)$ in terms of $\tan\theta$.

(b) Given that $\tan\left(\theta-\frac{\pi}{4}\right)=6\tan\theta$, find two possible values of $\tan\theta$.

(c) Hence solve the equation $\tan\left(\theta-\frac{\pi}{4}\right)=6\tan\theta$ for $0<\theta<\pi$. *[8 marks]*

6. Write each of the following as a single trigonometric function, and hence find the maximum value of each expression, and the smallest positive value of x for which it occurs.

(a) $\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}$

(b) $2\cos x\cos 25° + 2\sin x\sin 25°$ *[6 marks]*

7. Use the compound angle formula to prove the following identities:

(a) $\cos 0 = 1$ (b) $\sin 2\theta = 2\sin\theta\cos\theta$ (c) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

8. (a) Express $\cos 3A$ in terms $\cos A$.

(b) Express $\tan 3A$ in terms of $\tan A$. *[6 marks]*

9. (a) Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

(b) Hence solve the equation $\sin\left(x+\frac{\pi}{6}\right) + \sin\left(x-\frac{\pi}{6}\right) = 3\cos x$ for $0 \le x \le \pi$. *[7 marks]*

10. (a) Use the identities for $\sin(A-B)$ and $\cos(A-B)$ to show that $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$.

(b) Denote by θ the angle between the lines with equations $y = 2x$ and $y = 4x$. Find the exact value of $\tan\theta$. *[9 marks]*

11. (a) Show that $\cos(x+y) + \cos(x-y) = 2\cos x\cos y$.

(b) Hence solve the equation $\cos 3x + \cos x = 3\cos 2x$ for $x \in [0, 2\pi]$. *[7 marks]*

12. Find the exact value of:

(a) $\tan(\arctan(1.2) + \arctan(0.5))$

(b) $\tan\left(2\arctan\left(\frac{1}{3}\right)\right)$ *[7 marks]*

13. (a) Show that $\cos^2\left(\frac{1}{2}A\right) = \frac{1}{2}(1+\cos A)$.

(b) Hence show that $\cos\left(\frac{1}{2}\arccos x\right) = \sqrt{\frac{1}{2}(1+x)}$ for $-1 \le x \le 1$. *[6 marks]*

12C Functions of the form $a\sin x + b\cos x$

In this section we will look at a useful method for dealing with sums of trigonometric functions. To solve the equation $3\sin x + 4\cos x = 2$ exactly, we need to try to write everything in terms of just one trigonometric function, so we start by considering the identities met so far.

As the only identity we have linking sin and cos is $\sin^2 x + \cos^2 x = 1$, we cannot just replace $\sin x$ with a function of $\cos x$ (or $\cos x$ with a function of $\sin x$) and we have neither $\sin^2 x$ or $\cos^2 x$ in our equation. We could rearrange, take squares and then use the identity, but the calculation is arduous. Instead, we could use one of several options from the list of compound angle identities in Key point 12.4. We try one to see if it works.

Consider the first equation we met:

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y \\ &= \cos y \sin x + \sin y \cos x\end{aligned}$$

Now, compare this with the equation:

$$3\sin x + 4\cos x = 2$$

We can see that the LHS of the equation would be the same as the compound angle expression if we could find a value of y so that $\cos y = 3$ and $\sin y = 4$.

Since $\sin y$ and $\cos y$ are both less than 1, this is not possible and so we need to adjust our original identity by multiplying by a constant, R

$$R\sin(x+y) = (R\cos y)\sin x + (R\sin y)\cos x$$

Now we have:

$$R\cos y = 3 \qquad \text{and} \qquad R\sin y = 4$$

which constitutes a pair of simultaneous equations in two unknowns (R and y).

To find R, we can use $\sin^2\theta + \cos^2\theta = 1$:

$$(R\sin y)^2 + (R\cos y)^2 = 4^2 + 3^2$$

$$\Leftrightarrow \quad R^2(\sin^2 y + \cos^2 y) = 16 + 9$$

$$\Leftrightarrow \quad R^2 = 25 \quad \therefore \quad R = 5 \qquad \text{(always use } R > 0\text{)}$$

To find y we can eliminate R by dividing one equation by the other:

$$\frac{R\sin y}{R\cos y} = \frac{4}{3} \Leftrightarrow \tan y = \frac{4}{3}$$

$$\therefore y = \arctan\left(\frac{4}{3}\right) = 0.927 \text{ (3SF)}$$

So $3\sin x + 4\cos x$ can be written as $5\sin(x + 0.927)$ which allows us to rewrite the equation as:

$$5\sin(x + 0.927) = 2$$

This is now an equation that we know how to solve.

This whole procedure seems long and complicated but with practice you will find that you can complete it quite quickly. In the next example, we will see that we can follow the same

EXAM HINT

By taking $R > 0$ and combining the equations to get $\tan y$, we ensure the correct solution. Avoid using one of the initial equations with arcsin or arccos, as you may find a false solution, depending on the quadrant y lies in.

Worked example 12.10

(a) Write $3\sin x + 4\cos x$ in the form $R\cos(x-\alpha)$.

(b) Solve $3\sin x + 4\cos x = 2$ for $x \in [0, 2\pi]$.

First expand $R\cos(x-\alpha)$ using the compound angle identity

(a) $R\cos(x-\alpha) = R(\cos x\cos\alpha + \sin x\sin\alpha)$

$= (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$

$= (R\sin\alpha)\sin x + (R\cos\alpha)\cos x$

Compare with the original function to find the simultaneous equations in R and α

Comparing this with $3\sin x + 4\cos x$ we have

$R\sin\alpha = 3$, $R\cos\alpha = 4$

To find R use $\sin^2\theta + \cos^2\theta = 1$

$R^2(\cos^2\alpha + \sin^2\alpha) = 4^2 + 3^2$

$R^2 = 25 \Rightarrow R = 5$

Divide to find $\tan\alpha$ and hence α

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4}$

$\therefore \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 0.644$ (3SF)

$\therefore 3\sin x + 4\cos x = 5\cos(x - 0.644)$

Use the answer to (a) to give an equation to solve

(b) $3\sin x + 4\cos x = 2$

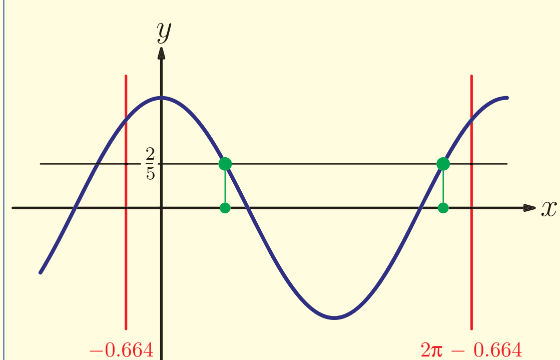
$\Rightarrow 5\cos(x-0.644) = 2$

$\Rightarrow \cos(x-0.644) = \frac{2}{5}$

Sketch the graph $y = \cos x$ to find all the solutions in the required range

Since $0 < x < 2\pi$

$-0.664 < x - 0.664 < 2\pi - 0.664$



$\arccos\left(\frac{2}{5}\right) = 1.159$

$x - 0.644 = 1.159, 5.124$

$x = 1.80, 5.77$ (3 SF)

procedure to express the same function in terms of cosine rather than sine, and then continue to solve the equation.

We can summarise the procedure as follows:

KEY POINT 12.6

To write $a\sin x \pm b\cos x$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$:

1. Expand the brackets using compound angle identities.
2. Equate coefficients of $\sin x$ and $\cos x$ to get equations for $R\sin\alpha$ and $R\cos\alpha$.
3. To find R, use $R^2 = a^2 + b^2$.
4. To find $\tan\alpha$, divide the $\sin\alpha$ equation by the $\cos\alpha$ equation.

EXAM HINT

In the examination, the question will always tell you which form to use.

Worked example 12.11

(a) Write $4\sin x - 6\cos x$ in the form $R\sin(x - \theta)$ where $R > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$.

(b) Hence find the maximum value of $4\sin x - 6\cos x$ and the smallest positive value of x for which it occurs.

Follow the standard procedure for this type of question and expand $R\sin(x-\theta)$

(a) $R\sin(x-\theta) = R(\sin x\cos\theta - \cos x\sin\theta)$

$= (R\cos\theta)\sin x - (R\sin\theta)\cos x$

Comparing this with

$4\sin x - 6\cos x$

we have

$R\cos\theta = 4, \quad R\sin\theta = 6$

Find R

$R^2 = 4^2 + 6^2 = 52$

$R = \sqrt{52} = 2\sqrt{13}$

Find θ

$\tan\theta = \frac{R\sin\theta}{R\cos\theta} = \frac{6}{4} = \frac{3}{2}$

$\theta = \arctan\left(\frac{3}{2}\right) = 0.983$ (3 SF)

Write the equation in the required form

$\therefore\ 4\sin x - 6\cos x = 2\sqrt{13}\sin(x - 0.983)$

continued . . .

Since sine has a maximum value of 1, the maximum value of this function must be R.

maximum value $= 2\sqrt{13}$

This value occurs when $\sin(x - 0.983) = 1$

The maximum value of $\sin y$ occurs when $y = \frac{\pi}{2}$

$$\Rightarrow \quad x - 0.983 = \frac{\pi}{2}$$

the smallest positive such value of x:

$$\Leftrightarrow \quad x = 0.983 + \frac{\pi}{2} = 2.55 \text{ (3 SF)}$$

The same method works for the difference of two trigonometric functions:

Exercise 12C

1. Express in the form $R\sin(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$:

(a) $4\sin x + 6\cos x$ (b) $\cos x + 3\sin x$

2. Express in the form $r\sin(\theta - a)$, where $r>0$ and $0<a<90°$:

(a) $2\sin\theta - 2\cos\theta$ (b) $\sin\theta - \sqrt{3}\cos\theta$

3. Express in the form $r\cos(x+\theta)$, where $r>0$ and $0<\theta<\frac{\pi}{2}$:

(a) $\sqrt{6}\cos x - \sqrt{2}\sin x$ (b) $5\cos x - 5\sin x$

4. Express in the form $R\cos(x-\theta)$, where $R>0$ and $0°<\theta<90°$:

(a) $7\cos x + 6\sin x$ (b) $5\sin x + 12\cos x$

5. (a) Express $5\sin x + 12\cos x$ in the form $R\sin(x+\theta)$.

(b) Hence give details of two successive transformations which transform the graph of $y = \sin x$ into the graph of $y = 5\sin x + 12\cos x$. *[6 marks]*

You may need to revise chapter 6 Transformations.

6. (a) Express $3\sin x - 7\cos x$ in the form $R\sin(x-\theta)$.

(b) Hence find the range of the function $f(x) = 3\sin x - 7\cos x$. *[6 marks]*

7. (a) Express $4\cos x - 5\sin x$ in the form $R\cos(x+\alpha)$.

(b) Hence find the smallest positive value of x for which $4\cos x - 5\sin x = 0$. *[6 marks]*

8. (a) Express $\sqrt{3}\sin x + \cos x$ in the form $R\cos(x-\theta)$.

(b) Hence find the coordinates of the minimum and maximum points on the graph of $y = \sqrt{3}\sin x + \cos x$ for $x \in [0, 2\pi]$.

[6 marks]

9. By expressing $\sin 2x + \cos 2x$ in the form $R\sin(2x + a)$, solve the equation $\sin 2x + \cos 2x = 1$ for $-\pi \le x \le \pi$. *[6 marks]*

12D Reciprocal trigonometric functions

You already know that $\tan x = \dfrac{\sin x}{\cos x}$, so we can do all the calculations we need with sine and cosine (only). But, having the notation for tan *x* can simplify many expressions.

Similarly, expressions of the form

$\dfrac{\cos x}{\sin x}\left(=\dfrac{1}{\tan x}\right)$, $\dfrac{1}{\sin x}$ and $\dfrac{1}{\cos x}$ often occur. This is part of the reason for introducing three additional trigonometric functions.

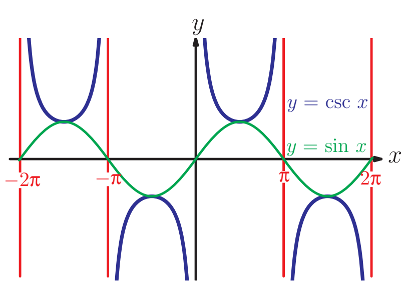
Is having names for the reciprocal functions useful? Does their definition add to our body of knowledge?

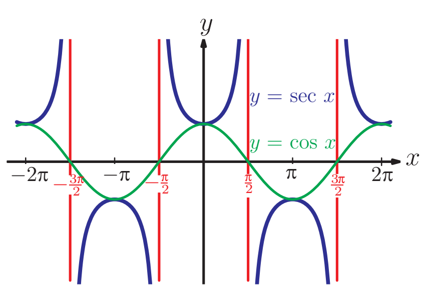
KEY POINT 12.7

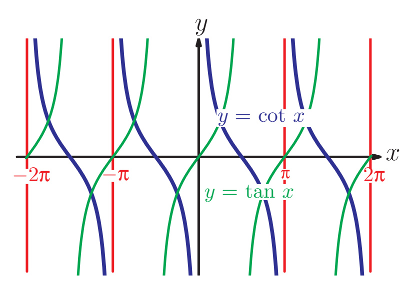
the secant function	the cosecant function	the cotangent function
$\sec x = \dfrac{1}{\cos x}$	$\csc x = \dfrac{1}{\sin x}$	$\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

We studied transformations of graphs in chapter 6.

Using the graphs of sine, cosine and tangent and what we know about reciprocal transformations, we can draw the graphs of $y = \sec x$, $y = \csc x$ and $y = \cot x$.







Perhaps the most common use of these functions is in the following identities. They can be deduced from $\sin^2 x + \cos^2 x = 1$ by dividing through by $\cos^2 x$ and $\sin^2 x$ respectively.

KEY POINT 12.8

Pythagorean identities:

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

($\cos^2 x + \sin^2 y = 1$ is also a pythagorean identity given in the formula booklet)

EXAM HINT

The reciprocal trigonometric functions follow the same conventions as the normal trigonometric functions so $\sec^2 x = (\sec x)^2$.

Worked example 12.12

Solve the equation $2\tan^2 x + \dfrac{3}{\cos x} = 0$ for $-\pi < x < \pi$.

	$2\tan^2 x + \dfrac{3}{\cos x} = 0$
Use $\dfrac{1}{\cos x} = \sec x$ to simplify the notation	$\Rightarrow 2\tan^2 x + 3\sec x = 0$
The equation should contain only one trig function, so we write $\tan^2 x$ in terms of $\sec^2 x$	$\Rightarrow 2(\sec^2 x - 1) + 3\sec x = 0$ (since $\tan^2 x = \sec^2 x - 1$) $\Rightarrow 2\sec^2 x + 3\sec x - 2 = 0$
Factorise the quadratic in $\sec x$	$\Rightarrow (2\sec x - 1)(\sec x + 2) = 0$ $\Rightarrow \sec x = \dfrac{1}{2}$ or $\sec x = -2$
We cannot find inverse sec, so write $\sec x$ in terms of $\cos x$	$\Rightarrow \cos x = 2$ or $\cos x = -\dfrac{1}{2}$
Finally we solve each equation separately	$\cos x = 2$: reject since $-1 \le \cos x \le 1$ When $\cos x = -\dfrac{1}{2}$: $\arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$ $\therefore x = \dfrac{2\pi}{3}$ or $-\dfrac{2\pi}{3}$

Would you have recognised the quadratic in the previous example so easily without the notation for $\sec x$? Do you think this justifies the introduction of extra notation such as $\sec x, \csc x, \cot x$?

Worked example 12.13

Show that $\sec(2\theta) = \dfrac{\sec^2\theta}{2-\sec^2\theta}$.

Introduce cosine with the aim of using the cosine double angle identity $\cos 2x = 2\cos^2 x - 1$

$$\sec(2\theta) = \frac{1}{\cos(2\theta)}$$

$$= \frac{1}{2\cos^2\theta - 1} \quad (\text{as } \cos(2\theta) = 2\cos^2\theta - 1)$$

Need $\sec\theta$ in the answer so change back from $\cos\theta$

$$= \frac{1}{\frac{2}{\sec^2\theta} - 1} \quad \left(\text{as } \cos\theta = \frac{1}{\sec\theta}\right)$$

Simplify by multiplying top and bottom of the fraction by $\sec^2\theta$

$$\therefore \sec(2\theta) = \frac{\sec^2\theta}{2-\sec^2\theta}$$

Exercise 12D

1. Giving your answers to 4 significant figures, find the value of:

(a) (i) $\sec 1.2$ (ii) $\csc 2.4$

(b) (i) $\cot(-3.5)$ (ii) $\sec(-0.6)$

(c) (i) $\csc\left(\dfrac{2\pi}{5}\right)$ (ii) $\cot\left(\dfrac{4\pi}{3}\right)$

2. Find the exact value of:

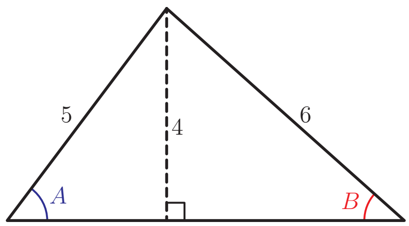
(a) (i) $\csc\left(\dfrac{\pi}{3}\right)$ (ii) $\csc\left(\dfrac{\pi}{4}\right)$

(b) (i) $\sec\left(\dfrac{3\pi}{4}\right)$ (ii) $\sec\left(\dfrac{5\pi}{6}\right)$

(c) (i) $\cot\left(-\dfrac{\pi}{4}\right)$ (ii) $\cot\left(-\dfrac{2\pi}{3}\right)$

(d) (i) $\csc\left(\dfrac{3\pi}{2}\right)$ (ii) $\cot\left(\dfrac{\pi}{2}\right)$

3. Find the values of $\csc A$ and $\sec B$:



4. Solve the following equations for $x \in [0, 2\pi]$, giving your answer to 3 significant figures.

(a) (i) $\sec x = 2$ (ii) $\sec x = 3$

(b) (i) $\csc x = 1.5$ (ii) $\csc x = 2.7$

(c) (i) $\cot x = 5$ (ii) $\cot x = 0.5$

(d) (i) $\sec 2x = 3$ (ii) $\csc 2x = 4$

5. Solve the following equations for $-\pi \le \theta \le \pi$:

(a) (i) $\csc\theta = -2$ (ii) $\csc\theta = -1$

(b) (i) $\cot\theta = \sqrt{3}$ (ii) $\cot\theta = 1$

(c) (i) $\sec\theta = 1$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$

(d) (i) $\cot\theta = 0$ (ii) $\cot\theta = -1$

6. (a) (i) Given that $\tan\theta = \frac{4}{3}$ and $0° < \theta < 90°$, find the exact value of $\sec\theta$.

(ii) Given that $\tan\theta = \frac{2}{5}$ and $0° < \theta < 90°$, find the exact value of $\sec\theta$.

(b) (i) Given that $\csc\theta = 5$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, find the exact value of $\cot\theta$.

(ii) Given that $\csc\theta = 3$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, find the exact value of $\cot\theta$.

(c) (i) Given that $\cot\theta = 3$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of $\sin\theta$.

(ii) Given that $\cot\theta = \frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of $\sin\theta$.

(d) (i) Given that $\sin\theta = \frac{\sqrt{2}}{3}$, find the possible values of $\sec\theta$.

(ii) Given that $\sin\theta = -\frac{1}{2}$, find the possible values of $\sec\theta$.

7. Prove that $\sin^2\theta + \cot^2\theta \sin^2\theta = 1$. *[5 marks]*

8. Solve the equation $\tan x + \sec x = 4$ for $0 \le x \le 2\pi$. *[5 marks]*

9. A function is defined by $f(x) = \tan x + \csc x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(a) Find the coordinates of the minimum and maximum points on the graph of $y = f(x)$.

(b) Hence write down the range of f. *[6 marks]*

10. Show that $\tan x + \cot x = \sec x \csc x$. *[5 marks]*

11. Prove that $\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta}{1+\cos\theta} = 2\cot\theta$. *[6 marks]*

12. (a) Given that $\sec^2 x - 3\tan x + 1 = 0$, show that $\tan^2 x - 3\tan x + 2 = 0$.

(b) Find the possible values of $\tan x$.

(c) Hence solve the equation $\sec^2 x - 3\tan x + 1 = 0$ for $x \in [0, 2\pi]$. *[8 marks]*

13. Prove that $\csc\ 2x = \frac{\sec x \csc x}{2}$. *[5 marks]*

14. Prove that $\cot\ 2x = \frac{\cot^2 x - 1}{2\cot x}$. *[5 marks]*

15. Find the inverse function of $\sec x$ in terms of the arccosine function. *[6 marks]*

Summary

- **Double angle** and **compound angle identities** are given in the Formula booklet (be careful to get the signs right!):

compound angle identities: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

double angle identities:

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\cos 2\theta = \begin{cases} 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \\ \cos^2\theta - \sin^2\theta \end{cases}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

- One particular application is to write $a \sin x \pm b \cos x$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$:
  1. Expand the brackets using compound angle identities.
  2. Equate coefficients of $\sin x$ and $\cos x$ to get equations for $R\sin\alpha$ and $R\cos\alpha$.
  3. $R^2 = a^2 + b^2$.
  4. To get $\tan\alpha$, divide the $\sin\alpha$ equation by the $\cos\alpha$ equation.
- Reciprocal trigonometric functions are defined by

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

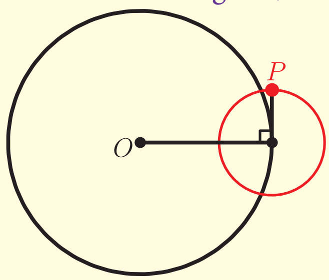
- Pythagorean identities are useful for reciprocal trigonometric functions:

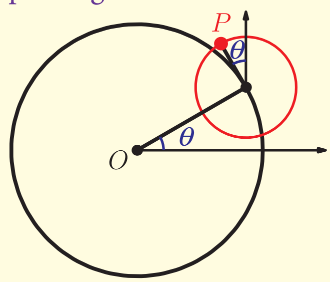
$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The two identities involving the reciprocal functions *sec*, *cot* and *csc* can be derived from the first function: $\cos^2\theta + \sin^2\theta = 1$.

Introductory problem revisited

A fairground ride consists of a vertical wheel with radius 10 m with smaller wheels of radius 2 m attached to it. The seats are arranged around the edges of the smaller wheels. The large wheel rotates at constant speed with period of 30 seconds, and the smaller wheels rotate with the same period around their own centres. The wheels start in the position shown in the diagram, and we consider the passenger in the seat labelled P.





The wheels are connected in such a way that when the large wheel has rotated through angle θ, the small wheel has also rotated through angle θ relative to its starting position, as shown in the second diagram.

The height, h, above ground is measured in metres (so that at the starting position, $\theta = 0$ and $h = 12$). Find the maximum height of the passenger above ground and the time he takes to return to the starting position.

The motion of the large wheel has amplitude 10 m and period 30 s, and when $t = 0$ the height above ground is 10 m. So the height of a point on the circumference is given by $h_1 = 10 + 10\sin\left(\frac{2\pi}{30}t\right)$. After time t the large wheel has rotated through angle $\theta = \frac{2\pi}{30}t$.
The small wheel has rotated through the same angle from its starting position, which was at right angles to the starting position of the large wheel. Hence the height of the point P above the centre of the small wheel is $h_2 = 2\sin\left(\frac{2\pi}{30}t + \frac{\pi}{2}\right)$.
Hence the height of the passenger is given by:

$$h = h_1 + h_2$$

$$h = 10 + 10\sin\left(\frac{2\pi}{30}t\right) + 2\sin\left(\frac{2\pi}{30}t + \frac{\pi}{2}\right)$$

We can then use the compound angle identity to show that the last term is $2\cos\left(\frac{2\pi}{30}t\right)$, and so the whole expression is of the form $10 + 10\sin\theta + 2\cos\theta$.

We know that this can be expressed in the form $10 + R\sin(\theta + \alpha)$; using the standard method we find that:

$$h = 10 + 2\sqrt{26}\sin\left(\frac{2\pi}{30}t + 11.3\right)$$

This tells us that the motion of the passenger follows another sine curve, with amplitude $2\sqrt{26}$ and the same period as the two wheels.

Hence the maximum height of the passenger is $10 + 2\sqrt{26} \approx 20.2\,\text{m}$ above the starting point, and he will return to the starting position after 30 seconds.

Mixed examination practice 12

Short questions

1. The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the *exact* value of $\sec\theta$.

[5 marks]

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2. (a) Write $\cos\left(x+\frac{\pi}{3}\right)$ in the form $a\cos x + b\sin x$.

(b) Hence find the exact values of $x \in [-2\pi, 2\pi]$ for which
$\cos\left(x+\frac{\pi}{3}\right) = \cos\left(x-\frac{\pi}{3}\right)$. *[6 marks]*

3. (a) Use the identity for $\cos(A+B)$ to show that $\cos 2\theta = 2\cos^2\theta - 1$.

(b) Solve the equation:

$$\frac{\sin\theta}{1+\cos\theta} = 3\cot\frac{\theta}{2} \quad \text{for} \quad \theta \in (0, 2\pi):$$ *[6 marks]*

4. The angle θ satisfies the equation $\tan\theta + \cot\theta = 3$, where θ is in degrees. Find all the possible values of θ lying in the interval $[0°, 90°]$.

[5 marks]

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5. (a) Express $\sqrt{15}\sin(2x) + \sqrt{5}\cos(2x)$ in the form $R\sin(2x+\alpha)$.

(b) The function f is defined by:

$$f(x) = \frac{2}{5+\sqrt{15}\sin(2x)+\sqrt{5}\cos(2x)}$$

Using your answer to part (a), find:

(i) the maximum value of $f(x)$, giving your answer in the form $p + q\sqrt{5}$ where $p, q \in \mathbb{Q}$

(ii) the smallest positive value of x for which this maximum occurs, giving your answer exactly, in terms of π. *[7 marks]*

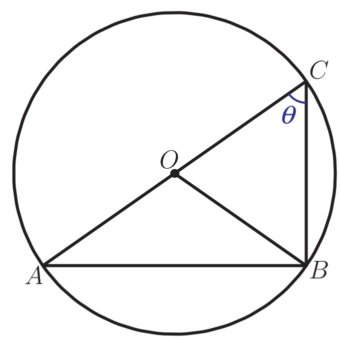
6. (a) Write down an expression for $\sin(\arcsin x)$.

(b) Show that $\sin(\arccos x) = \sqrt{1-x^2}$.

(c) Hence solve the equation $\arcsin x = \arccos x$ for $0 \le x \le 1$. *[6 marks]*

Long questions

1. The circle shown in the diagram has centre O and radius r.



(a) Write down the lengths of AB and BC in terms of r and θ.

(b) Write down an expression for the area of the triangle ABC.

(c) Write down an expression for the area of the triangle OBC.

(d) Hence find the ratio of the two areas in the form

$$\frac{\text{Area}(OBC)}{\text{Area}(ABC)} = k, \text{ where } k \in \mathbb{Q}.$$

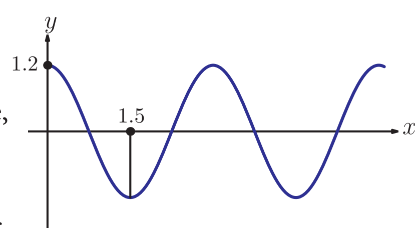
[10 marks]

2. (a) Use the identity for $\tan(A+B)$ to show that $\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$.

(b) Write down the value of $\tan 135°$.

(c) Hence find the exact value of $\tan 67.5°$. *[7 marks]*

3. A water wave has the profile shown in the graph, where y represents the height of the wave in metres, and x is the horizontal distance, also in metres.



(a) Given that the equation of the wave can be written as $y_1 = a\cos(px)$, find the values of a and p.

(b) A second wave has the profile given by the equation $y_2 = 0.9\sin\left(\frac{2\pi}{3}x\right)$.

Write down the amplitude and the period of the second wave.

When the two waves combine a new wave is formed, with the profile given by $y = y_1 + y_2$.

(c) Write the equation for y in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

(d) State the amplitude and the period of the combined wave.

(e) Find the smallest positive value of x for which the height of the combined wave is zero.

(f) Find the first two positive values of x for which the height of the combined wave is 1.3 m.

4. (a) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\alpha}{2}$, giving r and α as exact values.

(b) Hence, or otherwise, for $0 \leq \theta \leq 2\pi$, find the range of values of $\sqrt{3}\cos\theta - \sin\theta$.

(c) Solve $\sqrt{3}\cos\theta - \sin\theta = -1$, for $0 \leq \theta \leq 2\pi$, giving your answers as **exact** values. *[10 marks]*

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5. (a) Write $t^3 - 3t^2 - 3t + 1$ as a product of a linear and a quadratic factor.

(b) Show that $\tan(3A) = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

(c) Write down the exact value of $\tan 45°$.

(d) Hence find the exact values of $\tan 15°$ and $\tan 75°$. *[10 marks]*

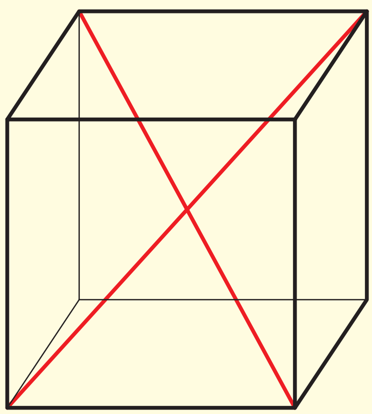
13 Vectors

In this chapter you will learn:

- to use vectors to represent displacements and positions in two and three dimensions
- to perform algebraic operations with vectors, and understand their geometric interpretation
- to calculate the distance between two points
- to use vectors to calculate the angle between two lines
- to use vectors to find areas of parallelograms and triangles
- to use two new operations on vectors, called scalar product and vector product.

Introductory problem

What is the angle between the diagonals of a cube?



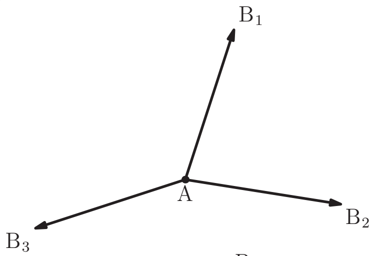
Solving problems in three dimensions can be difficult, as two-dimensional diagrams cannot always make clear what is happening. Vectors are a useful tool to describe geometrical properties using equations, which can often be analysed more easily. In this chapter we will develop techniques to calculate angles, distances and areas in two and three dimensions. We will apply those techniques to further geometrical problems in chapter 14.

Vectors are an example of abstraction in mathematics: a single concept that can be applied to many different situations. Force, velocity and displacements appear to have very little in common, yet they can all be described and manipulated using the rules of vectors. In the words of the French mathematician and physicist Henry Poincaré (1854–1912): 'Mathematics is the art of giving the same name to different things'.

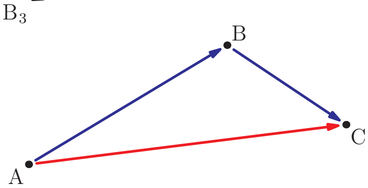
13A Positions and displacements

You may know from physics that **vectors** are used to represent quantities which have both magnitude (size) and direction, such as force or velocity. Vector quantities are different from **scalar** quantities which are fully described by a single number. In pure mathematics, vectors are used to represent displacements from one point to another, and so to describe geometrical figures.

If there is a fixed point A and a point B is 10 cm away from it, this information alone does not tell you where point B is.

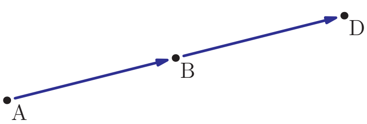


The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$. The vector contains both distance and direction information; it describes a way of getting from A to B.

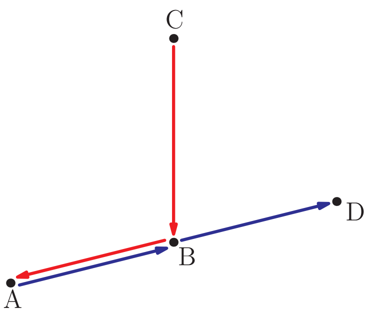


If we now add a third point, C, then there are two ways of getting from A to C: either directly, or via B. To express the second possibility using vectors, we use the addition sign to represent moving from A to B followed by moving from B to C: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Always remember that a vector represents a way of getting from one point to another, but it does not tell us anything about the actual position of the starting and the end points.



If getting from B to D involves moving the same distance and in the same direction as getting from A to B, then $\overrightarrow{BD} = \overrightarrow{AB}$.



To return from the end point to the starting point, we use the minus sign and so $\overrightarrow{BA} = -\overrightarrow{AB}$.

We can also use the subtraction sign with vectors:
$\overrightarrow{CB} - \overrightarrow{AB} = \overrightarrow{CB} + \overrightarrow{BA}$

To get from A to D we need to move in the same direction, but twice as far, as in getting from A to B. We can express this by writing $\overrightarrow{AD} = 2\overrightarrow{AB}$ or, equivalently, $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AD}$.

It is convenient to give vectors letters, as we do with variables in algebra. To emphasise that something is a vector, rather than a scalar (number) we use either bold type or an arrow on top. When writing by hand, we use underlining instead of bold type. For example, we can denote vector $\overrightarrow{AB}$ by $\boldsymbol{a}$ (or $\vec{a}$). Then in the diagrams above, $\overrightarrow{BD} = \boldsymbol{a}$, $\overrightarrow{BA} = -\boldsymbol{a}$ and $\overrightarrow{AD} = 2\boldsymbol{a}$.

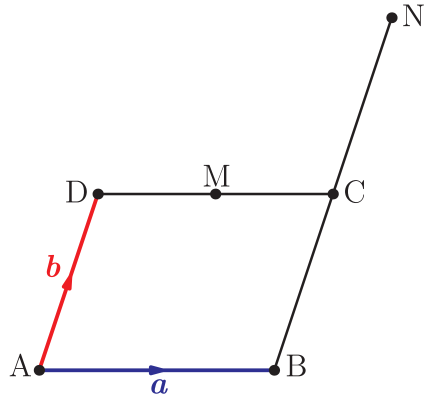
EXAM HINT

Fractions of a vector are usually written as multiples: $\frac{1}{2}\overrightarrow{AD}$, not $\frac{\overrightarrow{AD}}{2}$.

Worked example 13.1

The diagram shows a parallelogram $ABCD$.

Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of CD and N is the point on BC such that $CN = BC$.



Express vectors $\overrightarrow{CM}$, $\overrightarrow{BN}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

We can think of $\overrightarrow{CM}$ as describing a way of getting from C to M moving only along the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$

Going from C to M is the same as going half way from B to A, and $\overrightarrow{BA} = -\overrightarrow{AB}$

$$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA} = -\frac{1}{2}\boldsymbol{a}$$

Going from B to N is twice the distance and in the same direction as from B to C, and $\overrightarrow{BC} = \overrightarrow{AD}$

$$\overrightarrow{BN} = 2\overrightarrow{BC} = 2\boldsymbol{b}$$

To get from M to N we can go from M to C and then from C to M

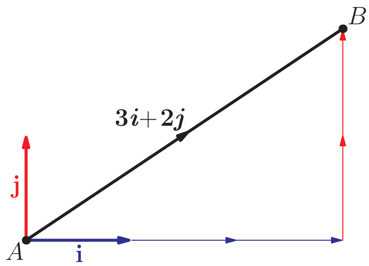
$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$$

$\overrightarrow{MC} = -\overrightarrow{CM}$ and $\overrightarrow{CN} = \overrightarrow{BC}$

$$= -\overrightarrow{CM} + \overrightarrow{BC}$$

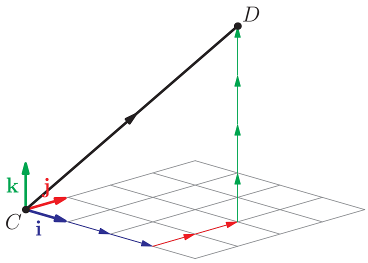
$$= \frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$$

To do further calculations with vectors we also need to describe them with numbers, not just diagrams. You are already familiar with coordinates, which are used to represent positions of points. A similar idea can be used to represent vectors.



Start in the two-dimensional plane and select two directions perpendicular to each other, and define vectors of length 1 in those two directions by $\boldsymbol{i}$ and $\boldsymbol{j}$. Then any vector in the plane can be expressed in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$, as shown in the diagram. $\boldsymbol{i}$ and $\boldsymbol{j}$ are called **base or unit vectors**.

To represent displacements in three-dimensional space, we need three base vectors, all perpendicular to each other. They are conventionally called $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.



$\vec{CD} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

An alternative notation is to use **column vectors**. In this notation, displacements shown above are written as:

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The numbers in the column are called the **components** of a vector.

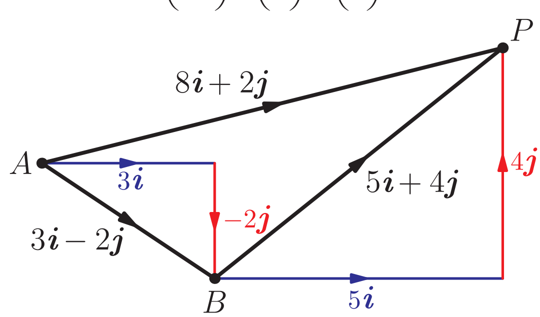
Using components in column vectors makes it easy to add displacements.

EXAM HINT

You must be familiar with both base vector and column vector notation, as both will be used in questions. When you write your answers, you can use whichever notation you prefer.

To get from A to B in the diagram below, we need to move 3 units in the $\boldsymbol{i}$ direction, and to get from B to P we need to move 5 units in the $\boldsymbol{i}$ direction; thus getting from A to P requires moving 8 units in the $\boldsymbol{i}$ direction. Similarly in the $\boldsymbol{j}$ direction we move –2 units from A to B and 4 units from B to P, making the total displacement from A to P equal to 2 units. As the total displacement from A to P is $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$, we can write it in component form as: $(3\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 4\boldsymbol{j}) = 8\boldsymbol{i} + 2\boldsymbol{j}$ or using the column vector notation as:

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$



EXAM HINT

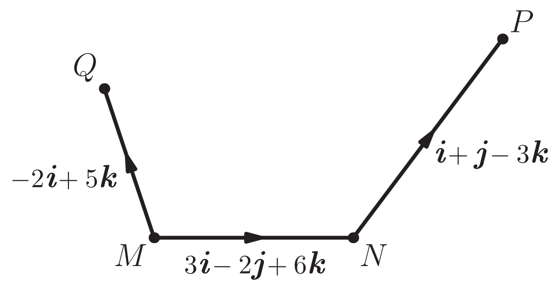
Vector diagrams do not have to be accurate or to scale to be useful. A two-dimensional sketch of a 3D situation is often enough to show you what is going on.

Reversing the direction of a vector is also simple: to get from B to A we need to move –3 units in the $\boldsymbol{i}$ direction and 2 units in the $\boldsymbol{j}$ direction and so $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

These rules for adding and subtracting vectors also apply in three dimensions.

Worked example 13.2

The diagram shows points M, N, P and Q such that $\overrightarrow{MN} = 3\boldsymbol{i}-2\boldsymbol{j}+6\boldsymbol{k}$, $\overrightarrow{NP} = \boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$ and $\overrightarrow{MQ} = -2\boldsymbol{j}+5\boldsymbol{k}$.



Write the following vectors in component form:

(a) $\overrightarrow{MP}$ (b) $\overrightarrow{PM}$ (c) $\overrightarrow{PQ}$

We can get from M to P via N

(a) $\overrightarrow{MP} = \overrightarrow{MN} + \overrightarrow{NP}$
$= (3\boldsymbol{i}-2\boldsymbol{j}+6\boldsymbol{k}) + (\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})$
$= 4\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

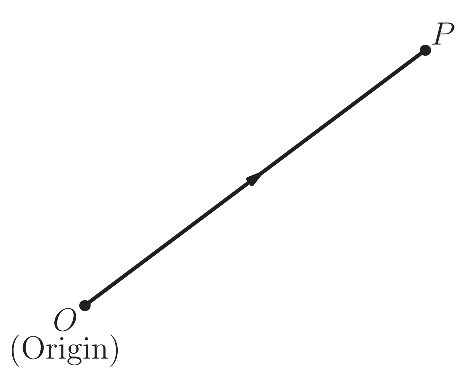
We have already found $\overrightarrow{MP}$

(b) $\overrightarrow{PM} = -\overrightarrow{MP} = -4\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$

We can get from P to Q via M using the answers to (a) and (b)

(c) $\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$
$= (-4\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}) + (-2\boldsymbol{j}+5\boldsymbol{k})$
$= -4\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$

Vectors represent displacements, but they can also represent the positions of points. If we use one fixed point, called the **origin**, then the position of a point can be described by its displacement from the origin. For example, the position of point P in the diagram can be represented by its **position vector**, $\overrightarrow{OP}$.



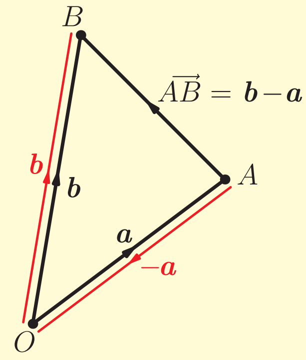
> **EXAM HINT**
> The position vector of point A is usually denoted by $\boldsymbol{a}$.

If we know position vectors of two points A and B we can find the displacement $\overrightarrow{AB}$ as shown in the diagram below: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

KEY POINT 13.1

If points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ then

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.



Position vectors are closely related to coordinates. If the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ have directions set along the coordinate axes, then the components of any position vector are simply the coordinates of the point.

Worked example 13.3

Points A and B have coordinates (3, –1, 2) and (5, 0, 3) respectively. Write as column vectors:

(a) the position vectors of A and B

(b) the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point

(a) $\underline{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$

Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

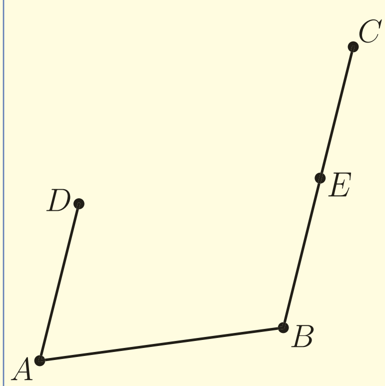
Worked example 13.4

Points A, B, C and D have position vectors $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 7 \\ 8 \\ -3 \end{pmatrix}$, $\boldsymbol{d} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

Point E is the midpoint of the line BC.

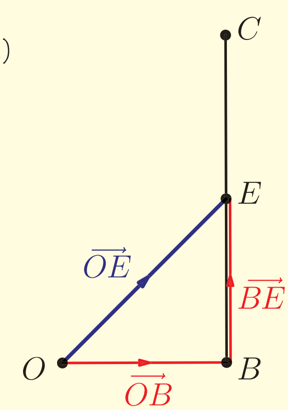
(a) Find the position vector of E.
(b) Show that $ABED$ is a parallelogram.

Draw a diagram to show what is going on



For this part, we only need to look at points B, C and E. It may help to show the origin on the diagram

(a)



$$\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE}$$
$$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$$

Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \underline{b} + \frac{1}{2}(\underline{c} - \underline{b})$$
$$= \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$$
$$= \begin{pmatrix} 2.5 \\ 0 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 4 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

continued . . .

In a parallelogram, opposite sides are equal length and parallel, which means that the vectors corresponding to those sides are equal

We need to show that $\overrightarrow{AD} = \overrightarrow{BE}$

(b) $\overrightarrow{AD} = \underline{d} - \underline{a}$

$$= \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{BE} = \underline{e} - \underline{b}$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{AD} = \overrightarrow{BE}$

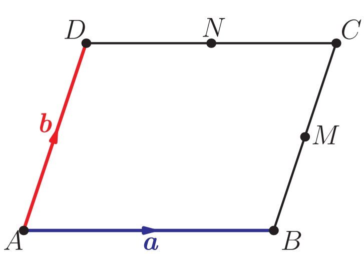
ABED is a parallelogram.

In Worked example 13.4a we derived a general formula for finding the position vector of a midpoint of a line segment.

KEY POINT 13.2

The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

Exercise 13A

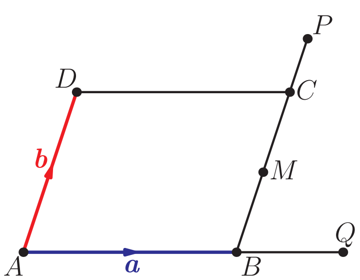


1. The diagram shows a parallelogram *ABCD* with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. *M* is the midpoint of *BC* and *N* is the midpoint of *CD*. Express the following vectors in terms of ***a*** and ***b***.

   (a) (i) $\overrightarrow{BC}$ (ii) $\overrightarrow{AC}$

   (b) (i) $\overrightarrow{CD}$ (ii) $\overrightarrow{ND}$

   (c) (i) $\overrightarrow{AM}$ (ii) $\overrightarrow{MN}$



2. In the parallelogram *ABCD*, $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. *M* is the midpoint of *BC*, *Q* is the point on *AB* such that $BQ = \frac{1}{2}AB$ and *P* is the point on the extended line *BC* such that $BC : CP = 3 : 1$, as shown on the diagram.

   Express the following vectors in terms of ***a*** and ***b***.

   (a) (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{AM}$

   (b) (i) $\overrightarrow{QD}$ (ii) $\overrightarrow{MQ}$

   (c) (i) $\overrightarrow{DQ}$ (ii) $\overrightarrow{PQ}$

3. Write the following vectors in column vector notation (in three dimensions):

(a) (i) $4\boldsymbol{i}$ (ii) $-5\boldsymbol{j}$

(b) (i) $3\boldsymbol{i}+\boldsymbol{k}$ (ii) $2\boldsymbol{j}-\boldsymbol{k}$

4. Three points O, A and B are given. Let $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$.

(a) Express $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) C is the midpoint of AB. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Point D lies on the line (AB) on the same side of B as A, so that $AD=3AB$. Express $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. *[5 marks]*

EXAM HINT

Remember that (AB) represents the infinite line through A and B, while $[AB]$ is the line segment (the part of the line between A and B).

5. Points A and B lie in a plane and have coordinates (3, 0) and (4, 2) respectively. C is the midpoint of $[AB]$.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors.

(b) Point D is such that $\overrightarrow{AD}=\begin{pmatrix}7\\-2\end{pmatrix}$.

Find the coordinates of D. *[5 marks]*

6. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$

and $\overrightarrow{OB}=\begin{pmatrix}4\\-2\\5\end{pmatrix}$.

(a) Write $\overrightarrow{AB}$ as a column vector.

(b) Find the position vector of the midpoint of $[AB]$. *[5 marks]*

7. Point A has position vector $\boldsymbol{a}=2\boldsymbol{i}-3\boldsymbol{j}$ and point D is such that $\overrightarrow{AD}=\boldsymbol{i}-\boldsymbol{j}$. Find the position vector of point D. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Point C lies on (AB) so that $AC:BC=2:3$. Find the position vector of C. *[5 marks]*

9. Points P and Q have position vectors $\boldsymbol{p}=2\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$ and $\boldsymbol{q}=\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}$.

(a) Find the position vector of the midpoint M of $[PQ]$.

(b) Point R lies on the line (PQ) such that $QR=QM$. Find the coordinates of R ($R\neq M$). *[6 marks]*

10. Points A, B and C have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find the position vector of point D such that $ABCD$ is a parallelogram. *[5 marks]*

13B Vector algebra

Although the idea of representing forces by directed line segments dates back to antiquity, the algebra of vectors was first developed in the 19th Century and was originally used to study complex numbers, which you will meet in chapter 15.

In the previous section we used vectors to describe positions and displacements of points in space, but vectors can represent quantities other than displacements; for example velocities or forces. Whatever the vectors represent, they always follow the same algebraic rules. In this section we will summarise those rules, which can be expressed using either diagrams or equations.

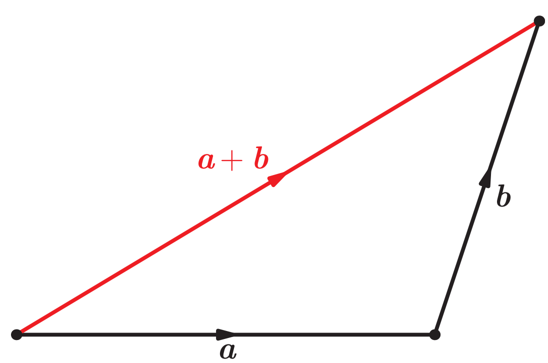
EXAM HINT

The ability to switch between diagrams and equations is essential for solving harder vector problems.

Vector addition can be done on a diagram by joining the starting point of the second vector to the end point of the first. In component form, it is carried out by adding corresponding components. When vectors represent displacements, vector addition represents one displacement followed by another.

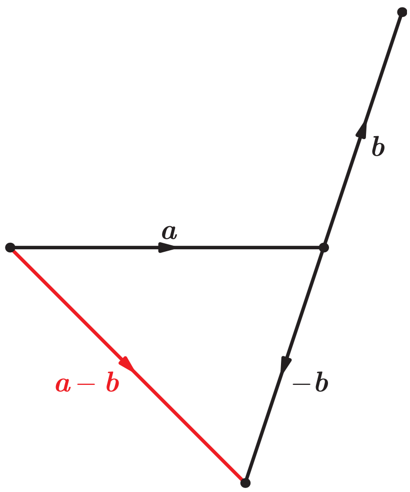
EXAM HINT

Remember that vectors only show relative positions of two points, they don't have a fixed starting point. So we are free to 'move' the second vector to the end point of the first.



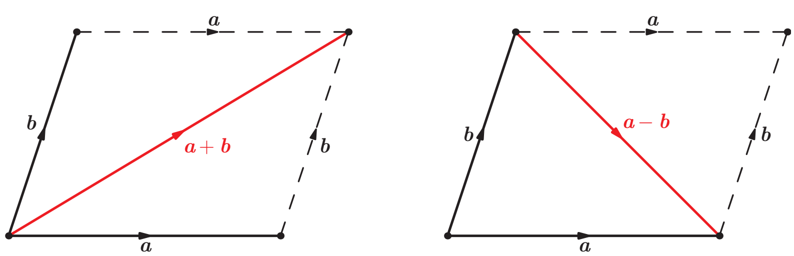
$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

Vector subtraction is the same as adding a negative vector. (−***a*** is the same length but the opposite direction to ***a***). In component form you simply subtract corresponding components.



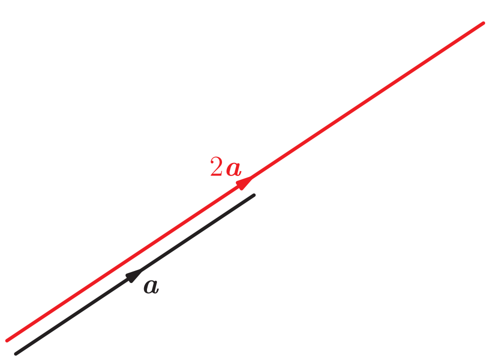
$$\begin{pmatrix}5\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\3\\-3\end{pmatrix}=\begin{pmatrix}5\\1\\-2\end{pmatrix}+\begin{pmatrix}-3\\-3\\3\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

You can also consider vector addition as the diagonal of the parallelogram formed by the two vectors. The difference of two vectors can be represented by the other diagonal of the parallelogram formed by the two vectors.

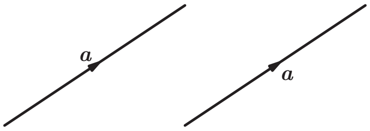


Scalar multiplication changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.

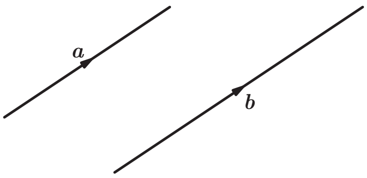
For any vector ***a***, *k**a*** represents a displacement in the same direction but with distance multiplied by *k*.



$$2\begin{pmatrix}3\\-5\\0\end{pmatrix}=\begin{pmatrix}6\\-10\\0\end{pmatrix}$$



Two vectors are **equal** if they have the same magnitude and direction. All their components are equal. They represent the same displacements but may have different start and end points.



If two vectors are in the same direction then they are **parallel**. Parallel vectors are scalar multiples of each other. This is because multiplying a vector by a scalar does not change its direction.

$$\begin{pmatrix}2\\-3\\1\end{pmatrix} \text{ is parallel to } \begin{pmatrix}6\\-9\\3\end{pmatrix} \text{ because } \begin{pmatrix}6\\-9\\3\end{pmatrix} = 3\begin{pmatrix}2\\-3\\1\end{pmatrix}.$$

KEY POINT 13.3

> If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.

The next example illustrates the vector operations we have just described.

Worked example 13.5

Given vectors $\boldsymbol{a} = \begin{pmatrix}1\\2\\7\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}-3\\4\\2\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}-2\\p\\q\end{pmatrix}$:

(a) Find $2\boldsymbol{a} - 3\boldsymbol{b}$.
(b) Find the values of p and q such that $\boldsymbol{c}$ is parallel to $\boldsymbol{a}$.
(c) Find the value of scalar k such that $\boldsymbol{a} + k\boldsymbol{b}$ is parallel to vector $\begin{pmatrix}0\\10\\23\end{pmatrix}$.

(a) $2\underline{a} - 3\underline{b} = 2\begin{pmatrix}1\\2\\7\end{pmatrix} - 3\begin{pmatrix}-3\\4\\2\end{pmatrix}$

$= \begin{pmatrix}2\\4\\14\end{pmatrix} - \begin{pmatrix}-9\\12\\6\end{pmatrix} = \begin{pmatrix}11\\-8\\8\end{pmatrix}$

continued . . .

If vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are parallel we can write $\boldsymbol{v}_2 = t\boldsymbol{v}_1$

(b) Write $\underline{c} = t\underline{a}$ for some scalar t

Then: $\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$

If two vectors are equal then all their components are equal

$$\Rightarrow \begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$\therefore p = -4,\ q = -14$

Write vector $\boldsymbol{a} + k\boldsymbol{b}$ in terms of k

(c) $\underline{a} + k\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -3k \\ 4k \\ 2k \end{pmatrix} = \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix}$

Then use $\boldsymbol{a} + k\boldsymbol{b} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

Parallel to $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

$$\Rightarrow \begin{cases} 1-3k = 0 & (1) \\ 2+4k = 10t & (2) \\ 7+2k = 23t & (3) \end{cases}$$

Find k from the first equation, but check that all three equations are satisfied

(1) $1-3k = 0 \Rightarrow k = \frac{1}{3}$

(2) $2+4\left(\frac{1}{3}\right) = 10t \Rightarrow t = \frac{1}{3}$

(3) $7+2\left(\frac{1}{3}\right) = 23\left(\frac{1}{3}\right)$ (correct)

$\therefore k = \frac{1}{3}$

Exercise 13B

1. Let $\boldsymbol{a} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Find the following vectors:

(a) (i) $3\boldsymbol{a}$ (ii) $4\boldsymbol{b}$

(b) (i) $\boldsymbol{a} - \boldsymbol{b}$ (ii) $\boldsymbol{b} + \boldsymbol{c}$

(c) (i) $2\boldsymbol{b} + \boldsymbol{c}$ (ii) $\boldsymbol{a} - 2\boldsymbol{b}$

(d) (i) $\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c}$ (ii) $3\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$

2. Let $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. Find the following vectors:

(a) (i) $-5\boldsymbol{b}$ (ii) $4\boldsymbol{a}$

(b) (i) $\boldsymbol{c} - \boldsymbol{a}$ (ii) $\boldsymbol{a} - \boldsymbol{b}$

(c) (i) $\boldsymbol{a} - \boldsymbol{b} + 2\boldsymbol{c}$ (ii) $4\boldsymbol{c} - 3\boldsymbol{b}$

3. Given that $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, find the vector $\boldsymbol{b}$ such that:

(a) $\boldsymbol{a} + \boldsymbol{b}$ is the zero vector (b) $2\boldsymbol{a} + 3\boldsymbol{b}$ is the zero vector

(c) $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{j}$ (d) $\boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{i}$

4. Given that $\boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$ find vector $\boldsymbol{x}$ such that

$3\boldsymbol{a} + 4\boldsymbol{x} = \boldsymbol{b}$. *[4 marks]*

5. Given that $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{k}$, find the value of the scalar t such that $\boldsymbol{a} + t\boldsymbol{b} = \boldsymbol{c}$. *[4 marks]*

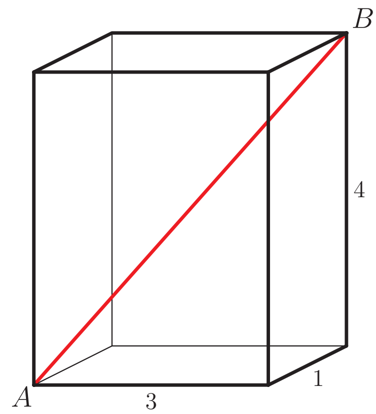
6. Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ find the value of the scalar p such that $\boldsymbol{a} + p\boldsymbol{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$. *[5 marks]*

7. Given that $\boldsymbol{x} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{y} = 4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ find the value of the scalar λ such that $\lambda x + y$ is parallel to vector $\mathbf{j}$. *[5 marks]*

8. Given that $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 2q\boldsymbol{i} + \boldsymbol{j} + q\boldsymbol{k}$ find the values of scalars p and q such that $p\boldsymbol{a} + \boldsymbol{b}$ is parallel to vector $\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$. *[6 marks]*

13C Distances

Geometry problems often involve finding distances between points. In this section we will see how to use vectors to do this.



Consider two points, A and B such that the displacement $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. The distance AB can be found by using Pythagoras' theorem in three dimensions: $AB = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$. This quantity is called the **magnitude** of $\overrightarrow{AB}$, and written as $\left|\overrightarrow{AB}\right|$.

To find the distance between A and B, using their position vectors, we first need to find the displacement vector $\overrightarrow{AB}$ and then calculate its magnitude.

Worked example 13.6

Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find the exact distance AB.

The distance is the magnitude of the displacement vector, so find $\overrightarrow{AB}$ first

$$\overrightarrow{AB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$$
$$= \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

Now use the formula for the magnitude

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$$

KEY POINT 13.4

The magnitude of a vector, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The distance between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{b} - \boldsymbol{a}|$.

EXAM HINT

Don't forget that squaring a negative number gives a positive value.

The symbol $\gtreqless$ means 'greater than, equal to or less than'. This may appear to be a useless symbol, but it highlights an important idea in vectors – they cannot be put into order. So while it is correct to say that $|v| \gtreqless |u|$ it is not possible to say the same about the vectors themselves.

We saw in Section 13B that multiplying a vector by a scalar produces a vector in the same direction but of different magnitude. In more advanced applications of vectors it is useful to be able to use vectors of length 1, called **unit vectors**. The base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are examples of unit vectors.

Worked example 13.7

(a) Find the unit vector in the same direction as $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 parallel to $\boldsymbol{a}$.

To produce a vector in the same direction but of different magnitude as $\boldsymbol{a}$, we need to multiply $\boldsymbol{a}$ by a scalar. We need to find the value of the scalar

(a) Call the required unit vector $\hat{a}$.

Then $\hat{a} = k\hat{a}$ and $\hat{a} = 1$

$|k\underline{a}| = k|\underline{a}| = 1$

$\Rightarrow k = \frac{1}{|\underline{a}|}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\therefore k = \frac{1}{3}$

Find the vector $\hat{\boldsymbol{a}}$

The unit vector is

$$\hat{a} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$$

To get a vector of magnitude 5 we need to multiply the unit vector by 5

(b) Let $\underline{b}$ be parallel to $\underline{a}$ and $|\underline{b}| = 5$

Then $\underline{b} = 5\hat{a}$

$$\therefore \underline{b} = \begin{pmatrix} \frac{10}{3} \\ \frac{-10}{3} \\ \frac{5}{3} \end{pmatrix}$$

EXAM HINT

Note that part (b) has two possible answers, as $\boldsymbol{b}$ could be in the opposite direction. To get the second answer we would take the scalar to be –5 instead of 5.

The last example showed the general method for finding the unit vector in a given direction.

KEY POINT 13.5

The unit vector in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 13C

1. Find the magnitude of the following vectors in two dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j}$ $\quad \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

2. Find the magnitude of the following vectors in three dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{d} = \boldsymbol{j} - \boldsymbol{k}$

3. Find the distance between the following pairs of points in the plane.

(a) (i) A (1, 2) and B (3, 7) (ii) C (2, 1) and D (1, 2)

(b) (i) P (–1, –5) and Q (–4, 2) (ii) M (1, 0) and N (0, –2)

4. Find the distance between the following pairs of points in three dimensions.

(a) (i) A (1, 0, 2) and B (2, 3, 5)

(ii) C (2, 1, 7) and D (1, 2, 1)

(b) (i) P (3, –1,–5) and Q (–1, –4, 2)

(ii) M (0, 0, 2) and N (0, –3, 0)

5. Find the distance between the points with the given position vectors.

(a) $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$

(b) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

(c) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

(d) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{j} - \boldsymbol{k}$

6. (a) (i) Find a unit vector parallel to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

(ii) Find a unit vector parallel to $6\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

(b) (i) Find a unit vector in the same direction as $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

(ii) Find a unit vector in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

7. Find the possible values of the constant c such that the vector $\begin{pmatrix} 2c \\ c \\ -c \end{pmatrix}$ has magnitude 12. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

C is the midpoint of $[AB]$. Find the exact distance AC. *[4 marks]*

9. Let $\boldsymbol{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$. *[6 marks]*

10. (a) Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 3 in the same direction as $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. *[6 marks]*

11. Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and

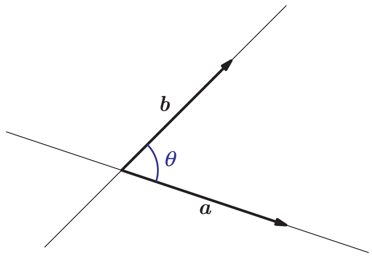
$\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ where O is the origin.

Find the possible values of t such that $AB = 3$. *[5 marks]*

12. Points P and Q have position vectors $\boldsymbol{p} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{q} = (2+t)\boldsymbol{i} + (1-t)\boldsymbol{j} + (1+t)\boldsymbol{k}$. Find the value of t for which the distance PQ is the minimum possible and find this minimum distance. *[6 marks]*

13D Angles

In geometry problems you are often asked to find angles between two lines. The diagram shows two lines with angle θ between them. $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in the directions of the two lines. Note that both arrows are pointing *away* from the intersection point. It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors. This result can be derived using the cosine rule. See Fill-in proof sheet 12 'Deriving scalar products' on the CD-ROM.



KEY POINT 13.6

If θ is the angle between vectors, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

The expression in the numerator of the above fraction has many important uses, and is called the **scalar product**.

KEY POINT 13.7

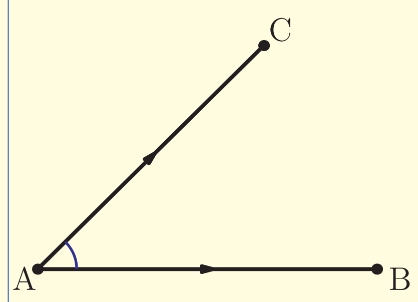
Scalar product

The quantity $a_1b_1 + a_2b_2 + a_3b_3$ is called the scalar product (**inner product**, or **dot product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a}\bullet\boldsymbol{b}$.

Worked example 13.8

Given points A (3, −5, 2), B (4, 1, 1) and C (−1, 1, 2) find the size of the angle $B\hat{A}C$ in degrees.

It's always a good idea to draw a diagram to be sure which vectors the angle lies between



The required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

We need the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix}4\\1\\1\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}1\\6\\-1\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}-4\\6\\0\end{pmatrix}$$

$$\cos\theta = \frac{1\times(-4)+6\times 6+(-1)\times 0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\therefore \theta = \arccos(0.7199) = 44.0°$$

It is very straightforward to check whether two vectors are perpendicular. If $\theta = 90°$ then $\cos\theta = 0$, so the top of the fraction in the formula for $\cos\theta$ must be zero. We do not even have to calculate the magnitudes of the two vectors.

KEY POINT 13.8

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Worked example 13.9

If $\boldsymbol{p} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ find the value of the scalar t such that $\boldsymbol{p} + t\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

Two vectors are perpendicular if their dot product equals 0

$$(\underline{p} + t\underline{q}) \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Find the components of $\boldsymbol{p} + t\boldsymbol{q}$ in terms of t

$$\underline{p} + t\underline{q} = \begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix}$$

So

$$\begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Form and solve the equation

$$\Leftrightarrow 3(4+2t) + 5(-1+t) + 1(2+t) = 0$$
$$\Leftrightarrow 9 + 12t = 0$$
$$\Leftrightarrow \quad t = -\frac{3}{4}$$

Exercise 13D

1. Calculate the angle between the following pairs of vectors, giving your answers in radians.

(a) (i) $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

(b) (i) $2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{i} - 2\boldsymbol{k}$

(c) (i) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{i} - \boldsymbol{j}$ and $2\boldsymbol{i} + 3\boldsymbol{j}$

2. The angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is θ. Find the exact value of $\cos\theta$ in the following cases:

(a) (i) $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

(ii) $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

(b) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(c) (i) $\boldsymbol{a} = -2\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i}$ (ii) $\boldsymbol{a} = 5\boldsymbol{i}$ and $\boldsymbol{b} = 3\boldsymbol{j}$

3. (a) The vertices of a triangle have position vectors

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{c} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.$$

Find, in degrees, the angles of the triangle.

(b) Find, in degrees, the angles of the triangle with vertices (2, 1, 2), (4, –1, 5) and (7, 1, –2).

4. Which of the following pairs of vectors are perpendicular?

(a) (i) $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

(b) (i) $5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$

(ii) $\boldsymbol{i} - 3\boldsymbol{k}$ and $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

5. Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$.

Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$. *[5 marks]*

6. Four points are given with coordinates A (2, –1, 3), B (1, 1, 2), C (6, –1, 2) and D (7, –3, 3).
Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$. *[5 marks]*

7. Four points have coordinates A (2, 4, 1), B (k, 4, $2k$), C ($k + 4$, $2k + 4$, $2k + 2$) and D (6, $2k + 4$, 3).

(a) Show that $ABCD$ is a parallelogram for all values of k.

(b) When $k = 1$ find the angles of the parallelogram.

(c) Find the value of k for which $ABCD$ is a rectangle.

[8 marks]

8. Vertices of a triangle have position vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{c} = 5\boldsymbol{i}$.

(a) Show that the triangle is right-angled.

(b) Calculate the other two angles of the triangle.

(c) Find the area of the triangle. *[8 marks]*

13E Properties of the scalar product

In the last section we defined the scalar product of vectors

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as

$$\boldsymbol{a} \bullet \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and saw that if θ is the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ then:

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

In this section we look at various properties of the scalar product in more detail; in particular, its algebraic rules. The scalar product has many properties similar to multiplication of numbers. These properties can be proved by using components of the vectors.

KEY POINT 13.9

Algebraic properties of the scalar product

$$\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$$

$$(-\boldsymbol{a}) \bullet \boldsymbol{b} = -(\boldsymbol{a} \bullet \boldsymbol{b})$$

$$(k\boldsymbol{a}) \bullet \boldsymbol{b} = k(\boldsymbol{a} \bullet \boldsymbol{b})$$

$$\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{b}) + (\boldsymbol{a} \bullet \boldsymbol{c})$$

But there are some properties of multiplication of numbers which do *not* apply to scalar product. For example, it is not possible to calculate the scalar product of three vectors: the expression $(\boldsymbol{a} \bullet \boldsymbol{b}) \bullet \boldsymbol{c}$ has no meaning, as $\boldsymbol{a} \bullet \boldsymbol{b}$ is a scalar (and so has no direction), and scalar product involves multiplying two vectors.

Two important properties of scalar product concern perpendicular and parallel vectors. These are very useful when solving geometry problems.

All the operations with vectors work in both two and three dimensions. If there were a fourth dimension, the position of each point could be described using four numbers. We could use analogous rules to calculate 'distances' and 'angles'. Does this mean that we can acquire knowledge about a four-dimensional world which we can't see, or even imagine?

EXAM HINT

These are not in the formula booklet!

KEY POINT 13.10

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors then $\boldsymbol{a}\cdot\boldsymbol{b}=0$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a}\cdot\boldsymbol{a}=|\boldsymbol{a}|^2$.

The following two examples show you how you can use these rules.

Worked example 13.10

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors such that $|\boldsymbol{a}|=5$ and $|\boldsymbol{b}|=3$, evaluate $(2\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}+4\boldsymbol{b})$.

Multiply out the brackets as we would with numbers $\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\bullet\boldsymbol{b}+\boldsymbol{a}\bullet\boldsymbol{c}$

$(2\underline{a}-\underline{b})\bullet(\underline{a}+4\underline{b})=2\underline{a}\bullet\underline{a}+8\underline{a}\bullet\underline{b}-\underline{b}\bullet\underline{a}-4\underline{b}\bullet\underline{b}$

$\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular so $\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{b}\bullet\boldsymbol{a}=0$

$=2\underline{a}\bullet\underline{a}-4\underline{b}\bullet\underline{b}$

Use the fact that $\boldsymbol{a}\bullet\boldsymbol{a}=|\boldsymbol{a}|^2$

$=2|\underline{a}|^2-4|\underline{b}|^2$

$=2\times5^2-4\times3^2$

$=14$

Worked example 13.11

Points A, B and C have position vectors $\boldsymbol{a}=k\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}3\\4\\-2\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}1\\1\\5\end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which (AB) is perpendicular to (BC).

Use $\overrightarrow{BC}=\boldsymbol{c}-\boldsymbol{b}$

(a) $\overrightarrow{BC}=\boldsymbol{c}-\boldsymbol{b}$

$=\begin{pmatrix}1\\1\\5\end{pmatrix}-\begin{pmatrix}3\\4\\-2\end{pmatrix}=\begin{pmatrix}-2\\-3\\7\end{pmatrix}$

Use $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$

(b) $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$

$=\begin{pmatrix}3\\4\\-2\end{pmatrix}-\begin{pmatrix}3k\\-k\\k\end{pmatrix}=\begin{pmatrix}3-3k\\4+k\\-2-k\end{pmatrix}$

continued . . .

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular then $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$

(c) $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$

$$\begin{pmatrix} 3-3k \\ 4+k \\ -2-2k \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = 0$$

$\Rightarrow -6+6k-12-3k-14-14k = 0$

$\Rightarrow$ $-11k = 32$

$k = -\frac{11}{32}$

Exercise 13E

1. Evaluate $\boldsymbol{a} \bullet \boldsymbol{b}$ in the following cases:

(a) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -12 \\ 4 \\ -8 \end{pmatrix}$

(b) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$

(c) (i) $\boldsymbol{a} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(d) (i) $\boldsymbol{a} = -3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{k}$

(ii) $\boldsymbol{a} = -3\boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{k}$

2. Given that θ is the angle between vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ find the exact value of $\cos\theta$.

(a) (i) $\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) (i) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

3. (a) Given that $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 5$ and $\boldsymbol{a} \bullet \boldsymbol{b} = 10$, find, in degrees, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Given that $|\boldsymbol{c}| = 9$, $|\boldsymbol{d}| = 12$ and $\boldsymbol{c} \bullet \boldsymbol{d} = -15$, find, in degrees, the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$.

4. (a) Given that $|\boldsymbol{a}|=6$, $|\boldsymbol{b}|=4$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $37°$, calculate $\boldsymbol{a}\cdot\boldsymbol{b}$.

(b) Given that $|\boldsymbol{a}|=8$, $\boldsymbol{a}\cdot\boldsymbol{b}=12$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, find the exact value of $|\boldsymbol{b}|$.

5. Given that $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$, $\boldsymbol{b}=\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{c}=5\boldsymbol{i}-3\boldsymbol{k}$ and $\boldsymbol{d}=-2\boldsymbol{j}+\boldsymbol{k}$ verify that:

(a) $\boldsymbol{b}\cdot\boldsymbol{d}=\boldsymbol{d}\cdot\boldsymbol{b}$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$

(c) $(\boldsymbol{c}-\boldsymbol{d})\cdot\boldsymbol{c}=|\boldsymbol{c}|^2-\boldsymbol{c}\cdot\boldsymbol{d}$

(d) $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}$

6. Find the values of t for which the following pairs of vectors are perpendicular.

(a) (i) $\begin{pmatrix}2t\\1\\-3t\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ (ii) $\begin{pmatrix}t+1\\2t-1\\2t\end{pmatrix}$ and $\begin{pmatrix}2\\6\\0\end{pmatrix}$

(b) (i) $5t\boldsymbol{i}-(2+t)\boldsymbol{j}+\boldsymbol{k}$ and $3\boldsymbol{i}+4\boldsymbol{j}-t\boldsymbol{k}$

(ii) $t\boldsymbol{i}-3\boldsymbol{k}$ and $2t\boldsymbol{i}+\boldsymbol{j}+t\boldsymbol{k}$

We will meet equations of this type in the next chapter. See Worked example 14.17.

7. In this question, we will introduce a method using scalar product to find x and y such that $\begin{pmatrix}4\\1\end{pmatrix}+\begin{pmatrix}2x\\-3x\end{pmatrix}=\begin{pmatrix}-2\\5\end{pmatrix}+\begin{pmatrix}3y\\y\end{pmatrix}$, and then use it to solve other similar equations.

(a) Use the usual method of simultaneous equations to find x and y.

(b) (i) Find the scalar product of both sides of the equation with $\begin{pmatrix}1\\-3\end{pmatrix}$ and hence find x.

(ii) Find the scalar product of both sides of the equation with $\begin{pmatrix}3\\2\end{pmatrix}$ and hence find y.

(iii) Can you see why the vectors $\begin{pmatrix}1\\-3\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$ were selected?

(c) (i) Find a vector perpendicular to $\begin{pmatrix}2\\-1\end{pmatrix}$ and a vector perpendicular to $\begin{pmatrix}5\\3\end{pmatrix}$.

(ii) Hence find x and y such that

$$\begin{pmatrix}1\\1\end{pmatrix}+x\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}+y\begin{pmatrix}5\\3\end{pmatrix}$$

8. Given that $\boldsymbol{a}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\boldsymbol{c}=\begin{pmatrix}3\\-5\\1\end{pmatrix}$ and $\boldsymbol{d}=\begin{pmatrix}3\\-3\\2\end{pmatrix}$: calculate:

(a) $\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c})$

(b) $(\boldsymbol{b}-\boldsymbol{a})\bullet(\boldsymbol{d}-\boldsymbol{c})$

(c) $(\boldsymbol{b}+\boldsymbol{d})\bullet(2\boldsymbol{a})$ *[7 marks]*

9. (a) If $\boldsymbol{a}$ is a unit vector perpendicular to $\boldsymbol{b}$, find the value of $\boldsymbol{a}\bullet(2\boldsymbol{a}-3\boldsymbol{b})$.

(b) If $\boldsymbol{p}$ is a unit vector making a 45° angle with vector $\boldsymbol{q}$ and $\boldsymbol{p}\bullet\boldsymbol{q}=3\sqrt{2}$, find $|\boldsymbol{q}|$. *[6 marks]*

10. (a) $\boldsymbol{a}$ is a vector of magnitude 3 and $\boldsymbol{b}$ makes an angle of 60° with $\boldsymbol{a}$. Given that $\boldsymbol{a}\bullet(\boldsymbol{a}-\boldsymbol{b})=\frac{1}{3}$, find the exact value of $|\boldsymbol{b}|$.

(b) Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors of equal magnitude such that $(3\boldsymbol{a}+\boldsymbol{b})$ is perpendicular to $(\boldsymbol{a}-3\boldsymbol{b})$, prove that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. *[6 marks]*

11. Points A, B and C have position vectors $\boldsymbol{a}=\boldsymbol{i}-19\boldsymbol{j}+5\boldsymbol{k}$, $\boldsymbol{b}=2\lambda\boldsymbol{i}+(\lambda+2)\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c}=-6\boldsymbol{i}-15\boldsymbol{j}+7\boldsymbol{k}$.

(a) Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above:

(b) find the angles of the triangle ABC

(c) find the area of the triangle ABC. *[8 marks]*

12. $ABCD$ is a parallelogram with AB parallel to DC. Let $\overrightarrow{AB}=\boldsymbol{a}$ and $\overrightarrow{AD}=\boldsymbol{b}$.

(a) Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Simplify $(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{b}-\boldsymbol{a})$.

(c) Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular. *[8 marks]*

13. Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

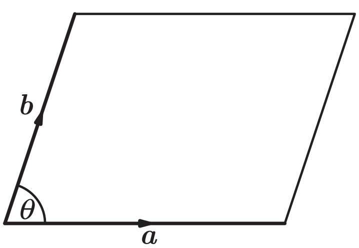
(a) Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

(b) Find the value of λ for which $C\hat{B}A$ is a right angle.

(c) For the value of λ found above, calculate the exact distance from C to the line OA. *[8 marks]*

13F Areas



Given the coordinates of the four vertices of a parallelogram, how can we calculate its area? The area of the parallelogram is given by $ab \sin\theta$ where a and b are the lengths of the sides and θ is the angle between them. We could use the coordinates of the vertices to find the lengths of the sides, and then use the cosine rule to find angle θ. However, using vectors gives a quicker way to calculate the area.

KEY POINT 13.11

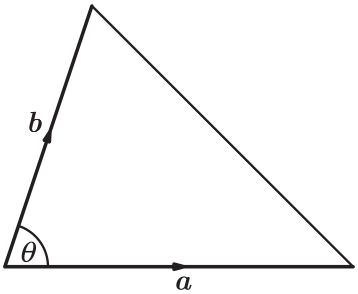
The area of the parallelogram with sides defined by vectors $\boldsymbol{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ is equal to the magnitude of the vector $\begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$.

This vector is called the **vector product** (or **cross product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \times \boldsymbol{b}$.

EXAM HINT

Notice that Notice that vectors a and b form two adjacent sides of the parallelogram. We can use any pair of adjacent sides.

A parallelogram can be divided in half to form two triangles, so we can also use the vector product to calculate the area of a triangle.



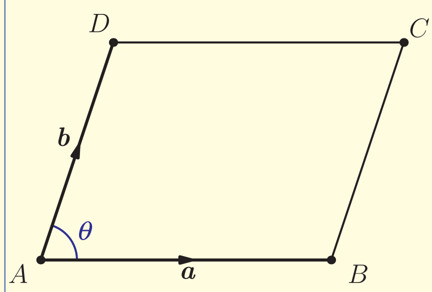
KEY POINT 13.12

The area of the triangle with two sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$.

Worked example 13.12

Find the area of the parallelogram with vertices $A\begin{pmatrix}1\\4\\2\end{pmatrix}$, $B\begin{pmatrix}3\\-3\\0\end{pmatrix}$, $C\begin{pmatrix}1\\1\\7\end{pmatrix}$, $D\begin{pmatrix}-1\\8\\9\end{pmatrix}$.

Draw a diagram to show which two vectors to use: we can choose any two adjacent sides of the parallelogram, e.g. *AB* and *AD*



$$\boldsymbol{a} = \overrightarrow{AB} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}$$

$$\boldsymbol{b} = \overrightarrow{AD} = \begin{pmatrix}-2\\4\\7\end{pmatrix}$$

Calculate $\boldsymbol{a}\times\boldsymbol{b}$ first, and then find its magnitude

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}\times\begin{pmatrix}-2\\4\\7\end{pmatrix}$$

$$= \begin{pmatrix}-49+8\\4-14\\8-14\end{pmatrix} = \begin{pmatrix}-41\\-10\\-6\end{pmatrix}$$

$$\text{Area} = |\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{41^2+10^2+6^2} = 55.4$$

Exercise 13F

1. Calculate $\boldsymbol{a}\times\boldsymbol{b}$ for the following pairs of vectors:

(a) (i) $\boldsymbol{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}5\\2\\2\end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}-12\\4\\-8\end{pmatrix}$

(b) (i) $\boldsymbol{a} = 4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

(ii) $\boldsymbol{a} = -3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i}-4\boldsymbol{k}$

2. Calculate the area of the triangle with vertices:

(a) (i) (1, 3, 3), (–1, 1, 2) and (1, –2, 4)
(ii) (3, –5, 1), (–1, 1, 3) and (–1, 5, 2)

(b) (i) (–3, –5, 1), (4, 7, 2) and (–1, 2, 2)
(ii) (4, 0, 2), (4, 1, 5) and (4, –3, 2)

3. Given points A, B and C with coordinates (3, –5, 1), (7, 7, 2) and (–1, 1, 3).

(a) calculate $\boldsymbol{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\boldsymbol{q} = \overrightarrow{BA} \times \overrightarrow{BC}$.

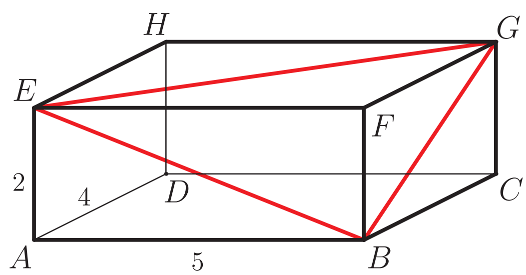
(b) What can you say about vectors $\boldsymbol{p}$ and $\boldsymbol{q}$?

4. The points A (3, 1, 2), B (–1, 1, 5) and C (7, 2, 3) are vertices of a parallelogram $ABCD$.

(a) Find the coordinates of D.

(b) Calculate the area of the parallelogram.

5. A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are A (0, 0, 0), B (5, 0, 0), C (5, 4, 0) and E (0, 0, 2).



(a) Find the coordinates of the remaining four vertices.

Face diagonals BE, BG and EG are drawn as shown.

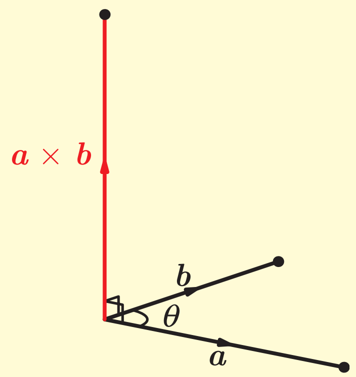
(b) Find the area of the triangle BEG.

13G Properties of the vector product

In this section we look at more properties of the vector product $\boldsymbol{a} \times \boldsymbol{b}$. We have already seen that the magnitude of this vector is equal to the area of the parallelogram defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

KEY POINT 13.13

The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in the diagram.



In component form, $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}$.

If you study physics, you may have come across the 'right-hand rule' for the direction of the magnetic field. This is just one example of application of vector product; others include circular motion, fluid dynamics and Maxwell's theory of electromagnetism.

EXAM HINT

The Formula booklet gives you the equations, but not the diagrams.

Worked example 13.13

Find the exact value of the sine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that

$|\boldsymbol{a}|=3$, $|\boldsymbol{b}|=2$ and $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}3\\6\\1\end{pmatrix}$.

We are not given the components of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ so need to use the definition of the vector product involving magnitudes

$$|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$$
$$\sqrt{3^2+6^2+1^2}=(3\times2)\sin\theta$$
$$\sqrt{46}=6\sin\theta$$
$$\therefore \sin\theta=\frac{\sqrt{46}}{6}$$

The fact that the vector product is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is very useful.

Worked example 13.14

Find a vector perpendicular to both $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$.

The vector product of two vectors is perpendicular to both of them

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix}$$

You may find the two different ways of multiplying vectors confusing. However, if you think about normal multiplication, you will realise that it can have at least two very different interpretations: It can be considered as repeated addition, taking two numbers and producing a third number as the answer; or the result can represent the area of a rectangle with given lengths of sides. The two 'types' of multiplication of numbers just happen to give the same numerical answer.

The vector product has many properties similar to multiplication of numbers, but the most important difference is that $\boldsymbol{a} \times \boldsymbol{b}$ is not the same as $\boldsymbol{b} \times \boldsymbol{a}$.

KEY POINT 13.14

Algebraic properties of vector product

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$
$$(k\boldsymbol{a}) \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b})$$
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$$

With the vector product it is possible to multiply three vectors together, but $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is not the same as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

Again, there are special results concerning parallel and perpendicular vectors.

KEY POINT 13.15

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = 0$. In particular, $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$.

Worked example 13.15

Given that $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, evaluate $|(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b})|$.

Expand the brackets as we would with numbers $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

$$(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b}) = 2\boldsymbol{a} \times \boldsymbol{a} + 6\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a} - 3\boldsymbol{b} \times \boldsymbol{b}$$

continued . . .

$\boldsymbol{a}\times\boldsymbol{a}=0$

$\boldsymbol{b}\times\boldsymbol{a}=-\boldsymbol{a}\times\boldsymbol{b}$

For perpendicular vectors, $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|$

$$=6\boldsymbol{a}\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{a}$$

$$=7\boldsymbol{a}\times\boldsymbol{b}$$

$$\therefore |(2\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}+3\boldsymbol{b})|=7|\boldsymbol{a}\times\boldsymbol{b}|$$
$$=7|\boldsymbol{a}||\boldsymbol{b}|=140$$

Exercise 13G

1. Find, in radians, the acute angle between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that:

(a) $|\boldsymbol{a}|=2,\ |\boldsymbol{b}|=5$ and $|\boldsymbol{a}\times\boldsymbol{b}|=7$

(b) $|\boldsymbol{a}|=12,\ |\boldsymbol{b}|=3$ and $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}2\\1\\4\end{pmatrix}$

(c) $|\boldsymbol{a}|=7,\ |\boldsymbol{b}|=1$ and $\boldsymbol{a}\times\boldsymbol{b}=2\boldsymbol{i}-3\boldsymbol{j}-2\boldsymbol{k}$

(d) $|\boldsymbol{a}|=4,\ |\boldsymbol{b}|=4$ and $|\boldsymbol{a}\times\boldsymbol{b}|=0$

2. Find a vector perpendicular to the following pairs of vectors:

(a) (i) $\begin{pmatrix}4\\1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$ (ii) $\begin{pmatrix}3\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\5\end{pmatrix}$

(b) (i) $\begin{pmatrix}1\\7\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\-3\end{pmatrix}$ (ii) $\begin{pmatrix}0\\0\\2\end{pmatrix}$ and $\begin{pmatrix}2\\0\\0\end{pmatrix}$

3. Find the unit vector perpendicular to the following pairs of vectors:

(a) $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\5\end{pmatrix}$ (b) $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$

4. Given that $|\boldsymbol{a}|=5,\ |\boldsymbol{b}|=7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$ find the exact value of $|\boldsymbol{a}\times\boldsymbol{b}|$. *[4 marks]*

5. (a) Prove that $(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}+\boldsymbol{b})=2\boldsymbol{a}\times\boldsymbol{b}$.

(b) Simplify $(2\boldsymbol{a}-3\boldsymbol{b})\times(3\boldsymbol{a}+2\boldsymbol{b})$. *[6 marks]*

6. (a) Explain why $(\boldsymbol{a}\times\boldsymbol{b})\bullet\boldsymbol{a}=0$.

(b) Evaluate $(\boldsymbol{a}\times\boldsymbol{b})\bullet(\boldsymbol{a}-\boldsymbol{b})$. *[5 marks]*

7. Prove that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $|\boldsymbol{a}\times\boldsymbol{b}|^2+(\boldsymbol{a}\bullet\boldsymbol{b})^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2$. *[5 marks]*

Summary

- The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$.
- Vectors can be expressed in terms of **base vectors** $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ or as **column vectors** using **components**. For example, $\overrightarrow{AB}$ can be represented by $(3\boldsymbol{i}+2\boldsymbol{j})$ or $\begin{pmatrix}3\\2\end{pmatrix}$. The numbers 3 and 2 are the components.
- Vectors can represent the position of points relative to the **origin**; the displacement of a point from the origin is the point's **position vector**.
- The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.
- The displacement between points A and B with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{b}-\boldsymbol{a}$.
- The distance between the points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{b}-\boldsymbol{a}|$.
- The **unit vector** in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}}=\frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b}=t\boldsymbol{a}$ for some scalar t.
- The magnitude of a vector $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ is $|\boldsymbol{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$.
- The angle θ, between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by: $\cos\theta=\frac{\boldsymbol{a}\bullet\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ where $\boldsymbol{a}\bullet\boldsymbol{b}$ is the **scalar product** (dot product), given in terms of the components by: $\boldsymbol{a}\bullet\boldsymbol{b}=a_1b_1+a_2b_2+a_3b_3$
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\bullet\boldsymbol{b}=0$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a}\bullet\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a}\bullet\boldsymbol{a}=|\boldsymbol{a}|^2$.
- Algebraic properties of scalar product:

$$\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{b}\bullet\boldsymbol{a}$$
$$(-\boldsymbol{a})\bullet\boldsymbol{b}=-(\boldsymbol{a}\bullet\boldsymbol{b})$$
$$(k\boldsymbol{a})\bullet\boldsymbol{b}=k(\boldsymbol{a}\bullet\boldsymbol{b})$$
$$\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\bullet\boldsymbol{b})+(\boldsymbol{a}\bullet\boldsymbol{c})$$

- The area of the parallelogram with sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the magnitude of the vector $\boldsymbol{a}\times\boldsymbol{b}$, which is called the **vector product** (or cross product). Its direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and it is given by:

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

- The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Algebraic properties of vector product:

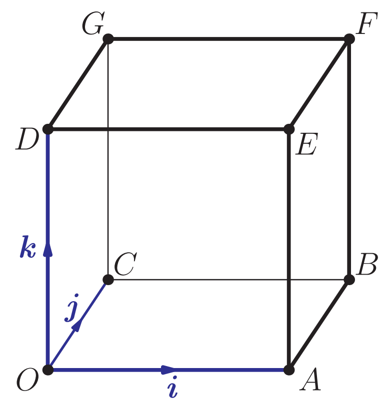
$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$
$$(k\boldsymbol{a})\times\boldsymbol{b} = k\ (\boldsymbol{a}\times\boldsymbol{b})$$
$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})$$

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{0}$. In particular, $\boldsymbol{a}\times\boldsymbol{a} = 0$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$.

Introductory problem revisited

What is the angle between the diagonals of a cube?

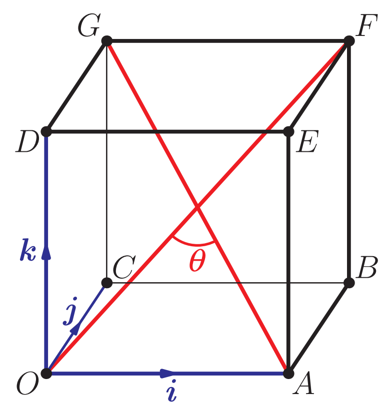
You can solve this problem by using the cosine rule in a triangle made by the diagonals and one side. However, using vectors gives a slightly faster solution, as we do not have to find the lengths of the sides of the triangle.



The angle between two lines can be found by using the vectors in the directions of two lines and the formula involving the scalar product. We do not know the actual positions of the vertices of the cube, or even the lengths of its sides, but as the answer does not depend on the size of the cube, we can look at the cube with side length 1, set with one vertex at the origin and sides parallel to the base vectors.

We want to find the angle between the diagonals OF and AG, so we need the coordinates of those four vertices. They are: $O(0, 0, 0)$, $A(1, 0, 0)$, $F(1, 1, 1)$, $G(0, 1, 1)$.

The required angle θ is between the lines OF and AG.



The corresponding vectors are: $\overrightarrow{OF} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad \overrightarrow{AG} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$

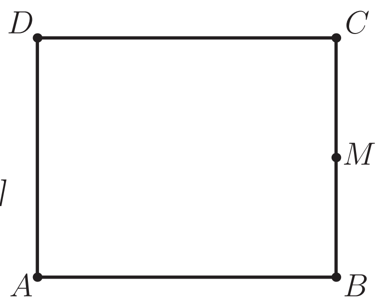
$$\cos\theta = \frac{\overrightarrow{OF}\bullet\overrightarrow{AG}}{|\overrightarrow{OF}||\overrightarrow{AG}|}$$
$$= \frac{-1+1+1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$
$$\therefore \theta = 70.5°$$

Mixed examination practice 13

Short questions

1. Given that $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$, find $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. *[6 marks]*

2. The diagram shows a rectangle $ABCD$. M is the midpoint of BC.



(a) Express MD in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

(b) Given that $AB = 6$ and $AD = 4$, show that $\overrightarrow{MD} \cdot \overrightarrow{MC} = 4$. *[5 marks]*

3. The position vectors of points N and L are:

$$\boldsymbol{n} = 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$$
$$\boldsymbol{l} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

(a) Find the vector product $\boldsymbol{n} \times \boldsymbol{l}$.

(b) Using your answer to part (a), or otherwise, find the area of the parallelogram with two sides $\overline{ON}$ and $\overline{OL}$. *[6 marks]*

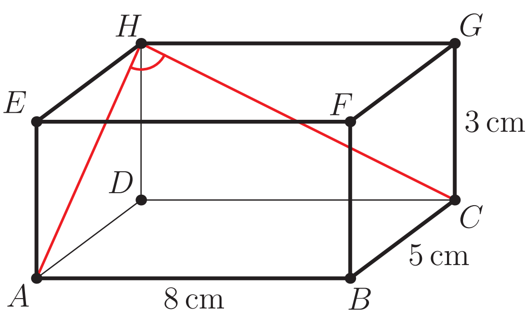
4. Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \\ p \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$.

(b) Find the value of p, given that $\boldsymbol{a} \times \boldsymbol{b}$ is parallel to $\boldsymbol{c}$. *[6 marks]*

5. The rectangle box shown in the diagram has dimensions 8 cm × 5 cm × 3 cm.

Find, correct to the nearest one-tenth of a degree, the size of the angle $A\hat{H}C$. *[6 marks]*



6. Let α be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ and $\boldsymbol{b} = (\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j}$ and $0 < \theta < \pi/4$. Express α in terms of θ.
(© *IB Organization 2000*) *[6 marks]*

7. Given two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}|$, find the value of $\boldsymbol{a}\bullet\boldsymbol{b}$.
(© *IB Organization 2002*) *[6 marks]*

8. (a) Show that $(\boldsymbol{b}-\boldsymbol{a})\bullet(\boldsymbol{b}-\boldsymbol{a}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a}\bullet\boldsymbol{b}$.

(b) In triangle MNP, $M\hat{P}N = \theta$. Let $\overrightarrow{PM} = \boldsymbol{a}$ and $\overrightarrow{PN} = \boldsymbol{b}$. Use the result from part (a) to prove the cosine rule: $MN^2 = PM^2 + PN^2 - 2PM\,PN\cos\theta$. *[6 marks]*

Long questions

1. Points A, B and D have coordinates (1, 1, 7), (–1, 6, 3) and (3, 1, k), respectively. AD is perpendicular to AB.

(a) Write down, in terms of $\boldsymbol{k}$, the vector $\overrightarrow{AD}$.

(b) Show that $\boldsymbol{k} = 6$.

Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.

(c) Find the coordinates of C.

(d) Find the exact value of $\cos(A\hat{D}C)$. *[10 marks]*

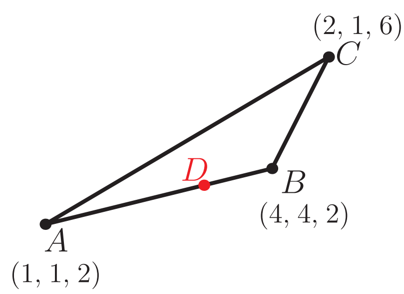
2. A triangle has vertices $A(1, 1, 2)$, $B(4, 4, 2)$ and $C(2, 1, 6)$. Point D lies on the side AB and $AD:DB = 1:k$.

(a) Find $\overrightarrow{CD}$ in terms of k.

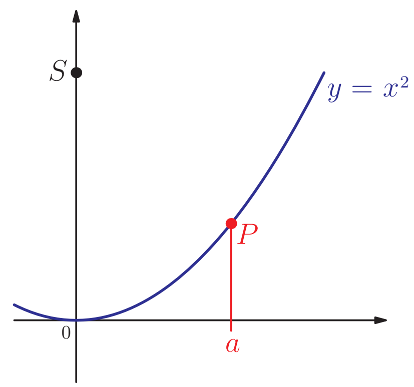
(b) Find the value of k such that CD is perpendicular to AB.

(c) For the above value of k, find the coordinates of D.

(d) Hence find the length of the altitude from vertex C. *[10 marks]*



3. Point P lies on the parabola $y = x^2$ and has x-coordinate a ($a > 0$).



(a) Write down, in terms of a, the coordinates of P.

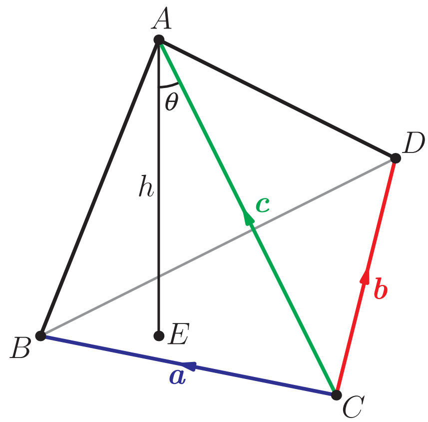
Point S has coordinates (0, 4) and O is the origin.

(b) Write down the vectors $\overrightarrow{PO}$ and $\overrightarrow{PS}$.

(c) Use scalar product to find the value of a for which OP is perpendicular to PS.

(d) For the value of a found above, calculate the exact area of the triangle OPS. *[10 marks]*

4. Consider the tetrahedron shown in the diagram and define vectors $\boldsymbol{a} = \overrightarrow{CB}$, $\boldsymbol{b} = \overrightarrow{CD}$ and $\boldsymbol{c} = \overrightarrow{CA}$.



(a) Write down an expression for the area of the base in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ only.

(b) AE is the height of the tetrahedron, $|AE| = h$ and $C\hat{A}E = \theta$. Express h in terms of $\boldsymbol{c}$ and θ.

(c) Use the results of (a) and (b) to prove that the volume of the tetrahedron is given by $\frac{1}{6}|\boldsymbol{a} \times \boldsymbol{b}| \boldsymbol{\cdot} \boldsymbol{c}$.

(d) Find the volume of the tetrahedron with vertices (0, 4, 0), (0, 6, 0) (1, 6, 1) and (3, −1, 2). *[14 marks]*

14 Lines and planes in space

In this chapter you will learn:

- to describe all points in space which lie on the same straight line
- to describe all points in space which lie in the same plane
- to solve three-dimensional problems involving intersections, distances and angles between lines and planes
- the connection between solving systems of linear equations and finding intersections of planes.

Introductory problem

Which is more stable (less wobbly): a three-legged stool or a four-legged stool?

Answering the above question involves thinking about whether given points lie in the same plane (on the same flat surface). In this chapter we will use the work on vectors from the last chapter to solve problems involving points, lines and planes in three-dimensional space. Such calculations are used in many areas such as design, navigation and computer games.

14A Vector equation of a line

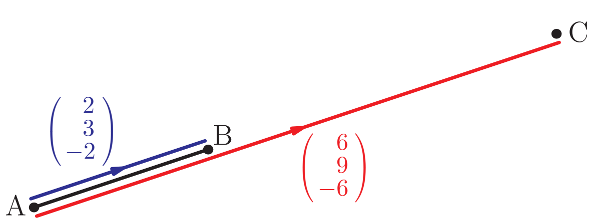
To solve problems involving lines in space we need a way of deciding whether a point lies on a given straight line.

Suppose we have two points, A (−1, 1, 4) and B (1, 4, 2). These two points determine a unique straight line (a 'straight line' extends indefinitely in both directions).

How can we check whether a third point lies on the same line?

We can use vectors to answer this question. For example, consider the point C (5, 10, −2).

Then $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 3\overrightarrow{AB}$.



EXAM HINT

Remember the IB notation for lines and line segments: (AB) means the (infinite) straight line through A and B, [AB] is the line segment (the part of the line between A and B), and AB is the length of [AB].

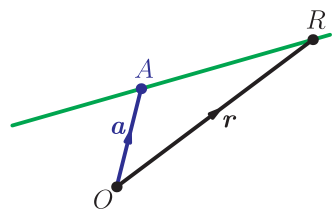
This means that (AC) is parallel to (AB). Since they both contain the point A, (AC) and (AB) must be the same straight line; in other words, C lies on the line (AB).

Next we ask how we can characterise all the points on the line (AB).

Using the above idea, we realise that a point R lies on (AB) if (AR) and (AB) are parallel. We know that this can be expressed using vectors by saying that $\overrightarrow{AR} = \lambda \overrightarrow{AB}$ for some value of the scalar λ (remember that a scalar means a number):

$$\overrightarrow{AR} = \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}.$$

We also know that $\overrightarrow{AR} = \boldsymbol{r} - \boldsymbol{a}$, where $\boldsymbol{r}$ and $\boldsymbol{a}$ are the position vectors of R and A, respectively.



See Section 13B for a reminder of vector algebra.

This means that $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}$ is the position vector of a general point on the line (AB). In other words, R has coordinates $(-1+2\lambda, 1+3\lambda, 4-2\lambda)$ for some value of λ. Different values of λ correspond to different points on the line; for example, $\lambda = 0$ corresponds to point A, $\lambda = 1$ to point B and $\lambda = 3$ to point C.

You will see on page 416 that there is more than one possible vector equation of a line.

The line is parallel to the vector $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, so this vector determines the direction of the line. The expression for the position vector of $\boldsymbol{r}$ is usually written as $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, so it is easy to identify the direction vector.

KEY POINT 14.1

> The expression $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$ is a **vector equation** of the line.
>
> The vector $\boldsymbol{d}$ is the direction vector of the line and $\boldsymbol{a}$ is the position vector of one point on the line.
>
> The vector $\boldsymbol{r}$ is the position vector of a general point on the line; different values of parameter λ give positions of different points on the line.

EXAM HINT

The formula booklet tells you the equation, but not what all the letters stand for.

Worked example 14.1

Write down a vector equation of the line passing through the point (–1, 1, 2) in the direction of the vector $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

The equation of the line is $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$, where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{d}$ is the direction vector

$$\underline{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$$

You know that, in two dimensions, a straight line is determined by its gradient and one point. The gradient is a value which determines the direction of the line. For example, for a line with gradient 3, an increase of 1 unit in x produces an increase of 3 units in y and so the line is in the direction of the vector $\begin{pmatrix}1\\3\end{pmatrix}$.

In three dimensions, a straight line is still determined by its direction and one point, but we cannot use a single number for the gradient. The line in Worked example 14.1 had a direction vector $\begin{pmatrix}2\\2\\1\end{pmatrix}$, so an increase of 2 units in x produces *both* an increase of 2 units in y *and* an increase of 1 unit in z.

Two points determine a straight line. The next example shows how to find a vector equation when two points on the line are given.

Worked example 14.2

Find a vector equation of the line through points A (–1, 1, 2) and B (3, 5, 4).

For a vector equation of the line, we need one point and the direction vector

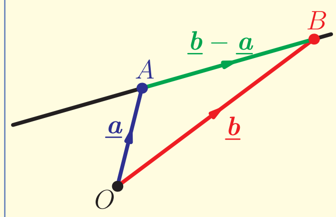
$$\underline{r} = \underline{a} + \lambda\underline{d}$$

The line passes through A (–1, 1, 2); we could use B (3, 5, 4) instead, but smaller values are preferable

$$\underline{a} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

continued . . .

The line is in the direction of $\overrightarrow{AB}$, as we can see by drawing a diagram.



$\underline{d} = \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

$\therefore \underline{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

What if we had used point B instead of A? Then we would have got the equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$. This equation represents the same line, but the value of the parameter λ corresponding to any particular point will be different. For example, with the first equation point A has $\lambda = 0$ and point B has $\lambda = 1$, while using the second equation point A has $\lambda = -1$ and point B has $\lambda = 0$.

Note that the direction vector is not unique either, since we are only interested in its direction and not its magnitude.

Hence $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$ could also be used as direction vectors for the above line, as they are all in the same direction. So yet another form of the equation of the same line would be

$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$ and with this equation, point A has $\lambda = 0$ and point B has $\lambda = -\frac{2}{3}$.

To simplify calculations, we usually choose the direction vector to be the one involving smallest possible integer values, although sometimes it is more convenient to use the corresponding unit vector.

See Key point 13.3 about parallel vectors.

Worked example 14.3

(a) Show that the equations $\mathbf{r}=\begin{pmatrix}-1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}5\\7\\5\end{pmatrix}+\mu\begin{pmatrix}6\\6\\3\end{pmatrix}$ both represent the same straight line.

(b) Show that the equation $\mathbf{r}=\begin{pmatrix}-5\\-3\\1\end{pmatrix}+t\begin{pmatrix}-4\\-4\\-2\end{pmatrix}$ represents a different straight line.

EXAM HINT

When a question asks for equations of several lines, then different letters should be used for the parameters. Commonly used letters are λ (lambda), μ (mu), t and s.

We need to show that the two lines have parallel direction vectors (they will then be parallel) and one common point (then they will be the same line)

Two vectors are parallel if one is a scalar multiple of the other

(a) Direction vectors are parallel, as

$$\begin{pmatrix}6\\6\\3\end{pmatrix}=3\begin{pmatrix}2\\2\\1\end{pmatrix}$$

The point (5, 7, 5) will lie on the first line if we can find the value of λ which gives this position vector

Show that (5, 7, 5) lies on the first line:

Find the value of λ which gives the first coordinate

$$-1+2\lambda=5$$
$$\therefore \lambda=3$$

This value of λ must give the other two coordinates

$$\begin{cases}1+3\times 2=7\\2+3\times 1=5\end{cases}$$

∴ (5, 7, 5) lies on the line.

Hence the two lines are the same.

Check whether the direction vectors are parallel

(b) $\begin{pmatrix}-4\\-4\\-2\end{pmatrix}=-2\begin{pmatrix}2\\2\\1\end{pmatrix}$

So the line is parallel to the other two.

Check whether (5, –3, 1) lies on the first line. Find the value of λ which gives the first coordinate

$$-1+2\lambda=-5$$
$$\therefore \lambda=-2$$

continued . . .

This value of λ must give the other two coordinates

$$\begin{cases} 1+(-2)\times 2=-3 \\ 2+(-2)\times 1=0\neq 1 \end{cases}$$

(5, −3, 1) does not lie on the line.

Hence the line is not the same as the first line.

In the above example we used the coordinates of the point to find the corresponding value of λ. However, sometimes we only know that a point lies on the line, but not its precise coordinates. The next example shows how we can work with a general point on the line (with an unknown value of λ).

Worked example 14.4

Point B (3, 5, 4) lies on the line with equation $r = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$. Find the possible positions of a point Q on the line such that $BQ = 15$.

We know that Q lies on the line, so it has the position vector $\begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ for some value of λ, and we need to find λ.

$$\underset{\sim}{q} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-1+2\lambda\\1+2\lambda\\2+\lambda\end{pmatrix}$$

We can express vector $\overrightarrow{BQ}$ in terms of λ and then set its magnitude equal to 15

$$\overrightarrow{BQ} = \underset{\sim}{q} - \underset{\sim}{b}$$

$$= \begin{pmatrix}-1+2\lambda\\1+2\lambda\\2+\lambda\end{pmatrix} - \begin{pmatrix}3\\5\\4\end{pmatrix} = \begin{pmatrix}2\lambda-4\\2\lambda-4\\\lambda-2\end{pmatrix}$$

$$\left|\overrightarrow{BQ}\right| = 15$$

It is easier to work without the square root, so we square the magnitude equation

$$\therefore (2\lambda-4)^2 + (2\lambda-4)^2 + (\lambda-2)^2 = 15^2$$

$$\Leftrightarrow \quad 9\lambda^2 - 36\lambda - 189 = 0$$

$$\Leftrightarrow \quad \lambda = -3 \text{ or } 7$$

We can now find the position vector of Q by substituting the values of λ in to the line equation

$$\therefore \underset{\sim}{q} = \begin{pmatrix}-7\\-5\\-1\end{pmatrix} \text{ or } \begin{pmatrix}13\\15\\9\end{pmatrix}$$

Exercise 14A

1. Find the vector equation of each line in the given direction through the given point.

   (a) (i) Direction $\begin{pmatrix}1\\4\end{pmatrix}$, point $(4,-1)$

   (ii) Direction $\begin{pmatrix}2\\-3\end{pmatrix}$, point $(4,1)$

   (b) (i) Point $(1,0,5)$, direction $\begin{pmatrix}1\\3\\-3\end{pmatrix}$

   (ii) Point $(-1,1,5)$, direction $\begin{pmatrix}3\\-2\\2\end{pmatrix}$

   (c) (i) Point $(4,0)$, direction $2\boldsymbol{i}+3\boldsymbol{j}$

   (ii) Point $(0,2)$, direction $\boldsymbol{i}-3\boldsymbol{j}$

   (d) (i) Direction $\boldsymbol{i}-3\boldsymbol{k}$, point $(0,2,3)$

   (ii) Direction $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, point $(4,-3,0)$

2. Find a vector equation of the line through each pair of points.

   (a) (i) $(4,1)$ and $(1,2)$ (ii) $(2,7)$ and $(4,-2)$

   (b) (i) $(-5,-2,3)$ and $(4,-2,3)$ (ii) $(1,1,3)$ and $(10,-5,0)$

3. Decide whether or not the given point lies on the given line.

   (a) (i) Line $\boldsymbol{r}=\begin{pmatrix}2\\1\\5\end{pmatrix}+t\begin{pmatrix}-1\\2\\2\end{pmatrix}$, point $(0,\ 5,\ 9)$

   (ii) Line $\boldsymbol{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\1\\5\end{pmatrix}$, point $(-1,\ 0,\ 3)$

   (b) (i) Line $\boldsymbol{r}=\begin{pmatrix}4\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\0\\3\end{pmatrix}$, point $(0,\ 0,\ 0)$

   (ii) Line $\boldsymbol{r}=\begin{pmatrix}-1\\5\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\7\end{pmatrix}$, point $(-1,\ 3,\ 8)$

4. (a) Show that the points A (4, −1, −8) and B (2, 1, −4) lie on the line l with equation $r = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Find the coordinates of the point C on the line l such that $AB = BC$. *[6 marks]*

5. (a) Find the vector equation of line l through points P (7, 1, 2) and Q (3, −1, 5).

(b) Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R. *[6 marks]*

6. (a) Write down the vector equation of the line l through the point A (2, 1, 4) parallel to the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(b) Calculate the magnitude of the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(c) Find the possible coordinates of point P on l such that $AP = 35$. *[8 marks]*

14B Solving problems with lines

In this section we will use vector equations of lines to solve problems involving angles and intersections.

We will also see how vector equations of lines can be used to describe paths of moving objects in mechanics.

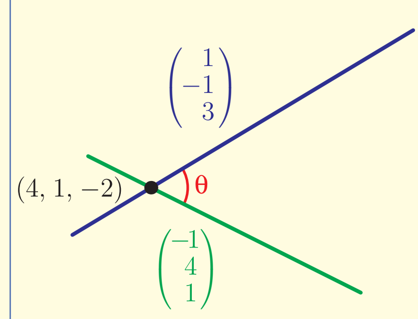
Worked example 14.5

Find the acute angle between lines with equations $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

We know the formula for the angle between two vectors (see Section 13D)

$$\cos\theta = \frac{\underline{a}\bullet\underline{b}}{|\underline{a}||\underline{b}|}$$

Draw a diagram to identify which two vectors ***a*** and ***b*** make the required angle



continued . . .

The two vectors are in the directions of the two lines. So we take $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the direction vectors of the two lines

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

We can now use the formula to calculate the angle

$$\therefore \quad \cos\theta = \frac{-1-4+3}{\sqrt{(1+1+9)}\sqrt{1+16+1}}$$

$$= -\frac{2}{\sqrt{11}\sqrt{18}}$$

$$\theta = 98.2°$$

The angle found is obtuse; the question asked for the acute angle

acute angle $= 180° - 98.2° = 81.8°$

The example above illustrates the general method for finding an angle between two lines.

KEY POINT 14.2

The angle between two lines is equal to the angle between their direction vectors.

Now that we know that the angle between two lines is the angle between their direction vectors, it is easy to identify parallel and perpendicular lines.

KEY POINT 14.3

Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are:

- parallel if $\boldsymbol{d}_1 = k\,\boldsymbol{d}_2$
- perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Parallel and perpendicular vectors were covered in Sections 13B and 13E.

Worked example 14.6

Decide whether the following pairs of lines are parallel, perpendicular, or neither:

(a) $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

(c) $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(a) If $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ then

$$\begin{cases} 4 = k \times 1 \Rightarrow k = 4 \\ -1 = k \times (-2) \Rightarrow k = \frac{1}{2} \end{cases}$$

$4 \neq \frac{1}{2}$

$\therefore$ They are not parallel.

Is $\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = 0$?

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 4 + 2 - 6 = 0$$

$\therefore$ The lines are perpendicular.

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(b) If $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ then

$$\begin{cases} 2 = k \times 1 \Rightarrow k = 2 \\ 1 = k \times 0 \text{ impossible} \end{cases}$$

$\therefore$ They are not parallel.

Is $\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = 0$?

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 2 + 0 + 6 = 8 \neq 0$$

$\therefore$ The lines are neither parallel nor perpendicular.

continued . . .

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(c) If $\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = k \begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$ then

$$\begin{cases} 4 = k \times (-10) \Rightarrow k = -\frac{2}{5} \\ -6 = k \times 15 \Rightarrow k = -\frac{2}{5} \\ 2 = k \times (-5) \Rightarrow k = -\frac{2}{5} \end{cases}$$

∴ The lines have parallel directions.

Check to see if they are the same line

Point $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ does not lie on the first line:

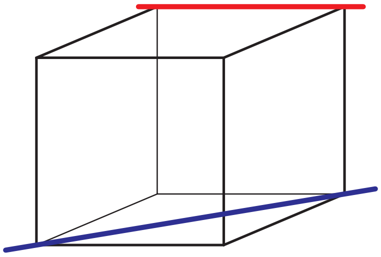
$$\begin{cases} 2 + 4t = -2 \Rightarrow t = -1 \\ -1 - 6t = 0 \Rightarrow t = -\frac{1}{6} \end{cases}$$

They are not the same line.

∴ The lines are parallel.

We will now see how to find the point of intersection of two lines. Suppose two lines have vector equations $\boldsymbol{r}_1 = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{b} + \mu \boldsymbol{d}_2$. If they intersect, then there must be a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\boldsymbol{r}$, this means that we need to find the values of λ and μ which make $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

In two dimensions, two straight lines either intersect or are parallel. However, in three dimensions it is possible to have two lines which are not parallel but do not intersect, as illustrated by the red and blue lines in the diagram. Such lines are called **skew lines**.



With skew lines we will see that we cannot find values of λ and μ such that $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

Worked example 14.7

Find the coordinates of the point of intersection of the following pairs of lines.

(a) $\boldsymbol{r} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$ (b) $\boldsymbol{r} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

We need to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

(a) $\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} 0+\lambda \\ -4+2\lambda \\ 1+\lambda \end{pmatrix} = \begin{pmatrix} 1+4\mu \\ 3-2\mu \\ 5-2\mu \end{pmatrix}$$

If two vectors are equal, then all their components are equal

$$\Rightarrow \begin{cases} 0+\lambda = 1+4\mu \\ -4+2\lambda = 3-2\mu \\ 1+\lambda = 5-2\mu \end{cases}$$

$$\Rightarrow \begin{cases} \lambda - 4\mu = 1 & (1) \\ 2\lambda + 2\mu = 7 & (2) \\ \lambda + 2\mu = 4 & (3) \end{cases}$$

Solve two simultaneous equations in two variables. Use eqn (1) and (3) (as subtracting them eliminates λ)

$(3)-(1)\quad 6\mu = 3$

$\therefore \mu = \frac{1}{2}, \lambda = 3$

We need to check that the values of λ and μ also satisfy the second equation otherwise the lines do not actually meet

$(2): 2\times 3 + 2\times\frac{1}{2} = 7$

$\therefore$ the lines intersect

The position of the intersection point is given by the vector $\boldsymbol{r}_1$ (or $\boldsymbol{r}_2$ - they should be the same)

$$\underline{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The lines intersect at the point $(3, 2, 4)$

Make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

(b) $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{cases} t - 2\lambda = 6 & (1) \\ t + 3\lambda = -2 & (2) \\ 4t - 2\lambda = -2 & (3) \end{cases}$$

continued . . .

We can find t and λ from eqn (1) and (2)

(1) and (2) $\Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$

We need to check that the values found also satisfy the third equation

(3) $4 \times \frac{14}{5} - 2 \times \left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$

This tells us that it is impossible to find t and λ to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

The two lines do not intersect.

EXAM HINT

You can use your calculator to solve simultaneous equations.
See Calculator sheet 6 on the CD-ROM.

Vector questions often ask you to find a point on a given line which satisfies certain conditions. We have already seen how we can use the position vector $\boldsymbol{r}$ for a general point on the line, and then use the condition to write an equation for λ.

See Worked example 14.4.

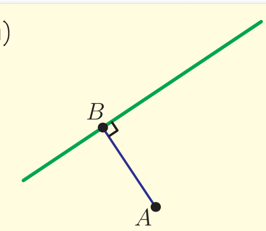
Worked example 14.8

Line l has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, –2).

(a) Find the coordinates of point B on l so that AB is perpendicular to l.
(b) Hence find the shortest distance from A to l.
(c) Find the coordinates of the reflection of the point A in l.

Draw a diagram. The line AB should be perpendicular to the direction vector of l

(a)



$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$ (1)

continued . . .

We know that B lies on l, so its position vector is given by the equation for $\boldsymbol{r}$

$$\overrightarrow{OB} = \boldsymbol{r} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix}$$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

Now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow (\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\Rightarrow \lambda = -4$$

Use value of λ in the equation of the line to give the position vector of B

$$r = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

$\therefore$ B has coordinates $(-1, 3, -4)$

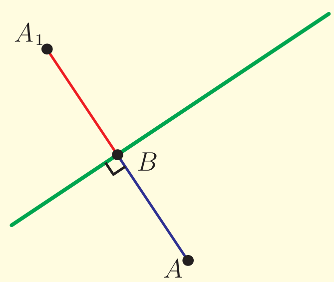
The shortest distance from a point to a line is the perpendicular distance AB. Again, use $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{16+36+4} = 2\sqrt{14}$$

The reflection A_1 lies on the line (AB). Since $BA_1 = AB$ and they are also in the same direction, $\overrightarrow{BA_1} = \overrightarrow{AB}$

(c)



$$\overrightarrow{BA_1} = \overrightarrow{AB}$$

$$\Rightarrow \underline{a}_1 - \underline{b} = \overrightarrow{AB}$$

$$\therefore \underline{a}_1 = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

So A_1 has coordinates $(-5, -3, -6)$

Worked example 14.8(c) illustrates the power of vectors. As vectors contain both distance and direction information, just one equation ($\overrightarrow{BA_1} = \overrightarrow{AB}$) was needed to express both the fact that A_1 lies on the line (AB) and that $BA_1 = AB$.

We have already mentioned that vectors have many applications, particularly in physics. One such application is describing positions, displacements and velocities. These are all vector quantities, since they have both magnitude and direction.

You are probably familiar with the rule that, for an object moving with constant velocity, displacement = velocity × time. If we are working in two or three dimensions, the positions of points also need to be described by vectors. Suppose an object has constant velocity $\boldsymbol{v}$ and in time t moves from the point with position $\boldsymbol{a}$ to the point with position $\boldsymbol{r}$. Then its displacement is $\boldsymbol{r} - \boldsymbol{a}$, so we can write:

$$\boldsymbol{r} - \boldsymbol{a} = \boldsymbol{v}t$$

This equation can be rearranged to $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$, which looks very much like a vector equation of a line with direction vector $\boldsymbol{v}$. This makes sense, as the object will move in the direction given by its velocity vector. As t changes, $\boldsymbol{r}$ gives position vectors of different points along the object's path.

Note that the speed is the magnitude of the velocity, $|\boldsymbol{v}|$, and the distance travelled is the magnitude of the displacement, $|\boldsymbol{r} - \boldsymbol{a}|$.

In 1896 the physicist Lord Kelvin wrote: "'vector' is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature".

(Quaternions are a special type of number linked to complex numbers.)

They are now one of the most important tools in physics. Even great mathematicians cannot always predict what will be useful!

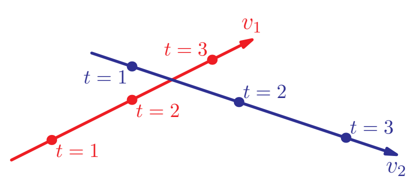
KEY POINT 14.4

For an object moving with constant velocity $\boldsymbol{v}$ from an initial position $\boldsymbol{a}$, the position at time t is given by $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$.

The object moves along the straight line with equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$.

The speed of the object is equal to $|\boldsymbol{v}|$.

When we wanted to find the intersection of two lines, we had to use different parameters (for example, λ and μ) in the two equations. If we have two objects, we can write an equation for $\boldsymbol{r}(t)$ for each of them. In this case, we should use the same t in both equations, as both objects are moving at the same time. For the two objects to meet, they need to be at the same place at the same time. Notice that it is possible for the objects' paths to cross without the objects themselves meeting, if they pass through the intersection point at different times.



Worked example 14.9

Two objects, A and B, have velocities $\boldsymbol{v}_A = 6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v}_B = -2\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$. Object A starts from the origin and object B from the point with position vector $13\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. Distance is measured in kilometres and time in hours.

(a) What is the speed of object B?

(b) Find the distance between the two objects after 5 hours.

(c) Show that the two objects do not meet.

Speed is the magnitude of velocity

(a)

$$|v_B| = \sqrt{2^2 + 1^2 + 7^2} = \sqrt{54}$$

So the speed of B is 7.35 km/h.

We need an equation for the position of each object in terms of t

(b)

Using $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$:

$$\boldsymbol{r}_A(t) = t(6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{r}_B(t) = 13\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k})$$

We can then find the position of each object when $t = 5$

When $t = 5$:

$$\boldsymbol{r}_A = 30\boldsymbol{i} + 15\boldsymbol{j} + 5\boldsymbol{k}$$

$$\boldsymbol{r}_B = 3\boldsymbol{i} + 4\boldsymbol{j} + 38\boldsymbol{k}$$

The distance is the magnitude of $\boldsymbol{r}_A - \boldsymbol{r}_B$

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = \sqrt{27^2 + 11^2 + 33^2}$$

$$= 44.0 \text{ km}$$

If the two objects meet then $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$

(c)

If $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$:

$$\begin{cases} 6t = 13 - 2t \Rightarrow t = \frac{13}{8} \\ 3t = -1 + t \Rightarrow t = -\frac{1}{2} \\ t = 3 + 7t \Rightarrow t = -\frac{1}{2} \end{cases}$$

The three coordinates are not equal at the same time, so the objects do not meet.

Exercise 14B

1. Find the acute angle between the following pairs of lines, giving your answer in degrees.

(a) (i) $\boldsymbol{r}=\begin{pmatrix}5\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}4\\-1\\3\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\mu\begin{pmatrix}-5\\1\\3\end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+t\begin{pmatrix}-1\\0\\0\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\3\\3\end{pmatrix}+s\begin{pmatrix}4\\0\\2\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}6\\6\\2\end{pmatrix}+t\begin{pmatrix}-1\\0\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+s\begin{pmatrix}4\\-1\\2\end{pmatrix}$

2. For each pair of lines, state whether they are parallel, perpendicular, the same line, or none of the above.

(a) $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\text{»}\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

(b) $\boldsymbol{r}=\begin{pmatrix}4\\1\\2\end{pmatrix}+s\begin{pmatrix}-1\\2\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}2\\-4\\-4\end{pmatrix}$

(c) $\boldsymbol{r}=\begin{pmatrix}1\\5\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\3\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}3\\3\\1\end{pmatrix}$

(d) $\boldsymbol{r}=\begin{pmatrix}2\\2\\1\end{pmatrix}+t\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}5\\-1\\10\end{pmatrix}+s\begin{pmatrix}1\\-1\\3\end{pmatrix}$

3. Determine whether the following pairs of lines intersect and, if they do, find the coordinates of the intersection point.

(a) (i) $\boldsymbol{r}=\begin{pmatrix}6\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\-14\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\3\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}6\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}3\\-4\\0\end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-4\\-4\\-11\end{pmatrix}+s\begin{pmatrix}5\\1\\2\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\0\\2\end{pmatrix}+t\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+s\begin{pmatrix}1\\-2\\-2\end{pmatrix}$

4. Line l has equation $\boldsymbol{r}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and point P has coordinates (7, 2, 3).

Point C lies on l and PC is perpendicular to l. Find the coordinates of C. *[6 marks]*

5. Find the shortest distance from the point (–1, 1, 2) to the line with equation $\boldsymbol{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}-3\\1\\1\end{pmatrix}$. *[6 marks]*

6. Two lines are given by $l_1:\boldsymbol{r}=\begin{pmatrix}-5\\1\\10\end{pmatrix}+\lambda\begin{pmatrix}-3\\0\\4\end{pmatrix}$ and

$l_2:\boldsymbol{r}=\begin{pmatrix}3\\0\\-9\end{pmatrix}+\mu\begin{pmatrix}1\\1\\7\end{pmatrix}$.

(a) l_1 and l_2 intersect at P, find the coordinates of P.

(b) Show that the point Q (5, 2, 5) lies on l_2.

(c) Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

(d) Find the area of the triangle PQM. *[10 marks]*

7. In this question, unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ point due East and North, respectively.

A port is located at the origin. One ship starts from the port and moves with velocity $\boldsymbol{v}_1 = (3\boldsymbol{i} + 4\boldsymbol{j})$ kmh^{-1}.

(a) Write down the position vector at time t hours.

At the same time, a second ship starts 18 km north of the port and moves with velocity $\boldsymbol{v}_2 = (3\boldsymbol{i} - 5\boldsymbol{j})$ kmh^{-1}.

(b) Write down the position vector of the second ship at time t hours.

(c) Show that after half an hour, the distance between the two ships is 13.5 km.

(d) Show that the ships meet, and find the time when this happens.

(e) How long after the meeting are the ships 18 km apart?

[12 marks]

8. At time $t = 0$, two aircraft have position vectors $5\boldsymbol{j}$ and $7\boldsymbol{k}$. The first moves with velocity $3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and the second with velocity $5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

(a) Write down the position vector of the first aircraft at time t.

(b) Show that at time t the distance, d, between the two aircraft is given by $d^2 = 44t^2 - 88t + 74$.

(c) Show that the two aircraft will not collide.

(d) Find the minimum distance between the two aircraft.

[12 marks]

9. Find the distance of the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

[7 marks]

10. Two lines with equations $l_1 : \boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ intersect at point P.

(a) Find the coordinates of P.

(b) Find, in degrees, the acute angle between the two lines.

Point Q has coordinates (−1, 5, 10).

(c) Show that Q lies on l_2.

(d) Find the distance PQ.

(e) Hence find the shortest distance from Q to the line l_1. *[12 marks]*

11. Given line $l: \boldsymbol{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and point P (21, 5, 10):

(a) Find the coordinates of point M on l such that PM is perpendicular to l.

(b) Show that the point Q (15, −14, 17) lies on l.

(c) Find the coordinates of point R on l_1 such that $|PR| = |PQ|$. *[10 marks]*

12. Two lines have equations $l_1: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and

$l_2: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and intersect at point P.

(a) Show that Q (5, 2, 6) lies on l_2.

(b) R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R. *[8 marks]*

14C Other forms of equation of a line

You know that in two dimensions, a straight line has equation of the form $y = mx + c$ or $ax + by = c$. How is this related to the vector equation of the line we introduced in this chapter?

Let us look at an example of a vector equation of a line in two dimensions. A line with direction vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ passing through the point $(1, 4)$ has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Vector $\boldsymbol{r}$ is the position vector of a point on the line; in other words, it gives coordinates (x, y) of a point on the line. So $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. This vector equation represents two equations:

$$\begin{cases} x = 1 + 3\lambda \\ y = 4 + 2\lambda \end{cases}$$

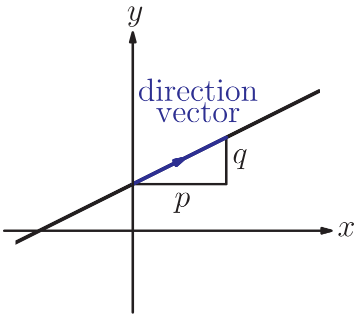
These are called **parametric equations**, because x and y are given in terms of a parameter λ. We can eliminate λ to obtain an equation relating x and y:

$$\lambda = \frac{x-1}{3}$$

$$\therefore y = 4 + \frac{2x-2}{3}$$

$$\Leftrightarrow y = \frac{2}{3}x + \frac{10}{3}$$

This is the more familiar **Cartesian equation** of the line.



It is easy to see how the gradient of line is related to the direction vector: if the direction vector is $\begin{pmatrix} p \\ q \end{pmatrix}$ then the gradient is $\frac{q}{p}$, as illustrated in the diagram.

What happens if we try to apply the same method to find a Cartesian equation of a line in three dimensions?

Consider the line with vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$.

The parametric equations of this line are

$$\begin{cases} x = 1 + 3\lambda \\ y = 4 + 2\lambda \\ z = -1 + 5\lambda \end{cases}$$

We can substitute λ from the first equation into the other two to express, for example, z in terms of x and y.

$$y = \frac{2}{3}x + \frac{10}{3}, \quad z = \frac{5}{3}x - \frac{8}{3}$$

It looks like we cannot obtain a single equation relating x, y and z.

This also means that there is no concept of a gradient in three dimensions, which is why we have introduced the notion of a direction vector.

A better way of writing one Cartesian equation for the line is to write three parametric equations with λ as the subject:

$$\begin{cases} \lambda = \dfrac{x-1}{3} \\ \lambda = \dfrac{y-4}{2} \\ \lambda = \dfrac{z+1}{5} \end{cases}$$

Equating all three expressions for λ gives another form of the Cartesian equation of the line:

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z+1}{5}$$

KEY POINT 14.5

To find the Cartesian equation of a line from its vector equation:

- Write $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of λ, giving three equations.
- Make λ the subject of each equation.
- Equate the three expressions for λ to get an equation of the form $\dfrac{x-x_0}{l} = \dfrac{y-y_0}{m} = \dfrac{z-z_0}{n}$.

EXAM HINT

The Formula booklet shows all three forms of equation of a line (vector, parametric and Cartesian), but it does not tell you how to change between them.

Sometimes a Cartesian equation cannot be written in the above form, as shown in the following example.

Worked example 14.10

Find the Cartesian equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ 5 \\ 0 \end{pmatrix}$.

$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so write an equation involving x, y and z

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ 5 \\ 0 \end{pmatrix}$$

continued . . .

Express λ in terms of x, y and z

$$\begin{cases} x = 1 + \frac{1}{3}\lambda \Rightarrow \lambda = \frac{x-1}{1/3} \\ y = \frac{1}{2} + 5\lambda \Rightarrow \lambda = \frac{y - \frac{1}{2}}{5} \\ z = -3 \end{cases}$$

Equate the expressions for λ from the first two equations. The third equation does not contain λ, so leave it as a separate equation

$$\frac{x-1}{1/3} = \frac{y - \frac{1}{2}}{5},\ z = -3$$

It will look neater if we rewrite the equation without 'fractions within fractions'

$$\Leftrightarrow \frac{3x-3}{1} = \frac{2y-1}{10},\ z = -3$$

EXAM HINT

The Cartesian equation can sometimes be 'read off' the vector equation; if the vector equation is $\boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} k \\ m \\ n \end{pmatrix}$ then the Cartesian equation is $\frac{x-a}{k} = \frac{y-b}{m} = \frac{z-c}{n}$.

However, if any of the components of the direction vector is 0, we have to complete the whole procedure described in the Key point 14.5.

We can reverse the above procedure to go from Cartesian to vector equation. Vector equations are convenient if we need to identify the direction vector of the line, or to use methods from Section 14B to solve problems involving lines.

KEY POINT 14.6

To find a vector equation of a line from a Cartesian equation in the form $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$:

- Set each of the three expressions equal to λ.
- Express x, y and z in terms of λ.
- Write $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to obtain $\boldsymbol{r}$ in terms of λ.

We can adapt this procedure for Cartesian equations that are not of the above form, as in the next example.

Worked example 14.11

Find a vector equation of the line with Cartesian equation $x = -2, \frac{3y+1}{4} = \frac{2-z}{5}$. Hence write down the direction vector of the line, making all its components integers.

Introduce a parameter λ. As the two expressions involving y and z are equal, set them both equal to λ

$$\begin{cases} \frac{3y+1}{4} = \lambda \\ \frac{2-z}{5} = \lambda \end{cases}$$

Now express x, y and z in terms of λ

$$\therefore \begin{cases} x = -2 \\ y = \frac{4\lambda - 1}{3} \\ z = 2 - 5\lambda \end{cases}$$

The vector equation is an equation for $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of λ. Separate the expression into a part without λ and a part involving λ

$$\therefore \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{1}{3} + \frac{4}{3}\lambda \\ 2 \quad -5\lambda \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ -\frac{1}{3} \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \frac{4}{3} \\ -5 \end{pmatrix}$$

Now identify the direction vector

The direction vector is $\begin{pmatrix} 0 \\ \frac{4}{3} \\ 5 \end{pmatrix}$

We can change the magnitude of the direction vector so that it does not contain fractions (multiply by 3 in this case)

or $\begin{pmatrix} 0 \\ 4 \\ -15 \end{pmatrix}$

You can solve problems involving lines given by Cartesian equations by changing into the vector equation and using methods from the previous section. However, there are some problems that can be solved more quickly by using the Cartesian equation directly.

Worked example 14.12

Does the point A (3, −2, 2) lie on the line with equation $\frac{x+1}{2}=\frac{4-y}{3}=\frac{2z}{3}$?

If the point lies on the line, the coordinates should satisfy the Cartesian equation.
This means that all three expressions should be equal

$$\begin{cases} \frac{x+1}{2}=\frac{3+1}{2}=2 \\ \frac{4-y}{3}=\frac{4+2}{3}=2 \\ \frac{2z}{3}=\frac{2\times 2}{3}=\frac{4}{3} \end{cases}$$

The second equality is not satisfied

$$2=2\neq\frac{4}{3}$$

∴ The point does not lie on the line.

Intersections of a line with the coordinate axes are also easy to find using the Cartesian equation.

Worked example 14.13

(a) Find the coordinates of the point where the line with equation $\frac{x-6}{2}=\frac{y+1}{7}=\frac{z+9}{-3}$ intersects the y-axis.

(b) Show that the line does not intersect the z-axis.

A point on the y-axis has $x = z = 0$

(a) A point on the y-axis has coordinates $(0, m, 0)$

Substitute coordinates into the Cartesian equation

$$\frac{0-6}{2}=\frac{m+1}{7}=\frac{0+9}{-3}$$

$$\Leftrightarrow \quad -3=\frac{m+1}{7}=-3$$

Find m

$m+1=-21$

$\therefore m=-22$

The point of intersection is $(0, -22, 0)$

continued . . .

A point on the z-axis has $x = y = 0$

(b) A point on the z-axis has coordinates $(0, 0, m)$

Substitute coordinates into the Cartesian equation

$$\frac{0-6}{2} \neq \frac{0+1}{7}$$

The first equality is not satisfied

The line does not intersect the z-axis.

Exercise 14C

1. (a) Write down the Cartesian equation of the line

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

(b) Write down the Cartesian equation of the line

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

(c) Write down a vector equation of the line with Cartesian equation $\dfrac{x-3}{2} = \dfrac{y+1}{-4} = \dfrac{z}{5}$.

(d) Write down a vector equation of the line with Cartesian equation $\dfrac{x+1}{5} = \dfrac{3-z}{2}, y = 1$.

2. Determine whether the following pairs of lines are parallel, perpendicular, the same line, or none of the above.

(a) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

(b) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -6 \\ 9 \\ 2 \end{pmatrix}$ and $\dfrac{2x-1}{4} = \dfrac{y-2}{-3} = \dfrac{6-3z}{2}$

(c) $\dfrac{x-5}{7} = \dfrac{y-2}{-1} = 4 - z$ and $x = 2\lambda + 1, y = 4, z = 5 - \lambda$

(d) $x = 2t + 1, y = 1 - 4t, z = 3$ and $\boldsymbol{r} = \begin{pmatrix} 8 \\ -13 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

3. (a) Find the Cartesian equation of the line with parametric equation $x = 3\lambda + 1, \quad y = 4 - 2\lambda, \quad z = 3\lambda - 1$.

(b) Find the unit vector in the direction of the line. *[5 marks]*

4. (a) Find a vector equation of the line with Cartesian equation $\frac{2x-1}{4} = \frac{y+2}{3} = \frac{4-3z}{6}$.

(b) Determine whether the line intersects the x-axis.

(c) Find the angle the line makes with the x-axis. *[8 marks]*

5. (a) Find, in degrees, the angle between the lines $\frac{x-3}{5} = y-2 = \frac{3-2z}{2}$ and $\frac{x+1}{3} = 3-z, y = 1$.

(b) Determine whether the lines intersect. *[7 marks]*

6. (a) Find the coordinates of the point of intersection of the lines with Cartesian equations $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+1}{1}$ and $5 - x = \frac{y+2}{-3} = \frac{z-7}{2}$.

(b) Show that the line with equation $\boldsymbol{r} = \begin{pmatrix} 7 \\ 8 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ passes through the above intersection point. *[6 marks]*

14D Equation of a plane

We next look at different ways of representing all points that lie in a given plane.

We will first try an approach like to the one used to derive a vector equation of a line, where we noted that every point on the line can be reached from the origin by going to one particular point on the line and then moving along the line using the direction vector. This gave us an equation for the position vector of any point on the line in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$.

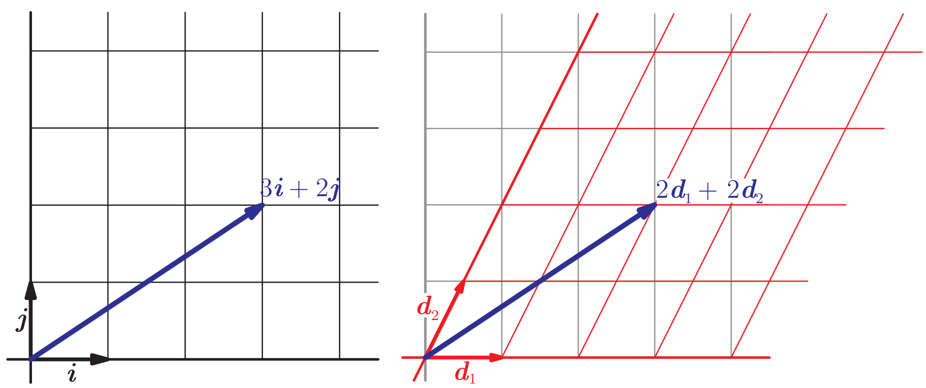
Can we similarly describe a way to get from the origin to any point in a plane?

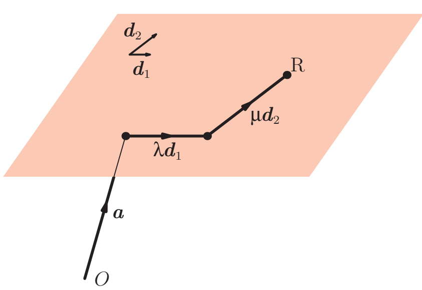
We describe positions of points in the x-y plane using coordinates. To reach the point with coordinates (p, q) from the origin, we move p units in x-direction and q units in y-direction.

We can also express this using unit vectors parallel to the x-and y-axes; the position vector of the point $P\ (2, 3)$ is $\boldsymbol{r}_p = 3\boldsymbol{i} + 2\boldsymbol{j}$.

However, it is possible to use directions other than those of $\boldsymbol{i}$ and $\boldsymbol{j}$. In the second diagram below, the same point P is reached from the origin by moving 2 units in the direction of vector $\boldsymbol{d}_1$ and 2 units in the direction of vector $\boldsymbol{d}_2$. Hence its position vector is $\boldsymbol{r}_p = 2\boldsymbol{d}_1 + 2\boldsymbol{d}_2$.

You can see that any point in the plane can be reached by going a certain number of units in the direction of $\boldsymbol{d}_1$ and a certain number of units in the direction of $\boldsymbol{d}_2$; hence every point in the plane has a position vector of the form $\lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$, where λ and μ are scalars.





Now consider a plane that does not pass through the origin. To reach a point in the plane starting from the origin, we go to some other point in the plane first, and then move along two directions which lie in the plane, as illustrated in the diagram alongside.

This means that every point in the plane has a position vector of the form $\boldsymbol{a} + \lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$, where $\boldsymbol{a}$ is the position vector of one point in the plane, and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are two vectors parallel to the plane (but not parallel to each other). $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ do not need to be perpendicular to each other.

KEY POINT 14.7

The plane containing point $\boldsymbol{a}$ and parallel to the directions of vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ has a vector equation:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1 + \mu \boldsymbol{d}_2$$

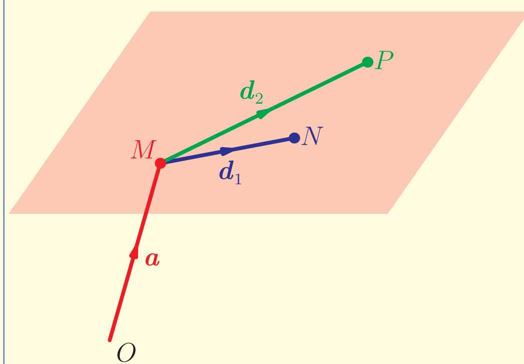
Worked example 14.14

Find a vector equation of the plane containing points $M\ (3, 4, -2)$, $N\ (1, -1, 3)$ and $P\ (5, 0, 2)$.

We need one point and two vectors parallel to the plane

$$\boldsymbol{r} = \underline{a} + \lambda \underline{d}_1 + \mu \underline{d}_2$$

Draw a diagram to see which vectors to use



Choose any of the three given points, as they all lie in the plane

$$\underline{a} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

Vectors $\overrightarrow{MN}$ and $\overrightarrow{MP}$ are parallel to the plane

$$\underline{d}_1 = \overrightarrow{MN} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$$

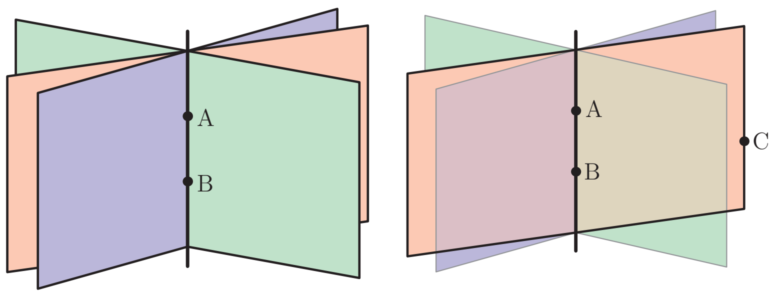
$$\underline{d}_2 = \overrightarrow{MP} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

We can now write down the equation

$$\therefore \underline{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

How many points are needed to determine a plane? In the above example, the plane was determined by three points. It should be clear from the diagram below that two points do not determine a plane: there is more than one plane containing the line determined by points *A* and *B*.

We can pick out one of these planes by requiring that it also passes through a third point which is not on the line (*AB*), as illustrated in the second diagram. This suggests that a plane can also be determined by a line and a point outside of that line.



Worked example 14.15

Find a vector equation of the plane containing the line $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ and point $A\,(4, -1, 2)$.

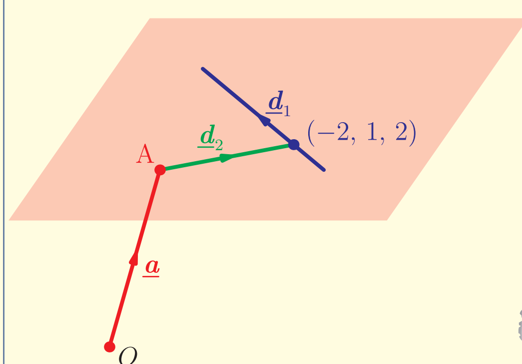
Point *A* lies in the plane

$$\underline{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

The direction vector of the line is parallel to the plane

$$\underline{d}_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

We need another vector parallel to the plane. We can use any vector between two points in the plane. One point in the plane is *A* and for the second point, we can pick any point on the line: for example, (–2, 1, 2)



continued . . .

$$\underline{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

We can now write down the equation of the plane

$$\therefore \underline{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

The following example looks at what happens if we have four points.

Worked example 14.16

Determine whether points A (2, −1, 3), B (4, 1, 1), C (3, 3, 2) and D (−3, 1, 5) lie in the same plane.

We know how to find an equation of the plane containing points A, B and C

Plane containing A, B and C

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

For D to also lie in the plane, we need values of λ and μ which make $\boldsymbol{r}$ equal to the position vector of D

$$\boldsymbol{r} = \overrightarrow{OD}:$$

$$\begin{cases} 2 + 2\lambda + \mu = -3 & (1) \\ -1 + 2\lambda + 4\mu = 1 & (2) \\ 3 - 2\lambda - \mu = 5 & (3) \end{cases}$$

Solve the first two equations

$$(1) \text{ and } (2) \quad \begin{cases} 2\lambda + \mu = -5 \\ 2\lambda + 4\mu = 2 \end{cases}$$

$$\Rightarrow \lambda = -\frac{11}{3}, \quad \mu = \frac{7}{3}$$

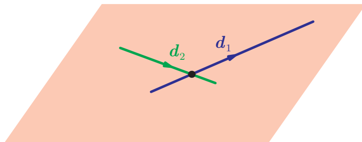
Then check whether the solutions satisfy the third

$$(3) \quad 3 - 2 \times \left(-\frac{11}{3}\right) - \frac{7}{3} = 8 \neq 5$$

There are no values of λ and μ which satisfy all three equations

D does not lie in the same plane as A, B and C

Worked example 14.16 shows that it is not always possible to find a plane containing four given points. However, we can always find a plane containing three given points, as long as they do not lie on the same straight line.

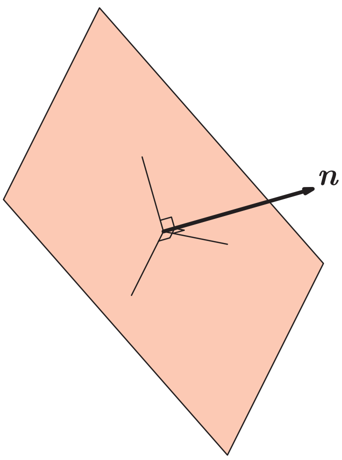


A plane can also be determined by two intersecting lines. In that case, vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ can be taken to be the direction vectors of the two lines. We will see how to find the equation of the plane in Section 14G.

KEY POINT 14.8

To uniquely determine a plane we need:

- three points, not on the same line, OR
- a line and a point outside that line, OR
- two intersecting lines.



A vector equation of the plane can be difficult to work with, as it contains two parameters. It is also difficult to see whether two equations actually describe the same plane, because there are many pairs of vectors parallel to the plane which can be used in the equation. So it is reasonable to ask: Is there a way we can determine the 'direction' of the plane using just one direction vector?

The diagram alongside shows a plane and a vector $\boldsymbol{n}$ which is perpendicular to it. This vector is perpendicular to every line in the plane, and it is called the **normal vector** of the plane.

If we know one point, A, in the plane and the normal vector, what can we say about the position vector of any other point, P, in the plane? The normal vector is perpendicular to the line (AP), so $\overrightarrow{AP}\bullet\boldsymbol{n} = 0$. This means that $(\boldsymbol{r} - \boldsymbol{a})\bullet\boldsymbol{n} = 0$, which gives us another form of an equation of the plane.

KEY POINT 14.9

A plane with a normal vector $\boldsymbol{n}$ and containing a point with position vector $\boldsymbol{a}$ has a **scalar product equation**:
$\boldsymbol{r}\bullet\boldsymbol{n} = \boldsymbol{a}\bullet\boldsymbol{n}$

Worked example 14.17

Vector $\boldsymbol{n} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ is perpendicular to the plane Π which contains point A (3, –5, 1).

(a) Write an equation of Π in the form $\boldsymbol{r}\cdot\boldsymbol{n} = d$.

(b) Find the Cartesian equation of the plane.

EXAM HINT

The letter Π (capital π) is often used as the name for a plane.

The equation of the plane is $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$

(a) $\underline{r}\cdot\underline{n} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

$= 6 - 20 - 1$

$\therefore \underline{r}\cdot\underline{n} = -15$

The Cartesian equation involves x, y and z (the coordinates of P), which are the components of the general position vector $\boldsymbol{r}$

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = -15$

$\Rightarrow 2x + 4y - z = -15$

KEY POINT 14.10

The Cartesian equation of a plane has the form

$$ax + by + cz = d$$

where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector of the plane.

The next example shows how to convert from vector to Cartesian equation of the plane.

Worked example 14.18

Find the Cartesian equation of the plane with vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$.

To find the Cartesian equation we need the normal vector and one point

$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

Point (1, –2, 5) lies in the plane

$\underline{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

$\boldsymbol{n}$ is perpendicular to all lines in the plane, so it is perpendicular to the two vectors $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ which are parallel to the plane. The cross product of two vectors is perpendicular to both of them

$\underline{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 14 \\ 1 \\ -5 \end{pmatrix}$

Cross product was introduced in Section 13G

To get the Cartesian equation, write $\boldsymbol{r}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ -5 \end{pmatrix} = 13$

$\Leftrightarrow \quad 14x + y - 5z = 13$

The Cartesian equation is very convenient for checking whether a point lies in the plane; we just need to check that the coordinates of the point satisfy the equation. In the next section we will see how to use it to examine the relationship between a line and a plane.

Exercise 14D

1. Write down the vector equation of the plane parallel to vectors ***a*** and ***b*** and containing point P.

(a) (i) $\boldsymbol{a} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, P\,(1, 0, 2)$

(ii) $\boldsymbol{a} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, P\,(0, 2, 0)$

(b) (i) $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{p} = \boldsymbol{j} + \boldsymbol{k}$

(ii) $\boldsymbol{a} = 5\boldsymbol{i} - 6\boldsymbol{j}, \boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{P} = \boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}$

2. Find a vector equation of the plane containing points A, B and C.

(a) (i) $A(3, -1, 3), B(1, 1, 2), C(4, -1, 2)$

(ii) $A(-1, -1, 5), B(4, 1, 2), C(-7, 1, 1)$

(b) (i) $A(9, 0, 0), B(-2, 1, 0), C(1, -1, 2)$

(ii) $A(11, -7, 3), B(1, 14, 2), C(-5, 10, 0)$

3. Find a vector equation of the plane containing line l and point P.

(a) (i) $l : \boldsymbol{r} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, P\,(-1, 4, 3)$

(ii) $l : \boldsymbol{r} = \begin{pmatrix} 9 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}, P\,(11, 12, 13)$

(b) (i) $l : \boldsymbol{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, P\,(-3, 1, 0)$

(ii) $l : \boldsymbol{r} = t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, P\,(4, 0, 2)$

4. A plane has normal vector $\boldsymbol{n}$ and contains point A. Find the equation of the plane in the form $\boldsymbol{r} \bullet \boldsymbol{n} = d$, and the Cartesian equation of the plane.

(a) (i) $\boldsymbol{n} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$, $A\ (3, 3, 1)$

(ii) $\boldsymbol{n} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$, $A\ (4, 3, -1)$

(b) (i) $\boldsymbol{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $A\ (-3, 0, 2)$

(ii) $\boldsymbol{n} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$, $A\ (0, 0, 2)$

5. Find a normal vector to the plane given by the vector equation:

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$

6. Find the equations of the planes from question 5 in the form $\boldsymbol{r} \bullet \boldsymbol{n} = d$.

7. Find the Cartesian equations of the planes from question 5.

8. Find the Cartesian equation of the plane containing points A, B and C.

(a) (i) $A\,(7, 1, 2)$, $B\,(-1, 4, 7)$, $C\,(5, 2, 3)$

(ii) $A\,(1, 1, 2)$, $B\,(4, -6, 2)$, $C\,(12, 12, 2)$

(b) (i) $A\,(12, 4, 10)$, $B\,(13, 4, 5)$, $C\,(15, -4, 0)$

(ii) $A\,(1, 0, 0)$, $B\,(0, 1, 0)$, $C\,(0, 0, 1)$

9. Show that point P lies in the plane Π.

(a) $P\ (-4, 8, 9),\ \Pi: \boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

(b) $P\ (4, 7, 5),\ \Pi: \boldsymbol{r} \bullet \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 19$

(c) $P\ (1, 1, -2),\ \Pi: 2x - 3y - 7z = 12$

10. Show that plane Π contains line l.

(a) $\Pi: x + 6y + 2z = 7,\ l: \boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(b) $\Pi: 5x + y - 2z = 15,\ l: \frac{x-4}{1} = \frac{y+1}{1} = \frac{z-2}{3}$

(c) $\Pi: \boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = -5,\ l: \boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$

(d) $\Pi: \boldsymbol{r} \bullet \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix},\ l: \frac{x-3}{2} = \frac{y}{3} = \frac{z+1}{2}$

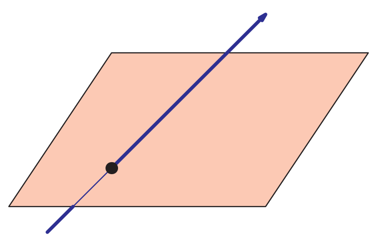
14E Angles and intersections between lines and planes

In this section we will look at angles and intersections between a line and a plane and between two planes. You will be expected to recall and use the four methods shown here. In the examination you will meet much longer and more complicated problems, where you are expected to combine these common techniques.

You saw how to find the angle and intersection of two lines in Section 14B.

We will discuss strategies for more complicated problems in the final section of this chapter.

When finding the intersection between a line and a plane, it is most convenient if the equation of the line is in the vector form and the equation of the plane in the Cartesian form. In all examples in this section the planes will be given by their Cartesian equations, but in the examination you may need to convert them into this form first. The key idea we use is that the coordinates of the intersection point (if there is one) must satisfy both the line equation and the plane equation.



Worked example 14.19

Find the intersection between the given line and plane, or show that they do not intersect.

(a) $\boldsymbol{r}=\begin{pmatrix}4\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\-2\end{pmatrix}$ and $2x-y+2z=7$ (b) $\frac{x-1}{-1}=\frac{y}{-3}=\frac{z+4}{2}$ and $x-3y-4z=12$

(c) $\boldsymbol{r}=\begin{pmatrix}3\\-1\\1\end{pmatrix}+t\begin{pmatrix}3\\1\\0\end{pmatrix}$ and $x-3y=6$

The coordinates of the intersection point must satisfy both equations. Remember that the coordinates of a point on the line are given by vector $r=\begin{pmatrix}x\\y\\z\end{pmatrix}$, so we can substitute x, y, z into the equation of the plane

(a) $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4+3\lambda\\-3\\1-2\lambda\end{pmatrix}$

$\Rightarrow 2(4+3\lambda)-(-3)+2(1-2\lambda)=7$

$\Leftrightarrow 2\lambda=-6$

$\Leftrightarrow \lambda=-3$

Now use this value of λ to find the coordinates

$\therefore \underline{r}=\begin{pmatrix}4\\-3\\1\end{pmatrix}-3\begin{pmatrix}3\\0\\-2\end{pmatrix}=\begin{pmatrix}-5\\-3\\7\end{pmatrix}$

∴ The intersection point is $(-5,-3,7)$

We would know how to do this if the equation of the line were in vector form. Set each expression = μ

(b) $\begin{cases}x-1=-\mu\\y=-3\mu\\z+4=2\mu\end{cases}$

$\Leftrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-\mu\\-3\mu\\-4+2\mu\end{pmatrix}$

Substitute x, y, z into the equation of the plane

$\therefore (1-\mu)-3(-3\mu)-4(-4+2\mu)=12$

$\Rightarrow 17=12$

It is impossible to find a value of μ for a point which satisfies both equations. This means that the line is parallel to the plane

Impossible to find μ.

∴ The line and plane do not intersect.

Substitute x, y, z into the equation of the plane

(c) $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3+3t\\-1+t\\1\end{pmatrix}$

$(3+3t)-3(-1+t)=6$

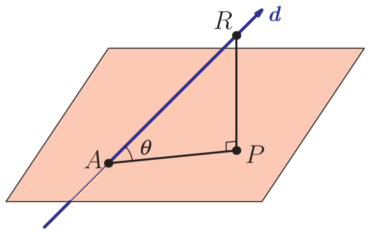
$\Rightarrow 6=6$

The equation is satisfied for all values of t. This means that every point on the line also lies in the plane

Every t is a solution.

∴ The line lies in the plane.

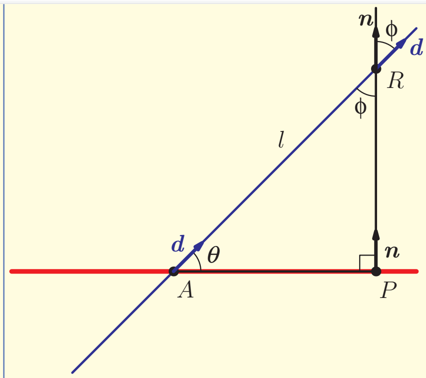
When a line intersects a plane, we can find the angle between them. First we need to decide which angle to find. If we take different lines in the plane, they will make different angles with the given line l. The smallest possible angle θ is with the line $[AP]$ shown in the diagram. Drawing a two-dimensional diagram of triangle APR makes it clearer what the angles are.



Worked example 14.20

Find the angle between line l with equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ and the plane with equation $5x - y + z = 7$.

We want to find the angle marked θ. We don't know the direction of the line (AP), but we do know that $\overrightarrow{AR}$ is in the direction of the line and $\overrightarrow{RP}$ is in the direction of the normal to the plane



Therefore we can find the angle marked ϕ

$$\cos\phi = \frac{\boldsymbol{l} \bullet \boldsymbol{n}}{|\boldsymbol{l}||\boldsymbol{n}|}$$

$$= \frac{\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{9+9+4}\sqrt{25+1+1}}$$

$$= \frac{14}{\sqrt{22}\sqrt{27}}$$

$\therefore \phi = 54.9°$

Then we use the fact that $A\hat{P}R$ is a right angle

$\theta = 90° - \phi = 35.1°$

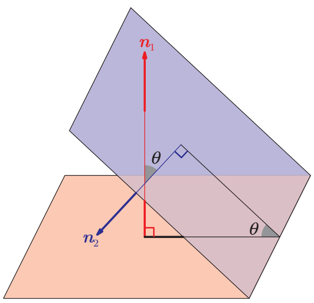
EXAM HINT

If the angle ϕ between the line and the normal is obtuse, find $\phi_1 = 180° - \phi$, and then $\theta = 90° - \phi_1$

The method shown in Worked example 14.20 can always be used to find the angle between a line and a plane.

KEY POINT 14.11

The angle between line with direction vector $\boldsymbol{d}$ and plane with normal $\boldsymbol{n}$ is $90° - \phi$, where ϕ is the acute angle between $\boldsymbol{d}$ and $\boldsymbol{n}$.



We can also find the angle between two planes. The diagram shows two planes and their normals. Using the fact that the sum of the angles in a quadrilateral is 360°, you can show that the two angles marked θ are equal.

KEY POINT 14.12

The angle between two planes is equal to the angle between their normals.

Worked example 14.21

Find the acute angle between planes with equations $4x - y + 5z = 11$ and $x + y - 3z = 3$.

We need to find the angle between the normals

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

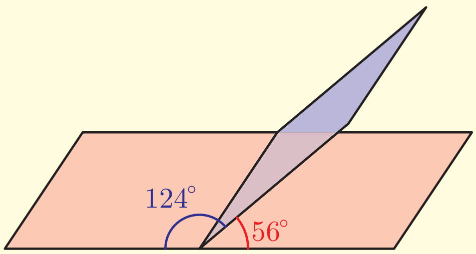
The components of the normal vectors are the coefficients in the Cartesian equations

$$= \frac{\begin{pmatrix}4\\-1\\5\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-3\end{pmatrix}}{\sqrt{16+1+25}\sqrt{1+1+9}}$$

$$= \frac{-12}{\sqrt{42}\sqrt{11}}$$

$$\Rightarrow \theta = 123.9°$$

We need the acute angle



$180° - 123.9° = 56.1°$

The angle between the planes is 56.1°

In our final example we look at the intersection of two planes. We will find the intersection of the planes with equations $4x + 5y - z = 7$ and $x - 4z = -7$.

Two planes intersect along a straight line. Every point on this line must satisfy equations of both planes, so it is a solution of a system of two equations with three unknowns. We saw in Worked example 4.13 that such a system has infinitely many solutions, and we can find the general solution using Gaussian elimination.

We covered Gaussian elimination in Section 4E.

Worked example 14.22

Find the line of intersection of planes with equations $4x + 5y - z = 7$ and $x - 4z = -7$.

The points on the intersection line satisfy both equation, so we need to solve the system using Gaussian elimination. We expect infinitely many solutions

$$\begin{cases} 4x + 5y - z = 7 & (1) \\ x - 4z = -7 & (2) \end{cases}$$

Note that there is no y in equation (2), so we can go straight to back substitution

$$(2) \Rightarrow z = t, x = 4t - 7$$

$$(1) \Rightarrow 5y = 7 + t - 4(4t - 7)$$
$$= 35 - 15t$$
$$\therefore y = 7 - 3t$$

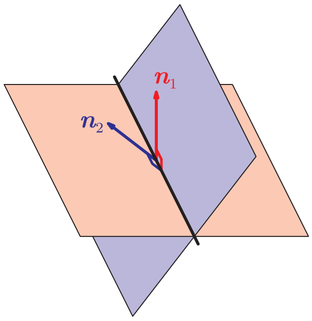
(x, y, z) are coordinates of any point on the line of intersection. To find the vector equation of the line we should write the coordinates as a position vector

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4t - 7 \\ 7 - 3t \\ t \end{pmatrix}$$
$$= \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 4t \\ -3t \\ t \end{pmatrix}$$

This is of the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$, which is a vector equation of a line

The equation of the line is

$$\underline{r} = \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$



It is worth noting that since the line of intersection lies in both planes, it must be perpendicular to the two normals. This can be used to find the direction of the line.

KEY POINT 14.13

The line of intersection of planes with normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ has direction $\boldsymbol{n}_1 \times \boldsymbol{n}_2$.

We can check in the above example that

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -20 \\ 15 \\ -5 \end{pmatrix},$$ which is parallel to the direction $\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ of the intersection line we found.

Worked example 14.23

Find the vector equation of the line of intersection of the planes with equations $3x - y + z = 0$ and $x - 3y - z = 0$.

Both planes contain the origin, so the line of intersection will also pass through the origin. So we only need to find the direction vector

The line passes through $\boldsymbol{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Direction vector:

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix}$$

$$\therefore \boldsymbol{d} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

Vector equation is $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{c}$

Line of intersection:

$$\underline{r} = \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

The only situation when the above methods will fail to find the intersection is if the two planes are parallel. In that case their normal vectors are parallel, and this is easy to see from the Cartesian equations. For example, the planes with equations $2x-4y+2z=5$ and $3x-6y+3z=1$ are parallel, since

$\begin{pmatrix}3\\-6\\3\end{pmatrix}=\frac{3}{2}\begin{pmatrix}2\\-4\\2\end{pmatrix}$. Note that before stating that the two planes:

are parallel you should check that the two equations do not represent the same plane. In this example they are not the same,

as multiplying the first equation by $\frac{3}{2}$ gives $3x-6y+3z=\frac{15}{2}$.

Exercise 14E

1. Find the coordinates of the point of intersection of line l and plane Π.

(a) (i) $l:\mathbf{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\0\\-1\end{pmatrix}$, $\Pi: 4x+2y-z=29$

(ii) $l:\mathbf{r}=\begin{pmatrix}-5\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}7\\3\\-3\end{pmatrix}$, $\Pi: x+y+5z=11$

(b) (i) $l:\frac{x-2}{5}=\frac{y+1}{2}=\frac{z}{6}$, $\Pi:\mathbf{r}\cdot\begin{pmatrix}1\\-4\\1\end{pmatrix}=4$

(ii) $l:\frac{x-5}{-1}=\frac{y-3}{2}=\frac{z-5}{1}$, $\Pi:\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=21$

2. Find the acute angle between line l and plane Π, correct to the nearest 0.1°.

(a) (i) $l:\mathbf{r}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\Pi:\mathbf{r}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix}=7$

(ii) $l:\mathbf{r}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\1\end{pmatrix}$, $\Pi:\mathbf{r}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=1$

(b) (i) $l: \frac{x}{2} = \frac{y-1}{5} = \frac{z-2}{5}$, $\Pi: x - y - 3z = 1$

(ii) $l: \frac{x+1}{-1} = \frac{y-3}{3} = \frac{z+2}{-3}$, $\Pi: 2x + y + z = 14$

3. Find the acute angle between the following pairs of planes:

(a) $3x - 7y + z = 4$ and $x + y - 4z = 5$

(b) $x - z = 4$ and $y + z = 1$

4. Find a vector equation of the line of intersection of the following pairs of planes:

(a) (i) $3x + y - z = 3$ and $x - 2y + 4z = -5$

(ii) $x + y = 3$ and $x - y = 5$

(b) (i) $2x - y = 4$ and $2y + z = 5$

(ii) $x + 2y - 5z = 6$ and $z = 0$

5. Plane Π_1 has Cartesian equation $3x - y + z = 7$.

(a) Write down a normal vector of Π_1.

Plane Π_2 has equation $x - 5y + 5z = 11$.

(b) Find, correct to the nearest degree, the acute angle between Π_1 and Π_2. *[6 marks]*

6. Find the coordinates of the point of intersection of line $\frac{x-2}{3} = \frac{y-1}{2} = z$ with the plane $2x - y - 2z = 5$. *[5 marks]*

7. Show that the lines $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ and $\frac{x-1}{4} = \frac{y+2}{3} = \frac{2z-1}{4}$ do not intersect. *[5 marks]*

8. The plane with equation $12x - 3y + 5z = 60$ intersects the x-, y- and z-axes at points P, Q and R respectively. *[7 marks]*

(a) Find the coordinates of P, Q and R.

(b) Find the area of the triangle PQR.

9. Plane Π has equation $5x - 3y - z = 1$.

(a) Show that point P (2, 1, 6) lies in Π.

(b) Point Q has coordinates (7, −1, 2). Find the exact value of the sine of the angle between (PQ) and Π.

(c) Find the exact distance PQ.

(d) Hence find the exact distance of Q from Π. *[10 marks]*

10. Two planes have equations:

$$\Pi_1 : 3x - y + z = 17$$

$$\Pi_2 : x + 2y - z = 4$$

(a) Calculate $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Show that Π_1 and Π_2 are perpendicular.

(c) Show that the point M (1, 1, 2) does not lie in either of the two planes.

(d) Find a vector equation of the line through M which is parallel to both planes. *[10 marks]*

11. (a) Calculate $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$.

(b) Planes Π_1 and Π_2 have equations:

$$\Pi_1 : x + 3y - 2z = 0$$

$$\Pi_2 : 3x + 5y - z = 0$$

l is the line of intersection of the two planes.

(i) Show that l passes through the origin.

(ii) Write down a vector equation for l.

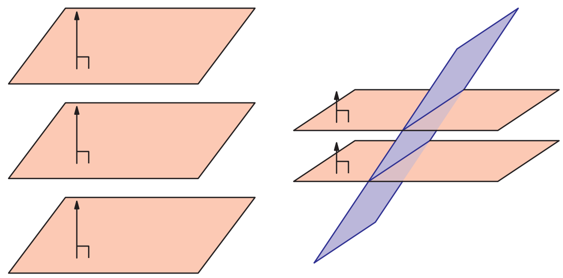
(c) A third plane has equation:

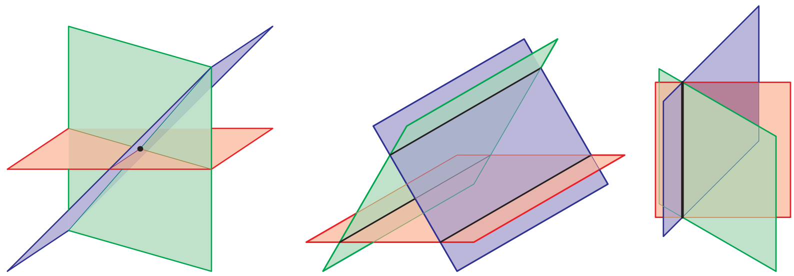
$$\Pi_3 : x - 5z + z = 8$$

Find the coordinates of the intersection of all three planes. *[10 marks]*

14F Intersection of three planes

In the last example of the previous section we saw how to find the line of intersection of two planes. Two different planes can either intersect or be parallel. When we have three planes there are many more possibilities.





Before looking through the examples you may wish to revisit Section 4E on solving systems of equations.

EXAM HINT

Remember that you can solve systems of equations either using the simultaneous equation solver, or perform Gaussian elimination using a matrix. See calculator skills sheets 5 and 6 on the CD ROM.

In the first two cases some of the planes are parallel. As we mentioned in the last section, this can be seen from the equations because one normal vector will be a multiple of another. When none of the planes are parallel, there are still three different possibilities for how they can intersect. To find out which of those is the case, we need to try solving the equations.

Any point which lies in the intersection of the three planes must satisfy all three equations. It is therefore a solution of the system of three equations with three unknowns. As we saw in Section 4E, such a system can have a unique solution, no solutions or infinitely many solutions. These three cases correspond to the three possibilities of how three planes can intersect.

Worked example 14.24

Find the intersection of the planes with equations:

$$3x - y + 4z = 7$$
$$x - 2y + z = 3$$
$$x - y + 4z = -5$$

continued . . .

Points of intersection will be the solutions of the system of equations. We can try solving them on the calculator

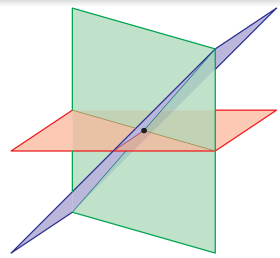
Using GDC:

$x=6, y=\frac{1}{7}, z=-\frac{19}{7}$

So the three planes intersect at the point $\left(6, \frac{1}{7}, -\frac{19}{7}\right)$.

In this example the system of equations has a unique solution, so the planes intersect at a single point, as in the third diagram.

In the next example the system of equations has no solutions.



Worked example 14.25

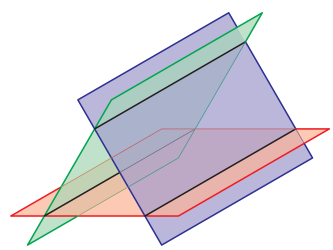
Show that the three planes with equations:

$x+2y+3z=10$

$2x+3y+2z=4$

$4x+7y+8z=7$

do not intersect.



We can attempt to solve the equations and show that it is impossible. Without a calculator, use Gaussian elimination

$$\begin{cases} x+2y+3z=10 & (1) \\ 2x+3y+2z=4 & (2) \\ 4x+7y+8z=7 & (3) \end{cases}$$

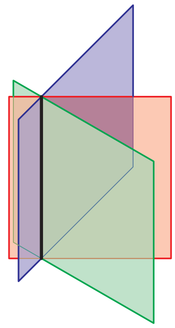
$$\begin{matrix} \\ 2\times(1)-(2) \\ 4\times(1)-(3) \end{matrix} \Rightarrow \begin{cases} x+2y+3z=10 & (1) \\ y+4z=16 & (4) \\ y+4z=23 & (5) \end{cases}$$

$$\begin{matrix} \\ \\ (5)-(4) \end{matrix} \Rightarrow \begin{cases} x+2y+3z=10 & (1) \\ y+4z=16 & (4) \\ 0z=-7 & (6) \end{cases}$$

The last equation is impossible, so we cannot find z

Equation (6) has no solutions, so the three planes do not intersect.

The result for the direction of the line of intersection of two planes was given in Key point 14.13.



In the above example none of the three planes are parallel, so each two planes intersect along a line. However, the line of intersection of any two planes is parallel to the third plane. For example, the line of intersection of the first two planes has direction $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}-5\\4\\-1\end{pmatrix}$. This is perpendicular to the normal of the third plane: $\begin{pmatrix}-5\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\7\\8\end{pmatrix}=-20+28-8=0$. Hence the line with direction $\begin{pmatrix}-5\\4\\-1\end{pmatrix}$ is parallel to the third plane.

The final possibility is for the three planes to intersect along a line, as in the next example. It also shows you a common type of exam question where you have to find an unknown parameter.

Worked example 14.26

Find the value of c for which the planes with equations

$$2x-y-3z=3$$
$$x+y-3z=0$$
$$x+2y-4z=c$$

intersect, and find the equation of the line of intersection.

This is the same as finding the value of the parameter for which the system of equations is consistent, as in Worked example 4.14.

Solve the equations using Gaussian elimination

$$\begin{cases}2x-y-3z=3 & (1)\\ x+y-3z=0 & (2)\\ x+2y-4z=c & (3)\end{cases}$$

$$\begin{matrix}\\ 2\times(2)-(1)\\ 2\times(3)-(1)\end{matrix}\Rightarrow\begin{cases}2x-y-3z-3 & (1)\\ 3y-3z=-3 & (4)\\ 5y-5z-2c & (5)\end{cases}$$

continued . . .

$$5\times(4)-3\times(5) \quad \Rightarrow \begin{cases} 2x-y-3z=3 & (1) \\ 3y-3z=-3 & (4) \\ 0z=-6c-6 & (6) \end{cases}$$

Equation (6) only has a solution when RHS=0

This has a solution when

$-6c-6=0$

$\therefore c=-1$

Then (6) says $0z = 0$, so z can be any number

For any $t\in\mathbb{R}$:

$z=t$

$(4)\Rightarrow 3y=-3+3t$

$\therefore y=-1+t$

$(1)\Rightarrow 2x=3+3t+(-1+t)$

$=2+4t$

$\therefore x=1+2t$

(x, y, z) are the coordinates of a point where the three planes intersect. There are infinitely many points, and we can write $\begin{pmatrix}x\\y\\z\end{pmatrix}$ as a vector equation of a line

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+\begin{pmatrix}2t\\t\\t\end{pmatrix}$$

So the equation of the line of intersection is

$$\underline{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Remember that your calculator can perform Gaussian elimination even when the equations do not have a unique solution. If the last equation is $0=0$ you can use the first two equations to find the general solution.

Exercise 14F

1. Find the coordinates of the point of intersection of the planes:

$$\begin{aligned}\Pi_1 &: \quad 3x + y + z = 8\\ \Pi_2 &: -7x + 3y + z = 2\\ \Pi_3 &: \quad x + y + 3z = 0\end{aligned}$$

[3 marks]

2. Find the coordinates of the point of intersection of the three planes with equations $x = 2,\ x + y - z = 7$ and $2x + y + z = 3.$ *[4 marks]*

3. Find the equation of the line of intersection of the planes $2x - z = 1,\ 4x + y - z = 5$ and $y + z = 3$. *[4 marks]*

4. Find the intersection of the planes:

$$\begin{aligned}x - 2y + z &= 5\\ 2x + y + z &= 1\\ x + 2y - z &= -2\end{aligned}$$

[5 marks]

5. Show that the planes with equations $2x - y + z = 6$, $3x + y + 5z = -7$ and $x - 3y - 3z = 8$ do not intersect. *[4 marks]*

6. Three planes have equations:

$$\Pi_1 : 2x + y - 2z = 0$$

$$\Pi_2 : x - 2y - z = 2$$

$$\Pi_3 : 3x + 4y - 3z = d$$

(a) Find the value of d for which the three planes intersect.

(b) For this value of d, find the equation of the line of intersection of the three planes. *[7 marks]*

7. Three planes have equations:

$$\begin{aligned}\Pi_1 &: x - y = 4\\ \Pi_2 &: y + z = 1\\ \Pi_3 &: x - z = d\end{aligned}$$

Find, in terms of d, the coordinates of the point of intersection of the three planes. *[5 marks]*

8. (a) Explain why the intersection of the planes

$$\Pi_1 : x + y = 0$$
$$\Pi_2 : x - 4y - 2z = 0$$
$$\Pi_3 : \frac{1}{2}x + 3y + z = 0$$

contains the origin.

(b) Show that the intersection of the three planes is a line and find its direction vector in the form $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, where $a, b, c \in \mathbb{Z}$. *[7 marks]*

9. (a) Find the value of a for which the three planes

$$\Pi_1 : x - 2y + z = 7$$
$$\Pi_2 : 2x + y - 3z = 9$$
$$\Pi_3 : x + y - az = 3$$

do not intersect.

(b) Find the Cartesian equation of the line of intersection of Π_1 and Π_2. *[9 marks]*

10. Three planes have equations

$$x - y - z = -2$$
$$2x + 3y - 7z = a + 4$$
$$x + 2y + pz = a^2$$

(a) Find the value of p and the two values of a for which the intersection of the three planes is a line.

(b) For the value of p and the larger value of a found above, find the equation of the line of intersection.

[12 marks]

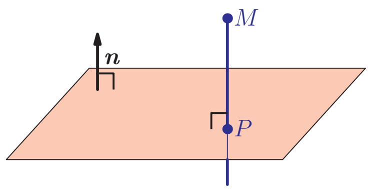
14G Strategies for solving problems with lines and planes

We now have all the tools we need to solve more complex problems involving lines and planes in space. We can find equations of lines and planes determined by points, intersections and angles between two lines, two planes, or a line and a plane. We also know how to calculate the distance between two points and areas of triangles.

Solving a more complex problem requires two things:

- a strategy saying what needs to be calculated
- being able to carry out all the calculations.

The second part is what we have been practising so far. In this section we look at strategies to solve the most common problems. There are no examples – you need to select the most appropriate strategy for each question. For each problem we explain the reasons behind the choice of strategy and then list the required steps.



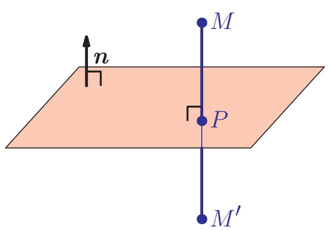
Distance of a point from a plane

Given a plane with equation $\boldsymbol{a}\cdot\boldsymbol{n} = d$ and a point M outside of the plane, the distance from M to the plane is equal to the distance MP, where the line (MP) is perpendicular to the plane. This means that the direction of (MP) is $\boldsymbol{n}$.

To find the distance MP:

- Write down the vector equation of the line with direction $\boldsymbol{n}$ through point M.
- Find the intersection, P, between the line and the plane.
- Calculate the distance MP.

The point P is called the **foot of the perpendicular** from the point to the plane.

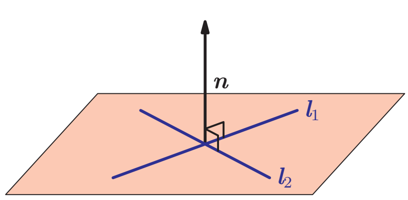


Reflection of a point in a plane

Given a plane Π with equation $\boldsymbol{a}\cdot\boldsymbol{n} = d$ and a point M which is not in the plane, the reflection of M in Π is the point M' such that MM' is perpendicular to the plane and the distance of M' from Π is the same as the distance of M from Π. Calculations with distances can be difficult, so instead we can use the fact that, since MPM' is a straight line, $\overrightarrow{PM'} = \overrightarrow{MP}$.

To find the coordinates of M':

- Write down the vector equation of the line with direction $\boldsymbol{n}$ through point M.
- Find the intersection, P, between the line and the plane.
- Find the point M' such that $\overrightarrow{PM'} = \overrightarrow{MP}$ by using position vectors: $\boldsymbol{m}' - \boldsymbol{p} = \boldsymbol{p} - \boldsymbol{m}$.



Equation of a plane determined by two intersecting lines

We noted in Key point 14.8 that a plane is uniquely determined by two intersecting lines. In other words, if we have equations of two lines that intersect, we should be able to find the equation of the plane containing both of them.

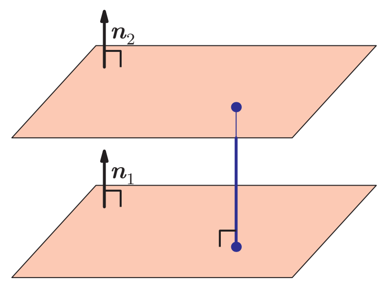
To do this, we note that the normal to the plane must be perpendicular to both lines (this is the definition of the normal;

a vector which is perpendicular to all the lines in the plane). But we know that the vector product of two vectors is perpendicular to *both* of them. So we can take the normal vector to be the vector product of the direction vectors of the two lines.
To complete the scalar product equation of the plane we also need one point. The intersection of the two lines clearly lies in the plane, so we can use this point, or we can use any point on either of the two lines, which may be easier!

So, to find the equation of the plane containing lines $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \lambda \boldsymbol{d}_2$:

- The normal vector is $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$.
- For a point P in the plane, pick any point on either of the two lines ($\boldsymbol{a}$, $\boldsymbol{b}$ or the intersection are some possible choices).
- The scalar product equation of the plane is $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{p} \bullet \boldsymbol{n}$.

Distance between parallel planes

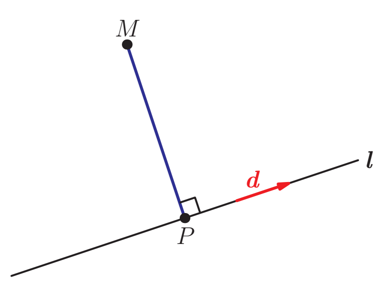


If two planes are parallel, we can find the perpendicular distance between them. To do this, we note that the perpendicular distance is measured in the direction of the normal vector of the two planes. (Since the planes are parallel, their normals are in the same direction!)

One possible strategy is as follows:

- Pick a point in the first plane.
- Write down the equation of the line in the direction of the normal passing through this point.
- Find the intersection point of this line and the second plane.
- Find the distance between the two points.

Distance from a point to a line



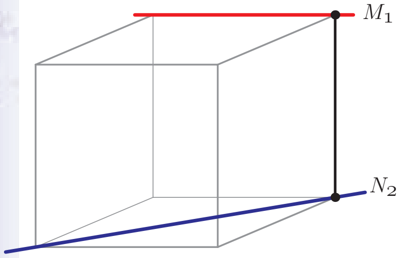
We have already seen an example of this in Worked example 14.8. The strategy is based on the fact that the distance is measured along a direction perpendicular to the line. However, there is more than one direction perpendicular to any given line so we cannot not just write down the required direction. Instead, we use a general point, P, on the line (given by the position vector $\boldsymbol{r}$) and use scalar product to express the fact that MP is perpendicular to the line.

To find the shortest distance from point M to a line l given by $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$:

- Form the vector $\overrightarrow{MP} = \boldsymbol{r} - \boldsymbol{m}$; it will be in terms of λ.
- $[MP]$ is perpendicular to the line: $\overrightarrow{MP} \bullet \boldsymbol{d} = 0$.
- This is an equation for λ; solve it.

- Use this value of λ to find $\overrightarrow{MP}$.
- The required distance is $\left|\overrightarrow{MP}\right|$.

Distance between two skew lines



Consider points M and N moving along two skew lines, l_1 and l_2 respectively. The distance between them is minimum possible when $[MN]$ is perpendicular to both lines. It may not be immediately obvious that such a position of M and N always exists, but it does. When you sketch a diagram of this, it is useful to imagine a cuboid, where one line runs along an upper edge, and the other runs along the diagonal of the base, as shown. The shortest distance between the two is then the height of the cuboid.

Suppose the two skew lines have equations $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \mu \boldsymbol{d}_2$. The strategy is similar to finding the distance between a line and a plane, except now we have two general points, one on each line.

- Write down position vectors of two general points M and N, one on each line, using the equations for $\boldsymbol{r}$.
- Form the vector $\overrightarrow{MN}$; this will be in terms of both λ and μ.
- Write down two equations: $\overrightarrow{MN} \bullet \boldsymbol{d}_1 = 0$ and $\overrightarrow{MN} \bullet \boldsymbol{d}_2 = 0$.
- These are simultaneous equations for λ and μ; solve them.
- Use the values of λ and μ to find $\overrightarrow{MN}$.
- The required distance is $\left|\overrightarrow{MN}\right|$.

Exam questions usually give you hints to help solve any of the above problems. (Any question that does not would definitely be difficult!) However, questions often ask you to carry out the required calculations, but not tell you how to fit them together to solve the final part. This is why it is extremely useful to draw a diagram and label everything you have found. Remember that the diagrams are just sketches showing relative positions of points, lines and planes – they do not have to be accurate.

The exam-style questions in the following exercise are intended give you an idea how much guidance you can expect to get. Use the strategies described in this section to help you. Some questions will not use any of the above strategies, but you will be given hints.

Exercise 14G

1. Plane Π has equation $2x+2y-z=11$. Line l is perpendicular to Π and passes through the point $P(-3, -3, 4)$.

(a) Find the equation of l.

(b) Find the coordinates of the point Q where l intersects Π.

(c) Find the shortest distance from P to Π. *[8 marks]*

2. Two planes have equations:

$$\Pi_1 : x - 3y + z = 6$$

$$\Pi_2 : 3x - 9y + 3z = 0$$

(a) Show that Π_1 and Π_2 are parallel.

(b) Show that Π_2 passes through the origin.

(c) Write down the equation of the line through the origin which is perpendicular to Π_2.

(d) Hence find the distance between the planes Π_1 and Π_2. *[10 marks]*

3. (a) Calculate $\begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\1\\3\end{pmatrix}$.

(b) Two lines have equations:

$$l_1 : \boldsymbol{r} = \begin{pmatrix}7\\-3\\2\end{pmatrix} + t\begin{pmatrix}-1\\0\\2\end{pmatrix} \text{ and } l_2 : \boldsymbol{r} = \begin{pmatrix}1\\1\\26\end{pmatrix} + s\begin{pmatrix}0\\1\\3\end{pmatrix}.$$

(i) Show that l_1 and l_2 intersect.

(ii) Find the coordinates of the point of intersection.

(c) Plane Π contains lines l_1 and l_2 Find the Cartesian equation of Π. *[11 marks]*

4. Four points have coordinates $A(7, 0, 1)$ $B(8, -1, 4)$, $C(9, 0, 2)$ and $D(6, 5, 3)$

(a) Show that $\overrightarrow{AD}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Write down the equation of the plane Π containing the points A, B and C in the form $\boldsymbol{r} \bullet \boldsymbol{n} = k$.

(c) Find the exact distance of point D from plane Π.

(d) Point D_1 is the reflection of D in Π. Find the coordinates of D_1. *[10 marks]*

5. Two lines are given by Cartesian equations:

$$l_1 : \frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-2}{1}$$

$$l_2 : \frac{x-5}{3} = 1 - y = z + 4$$

(a) Show that l_1 and l_2 are parallel.

(b) Show that the point $A\ (14, -5, 6)$ lies on l_1.

(c) Find the coordinate of point B on l_2 such that (AB) is perpendicular to the two lines.

(d) Hence find the distance between l_1 and l_2, giving your answer to 3 significant figures. *[10 marks]*

6. (a) Find the coordinates of the point of intersection of lines

$$l_1 : \frac{x-1}{3} = \frac{y+1}{4} = \frac{3-z}{3} \text{ and } l_2 : \frac{x+12}{2} = \frac{y}{1} = \frac{z+17}{1}.$$

(b) Find a vector perpendicular to both lines.

(c) Hence find the Cartesian equation of the plane containing l_1 and l_2. *[13 marks]*

7. Points $A\ (8,\ 0,\ 4)$, $B\ (12, -1,\ 5)$ and $C\ (10,\ 0,\ 7)$ lie in the plane Π.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Hence find the area of the triangle ABC, correct to 3 significant figures.

(c) Find the Cartesian equation of Π.

Point D has coordinates $(-7, -28, 11)$.

(d) Find a vector equation of the line through D perpendicular to the plane.

(e) Find the intersection of this line with Π, and hence find the perpendicular distance of D from Π.

(f) Find the volume of the pyramid $ABCD$. *[16 marks]*

8. Line l passes through point $A\ (-1, 1, 4)$ and has direction vector $\boldsymbol{d} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$. Point B has coordinates $(3, 3, 1)$. Plane Π has normal vector $\boldsymbol{n}$, and contains the line l and the point B.

(a) Write down a vector equation for l.

(b) Explain why $\overrightarrow{AB}$ and $\boldsymbol{d}$ are both perpendicular to $\boldsymbol{n}$.

(c) Hence find one possible vector $\boldsymbol{n}$.

(d) Find the Cartesian equation of plane Π. *[10 marks]*

9. Plane Π has equation $6x - 2y + z = 16$. Line l is perpendicular to Π and passes through the origin.

(a) Find the coordinates of the foot of the perpendicular from the origin to Π.

(b) Find the shortest distance of Π from the origin, giving your answer in exact form. *[8 marks]*

10. (a) Show that the planes $\Pi_1 : x - z = 4$ and $\Pi_2 : z - x = 8$ are parallel.

(b) Write down a vector equation of the line through the origin which is perpendicular to the two planes.

(c) (i) Find the coordinates of the foot of the perpendicular from the origin to Π_1.

(ii) Find the coordinates of the foot of the perpendicular from the origin to Π_2.

(d) Use your answers from part (c) to find the exact distance between the two planes. *[11 marks]*

Summary

- Vector equations give position vectors of points on a line or a plane.
- The **vector equation** of a line has the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$, where $\boldsymbol{d}$ is a vector in the direction of the line and $\boldsymbol{a}$ is the position vector of one point on the line. $\boldsymbol{r}$ is the position vector of a general point on the line and the parameter λ gives positions of different points on the line.
- Vector equation of a plane has the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$, where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are two vectors parallel to the plane and $\boldsymbol{a}$ is the position vector of one point in the plane.
- Cartesian equations are equations satisfied by the coordinates of a point on the line or in the plane.
- To uniquely determine a plane we need three points, not on the same line, OR a line and a point outside that line, OR two intersecting lines.
- **Cartesian equation** of a line has the form $\frac{x-a}{k} = \frac{y-b}{m} = \frac{z-c}{n}$, and can be derived from the vector equation by writing three equations for λ in terms of x, y and z. If we express x, y and z in terms of λ instead, we obtain **parametric equations** of the line.
- The Cartesian equation of a plane has the form $n_1x + n_2y + n_3z = k$. This can also be written in the scalar product form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, where $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is the **normal vector** of the plane, which is perpendicular to every line in the plane. To derive the Cartesian equation from a vector equation, use $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$.

- The angle between two lines is the angle between their direction vectors.
- Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel if $\boldsymbol{d}_1 = k\boldsymbol{d}_2$, perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.
- The angle between two planes is the angle between their normals.
- The angle between a line and a plane is $90° - \theta$, where θ is the angle between the line direction vector and the plane's normal.
- To find the intersection of two lines, set the two position vectors equal to each other and use two of the equations to find λ and μ. If these values do not satisfy the third equation, the lines are **skew lines**.
- To find the intersection between a line and a plane, express x, y and z for the line in terms of λ and substitute into the Cartesian equation of the plane.
- The line of intersection of two planes has direction parallel to $\boldsymbol{n}_1 \times \boldsymbol{n}_2$, and we can use any point which satisfies both plane equations.
- The intersection of two planes or three planes can be found by solving the system of equations given by the Cartesian equations of the planes.
- Three distinct planes may intersect at a single point, along a straight line, or have no intersection at all. These cases correspond to the different possibilities for the solutions of a system of three equations. When the solution is not unique, the straight line corresponds to the general solution of the system.
- The vector equation of a line can be used to describe the path of an object moving with constant velocity. For an object moving with constant velocity $\boldsymbol{v}$ from an initial position $\boldsymbol{a}$, the direction vector of the line can be taken to be the velocity vector, and the position of time $\boldsymbol{t}$ is given by $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$. The object moves along the straight line with equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$. The speed is equal to $|v|$.

In solving problems with lines and planes we often need to set up and solve equations. In doing so we use properties of vectors, in particular the fact that the magnitude of a vector represents distance, and that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for perpendicular vectors. We also need to use diagrams, often to identify right angled triangles. In longer questions, we can combine several answers to solve the last part.

Introductory problem revisited

Which is more stable (less wobbly): a three-legged stool or a four-legged stool?

A stool will be stable if the end points of all the legs lie in the same plane. As we have seen, we can always find a plane containing three points, so a three-legged stool is stable, it never wobbles. This is why photographers place their cameras on tripods.

If there are four points, it is possible that the fourth one does not lie in the same plane as the other three. So if the legs are not all of the same length, the four end points could determine four different planes. Equally, if the floor is slightly uneven, only a three-legged stool can be relied upon to be stable. This is why four-legged chairs and tables often wobble. However, we also want furniture that is not easily knocked over – can you see why stool legs which form a square at the base might be better than legs that form a triangle in this respect?

Mixed examination practice 14

Short questions

1. Find a vector equation of the line passing through points $(3, -1, 1)$ and $(6, 0, 1)$. *[4 marks]*

2. The point $(3, -1, 2)$ lies on the line with equation $\frac{x+3}{2} = \frac{y-8}{-3} = \frac{z+13}{p}$.

   Find the value of p. *[4 marks]*

3. The vector $\boldsymbol{n} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is normal to a plane which passes through the point $(3, -1, 2)$.

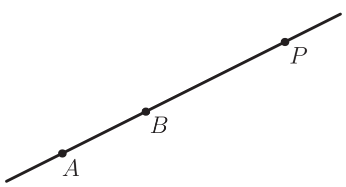
   (a) Find an equation for the plane.

   (b) Find a if the point $(a, 2a, a-1)$ lies on the plane. *[6 marks]*

4. Find the coordinates of the point of intersection of the planes with equations $x - 2y + z = 5$, $2x + y + z = 1$ and $x + 2y - z = -2$. *[6 marks]*

5. Points A $(-1, 1, 2)$ and B $(3, 5, 4)$ lie on the line with equation $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

   Find the coordinates of point P on the same line such that $AP = 3AB$, as shown in the diagram.



   [5 marks]

6. Point A $(-3, 0, 4)$ lies on the line $\boldsymbol{r} = -3\boldsymbol{i} + 4\boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$, where λ is a real parameter. Find the coordinates of one point on the line which is 10 units from A. *[6 marks]*

7. Points A $(4, 1, 12)$ and B $(8, -11, 20)$ lie on the line l.

   (a) Find an equation of line l, giving the answer in parametric form.

   (b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P. *[6 marks]*

8. (a) Given that $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$, show that $\boldsymbol{b} \times \boldsymbol{a} = 3\boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k}$.

   Two planes have equations $\boldsymbol{r} \cdot \boldsymbol{a} = 5$ and $\boldsymbol{r} \cdot \boldsymbol{b} = 12$.

   (b) Show that the point $(2, 2, 3)$ lies in both planes.

   (c) Write down the Cartesian equation of the line of intersection of the two planes. *[6 marks]*

9. The plane $3x+2y-z=2$ contains the line $x-3=\frac{2y+2}{5}=\frac{z-5}{k}$. Find the value of k. *[6 marks]*

10. (a) If $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{v}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ show that $\boldsymbol{u}\times\boldsymbol{v}=7\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}$.

(b) Let $\boldsymbol{w}=\lambda\boldsymbol{u}+\mu\boldsymbol{v}$ where λ and μ are scalars. Show that $\boldsymbol{w}$ is perpendicular to the line of intersection of the planes $x+2y+3z=5$ and $2x-y+2z=7$ for all values of λ and μ.

[8 marks]

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11. Find the Cartesian equation of the plane containing the two lines

$$x=\frac{3-y}{2}=z-1 \text{ and } \frac{x-2}{3}=\frac{y+1}{-3}=\frac{z-3}{5}.$$ *[8 marks]*

Long questions

1. Points A and B have coordinates $(4, 1, 2)$ and $(0, 5, 1)$. Line l_1 passes through A and has equation $\boldsymbol{r}_1=\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$. Line l_2 passes through B and has equation $\boldsymbol{r}_2=\begin{pmatrix}0\\5\\1\end{pmatrix}+t\begin{pmatrix}4\\-4\\1\end{pmatrix}$.

(a) Show that the line l_2 also passes through A.

(b) Calculate the distance AB.

(c) Find the angle between l_1 and l_2 in degrees.

(d) Hence find the shortest distance from B to l_1. *[10 marks]*

2. (a) Show that the lines $l_1:\boldsymbol{r}=\begin{pmatrix}-3\\3\\18\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-8\end{pmatrix}$ and $l_2:\boldsymbol{r}=\begin{pmatrix}5\\0\\2\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-1\end{pmatrix}$ do not intersect.

(b) Points P and Q lie on l_1 and l_2 respectively, such that (PQ) is perpendicular to both lines.

(i) Write down $\overrightarrow{PQ}$ in terms of λ and μ.

(ii) Show that $9\mu-69\lambda+147=0$.

(iii) Find a second equation for λ and μ.

(iv) Find the coordinates of P and the coordinates of Q.

(v) Hence find the shortest distance between l_1 and l_2. *[14 marks]*

3. Plane Π has equation $x-2y+z=20$ and point A has coordinates $(4, -1, 2)$.

(a) Write down the vector equation of the line l through A which is perpendicular to Π.

(b) Find the coordinates of the point of intersection of line l and plane Π.

(c) Hence find the shortest distance from point A to plane Π. *[10 marks]*

4. In this question, unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ point East and North, and unit vector $\boldsymbol{k}$ is vertically up. The time (t) is measured in minutes and the distance in kilometres.

Two aircraft move with constant velocities $\boldsymbol{v}_1 = (7\boldsymbol{i}+10\boldsymbol{j}+3\boldsymbol{k})$ km/min and $\boldsymbol{v}_2 = (3\boldsymbol{i}-8\boldsymbol{j}-4\boldsymbol{k})$ km/min. At $t=0$, the first aircraft is at the point with coordinates (16, 30, 3) and the second aircraft at the point with coordinates (24, 66, 12).

(a) Calculate the speed of the first aircraft.

(b) Write down the position vector of the second aircraft at the time t minutes.

(c) Find the distance between the aircraft after 3 minutes.

(d) Show that there is a time when the first aircraft is vertically above the second one, and find the distance between them at that time.

5. Line L_1 has equation $\boldsymbol{r} = \begin{pmatrix}5\\1\\2\end{pmatrix} + t\begin{pmatrix}-1\\1\\3\end{pmatrix}$ and line L_2 has equation $\boldsymbol{r} = \begin{pmatrix}5\\4\\9\end{pmatrix} + s\begin{pmatrix}2\\1\\1\end{pmatrix}$.

(a) Find $\begin{pmatrix}-1\\1\\3\end{pmatrix} \times \begin{pmatrix}2\\1\\1\end{pmatrix}$.

(b) Find the coordinates of the point of intersection of the two lines.

(c) Write down a vector perpendicular to the plane containing the two lines.

(d) Hence find the Cartesian equation of the plane containing the two lines. *[10 marks]*

6. Three planes have equations:

$$\Pi_1 : 3x - y + z = 2$$

$$\Pi_2 : x + 2y - z = -1$$

$$\Pi_3 : 5x - 4y + dz = 3$$

(a) Find the value of d for which the three planes do not intersect.

(b) Find the vector equation of the line l_1 of intersection of Π_1 and Π_2.

(c) For the value of d found in part (a):

(i) Find the value of p so that the point $A\,(0, 1, p)$ lines on l_1.

(ii) Find the vector equation of the line l_2 through A perpendicular to Π_3.

(iii) Hence find the distance between l_1 and Π_3. *[17 marks]*

7. Line l_1 has Cartesian equation $\frac{x-2}{4} = \frac{y+1}{-3} = \frac{z}{3}$. Line l_2 is parallel to l_1 and passes through point $A\ (0,-1,\ 2)$.

(a) Write down a vector equation of l_2.

(b) Find the coordinates of the point B on l_1 such that (AB) is perpendicular to l_1.

(c) Hence find, to three significant figures, the shortest distance between the two lines. *[9 marks]*

8. Line L has equation $\frac{x+5}{3} = \frac{y-1}{3} = \frac{z-2}{-1}$.

(a) Show that the point A with coordinates $(4,10,-1)$ lies on L.

(b) Given that point B has coordinates $(2,1,2)$, calculate the distance AB.

(c) Find the acute angle between L and (AB) in radians.

(d) Find the shortest distance of B from L. *[12 marks]*

9. (a) The plane Π_1 has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane Π_2 has the equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) For points which lie on Π_1 and Π_2, show that $\lambda = \mu$.

(ii) Hence, or otherwise, find a vector equation of the line of intersection of Π_1 and Π_2.

(b) The plane Π_3 contains the line $\frac{2-x}{3} = \frac{y}{-4} = z+1$ and is perpendicular to $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. Find the cartesian equation of Π_3.

(c) Find the intersection of Π_1, Π_2 and Π_3. *[12 marks]*

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10. (a) Find the vector equation of the line L through point $A\left(-2, 4, 2\right)$ parallel to the vector $\boldsymbol{l} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(b) Point B has coordinates $(2,\ 3,\ 3)$. Find the cosine of the angle between (AB) and the line L.

(c) Calculate the distance AB.

(d) Point C lies on L and BC is perpendicular to L. Find the exact distance AC. *[10 marks]*

11. Plane Π has equation $x - 4y + 2z = 7$ and point P has coordinates $(9, -7, 6)$.

(a) Show that point R $(5, 1, 3)$ lies in the plane Π.

(b) Find the vector equation of the line (PR).

(c) Write down the vector equation of the line through P perpendicular to Π.

(d) N is the foot of the perpendicular from P to Π. Find the coordinates of N.

(e) Find the exact distance of point P from the plane Π. *[14 marks]*

12. Point A $(3,\ 1, -4)$ lies on line L which is perpendicular to plane $\Pi : 3x - y - z = 1$.

(a) Find the Cartesian equation of L.

(b) Find the intersection of the line L and plane Π.

(c) Point A is reflected in Π. Find the coordinates of the image of A.

(d) Point B has coordinates $(1, 1, 1)$. Show that B lies in Π.

(e) Find the distance between B and L. *[14 marks]*

13. (a) Calculate $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

(b) Plane Π_1 has normal vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and contains point A $(3, 4, -2)$.

Find the Cartesian equation of the plane.

(c) Plane Π_2 has equation $3x + y - z = 15$. Show that Π_2 contains point A.

(d) Write down the vector equation of the line of intersection of the two planes.

(e) A third plane, Π_3, has equation $\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 12$. Find the coordinates of the point of intersection of all three planes.

(f) Find the angle between Π_1 and Π_3 in degrees. *[17 marks]*

In this chapter you will learn:

- about a new set of numbers called the complex numbers, some of which square to give a negative number
- how to do arithmetic with complex numbers
- a geometric interpretation of complex numbers
- why we put complex numbers into pairs called conjugate pairs
- how conjugate pairs are useful when solving polynomial equations
- how you can find the sum and the product of all the roots of a polynomial equation without solving the equation
- how some arithmetic with complex numbers can be interpreted as rotation
- why it is sometimes convenient to describe complex numbers as exponentials
- how to find roots of complex numbers
- how complex numbers can be used to establish trigonometric identities.

15 Complex numbers

Introductory problem

Express $\sin 5x$ in terms of $\sin x$.

Very few topics in mathematics cause as much debate as complex numbers. This may be the first time that you meet a very common practice in research-level mathematics: first inventing a mathematical idea and then investigating the properties of it. With complex numbers, we start from the idea that there exists a number which squares to −1. We do not claim that this number is real, in fact we call it an **imaginary number**. You may wonder why mathematicians do this, but by the end of this chapter you should see that, in this case, there are many useful applications within the real world!

Is it more important for mathematics to be real or useful?

15A Definition and basic arithmetic of i

Although there is no real number that squares to give −1, we can imagine a number that does. Conventionally it is called i.

KEY POINT 15.1

$$\mathrm{i}^2 = -1$$

In all other ways, i acts just like a normal constant.

You may be sceptical about the idea of 'imagining' numbers. However, when negative numbers were introduced by the Indian mathematician Brahmagupta in the 7th Century there was just as much scepticism. 'How can you have –2 carrots?' they would ask. In Europe, it took until the 17th Century for negative numbers to be accepted. Mathematicians had worked successfully for thousands of years without using negative numbers, treating equations like $x + 3 = 0$ as having no solution. But once negative numbers were 'invented' it took only a hundred years or so to accept that equations such as $x^2 + 3 = 0$ also have a solution.

You need to be familiar with some common terminology for imaginary numbers. It does not give you any great insight into complex numbers but it does need to be learnt.

- A **complex number** is one that can be written as $x + iy$ where x and y are real. Commonly z is used to denote an unknown complex number and $\mathbb{C}$ is used for the set of all complex numbers.
- In the above definition x is the **real part** of z and it is given the symbol $\text{Re}(z)$; y is the **imaginary part** of z and it is given the symbol $\text{Im}(z)$. So for example $\text{Re}(3 - i)$ is 3 and $\text{Im}(3 - i)$ is -1.

EXAM HINT

Remember that the imaginary part is itself real: 'y' not 'iy'

We can now do some arithmetic with complex numbers.

Worked example 15.1

If $z = 3 + i$ and $w = 5 - 2i$ find:

(a) $z + w$ (b) $z - w$ (c) zw (d) $w \div 2$

Group real and imaginary parts

(a) $(3 + i) + (5 - 2i) = (3 + 5) + (i - 2i)$

Write in complex form

$= 8 - i$

Group real and imaginary parts

(b) $(3 + i) - (5 - 2i) = (3 - 5) + (i + 2i)$

Write in complex form

$= -2 + 3i$

(c) $(3 \times i) \times (5 - 2i)$

Multiply out the brackets

$= (3 \times 5) + (3 \times -2i) + (i \times 5) + (i \times -2i)$

$= 15 - 6i + 5i - 2i^2$

Remember $i^2 = -1$

$= 15 - i - (2 \times -1)$

$= 17 - i$

→

continued . . .

Use rules of fractions to separate real and imaginary parts

(d) $\frac{5-2i}{2} = \frac{5}{2} - \frac{2i}{2}$

$= \frac{5}{2} - i$

EXAM HINT

Do not forget that most graphical calculators can do complex arithmetic.

See Calculator skills sheet 8.

You must be able to work quickly and accurately both with and without a calculator.

The original purpose of introducing complex numbers was to solve quadratic equations with a negative discriminant. The next example shows you how to do this.

Worked example 15.2

(a) Find $\sqrt{-4}$. (b) Hence solve the equation $x^2 - 4x + 5 = 0$.

Use standard rules of square roots

(a) $\sqrt{-4} = \sqrt{4}\sqrt{-1}$

Use $i = \sqrt{-1}$

$\sqrt{-4} = 2i$

(b) $x = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2}$

Use the quadratic formula

$x = \frac{4 \pm \sqrt{-4}}{2}$

continued . . .

Use part (a)

$x = \dfrac{4 \pm 2i}{2}$

Give the answer in standard complex form

$x = 2 \pm i$

When we look at new number systems, one of the most important questions to ask is: when are two numbers equal? With complex numbers the answer is what you would expect, but despite its apparent simplicity, it has some remarkably powerful uses.

Although this may seem trivial, in mathematics it pays to be careful. If two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal, does that mean that $a = c$ and $b = d$?

KEY POINT 15.2

If two complex numbers are equal, their real parts are the same and their imaginary parts are the same.

Worked example 15.3

Solve $z^2 = i$.

Write z as $x + iy$

$(x + iy)^2 = i$

Multiply out brackets remembering that $i^2 = -1$

$x^2 + 2ixy - y^2 = i$

Compare real and imaginary parts

Comparing real parts:

$x^2 - y^2 = 0$

(Divide both sides by i)

Comparing imaginary parts:

$2xy = 1$

We have two equations with two unknowns, so we can solve them by substitution

$\Leftrightarrow y = \dfrac{1}{2x}$

$\therefore x^2 - \dfrac{1}{4x^2} = 0$

$\Leftrightarrow 4x^4 - 1 = 0$

$\Leftrightarrow (2x^2 - 1)(2x^2 + 1) = 0$

Remember that x is a real number

$\therefore x^2 = \dfrac{1}{2}$ or $-\dfrac{1}{2}$ (impossible)

$\therefore x = \pm\dfrac{1}{\sqrt{2}},\ y = \dfrac{1}{2x} = \pm\dfrac{1}{\sqrt{2}}$

Answer the original question

So $z = \pm\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)$

We looked at solving simultaneous equations by substitution in Section 4B.

Notice that there are two numbers which both have the same square; this is consistent with what you have already seen with real numbers.

Exercise 15A

1. State the imaginary part of each of the following complex numbers:

(a) (i) $-3+5\text{i}$ (ii) $8-2\text{i}$

(b) (i) $6+\text{i}$ (ii) $19-\text{i}$

(c) (i) $2\text{i}-8$ (ii) $7\text{i}-2$

(d) (i) 15 (ii) -3i

(e) (i) i^2 (ii) $(1+\text{i})-\text{i}$

(f) (i) $1+a\text{i}+b-\text{i},\ a,b\in\mathbb{R}$ (ii) $2-4\text{i}-(b\text{i}-a),\ a,b\in\mathbb{R}$

2. Evaluate, giving your answer in the form $x+\text{i}y$:

(a) (i) $2\text{i}+3\text{i}$ (ii) $\text{i}-9\text{i}$

(b) (i) 5i^2 (ii) i^2

(c) (i) $(-3\text{i})^2$ (ii) $(4\text{i})^2$

(d) (i) $(4\text{i}+3)-(6\text{i}-2)$ (ii) $2(2\text{i}-1)-3(4-2\text{i})$

3. Evaluate, giving your answer in the form $x+\text{i}y$:

(a) (i) $\text{i}(1+\text{i})$ (ii) $3\text{i}(2-5\text{i})$

(b) (i) $(2+\text{i})(1+2\text{i})$ (ii) $(5+2\text{i})(4+3\text{i})$

(c) (i) $(2+3\text{i})(1-2\text{i})$ (ii) $(3+\text{i})(5-\text{i})$

(d) (i) $(3+\text{i})^2$ (ii) $(4-3\text{i})^2$

(e) (i) $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\text{i}\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\text{i}\right)$ (ii) $(3+2\text{i})(3-2\text{i})$

4. Evaluate, giving your answer in the form $x+\text{i}y$:

(a) (i) $\frac{6+8\text{i}}{2}$ (ii) $\frac{9-3\text{i}}{3}$

(b) (i) $\frac{5+2\text{i}}{10}$ (ii) $\frac{\text{i}-4}{8}$

(c) (i) $\frac{3+\text{i}}{2}+\text{i}$ (ii) $9\text{i}-\frac{6-4\text{i}}{2}$

5. Evaluate, giving your answer in the form $x+\mathrm{i}y$:

(a) (i) $\sqrt{-4}$ (ii) $\sqrt{-49}$

(b) (i) $\sqrt{-8}$ (ii) $\sqrt{-50}$

(c) (i) $\dfrac{4-\sqrt{-36}}{3}$ (ii) $\dfrac{-1+\sqrt{-25}}{3}$

(d) (i) $\dfrac{2+\sqrt{16-25}}{6}$ (ii) $\dfrac{5-2\sqrt{4-9}}{4}$

6. Solve the equations below, simplifying your answers:

(a) (i) $x^2+9=0$ (ii) $x^2+36=0$

(b) (i) $x^2=-10$ (ii) $x^2=-13$

(c) (i) $x^2-2x+5=0$ (ii) $x^2-x+10=0$

(d) (i) $3x^2+20=6x$ (ii) $6x+5=-5x^2$

7. Evaluate, simplifying your answers:

(a) (i) i^3 (ii) i^4

(b) (i) $(-2\mathrm{i})^4$ (ii) $(-5\mathrm{i})^3$

(c) (i) $\left(1-\sqrt{3}\,\mathrm{i}\right)^3$ (ii) $\left(\sqrt{3}+\mathrm{i}\right)^3$

(d) (i) $\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)^3$ (ii) $\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)^3$

8. By expanding and comparing real and imaginary parts, find real numbers a and b such that:

(a) (i) $(a+b\mathrm{i})(3-2\mathrm{i})=5\mathrm{i}+1$ (ii) $(6+\mathrm{i})(a+b\mathrm{i})=2$

(b) (i) $(a+2\mathrm{i})(1+2\mathrm{i})=4-b\mathrm{i}$ (ii) $(1+a\mathrm{i})(1+\mathrm{i})=b+2\mathrm{i}$

(c) (i) $(a+b\mathrm{i})(2+\mathrm{i})=2a-(b-1)\mathrm{i}$ (ii) $\mathrm{i}(a+b\mathrm{i})=a-6\mathrm{i}$

9. By writing $z=x+y\mathrm{i}$, solve the following equations:

(a) (i) $z^2=-4\mathrm{i}$ (ii) $z^2=9\mathrm{i}$

(b) (i) $z^2=2+2\sqrt{3}\mathrm{i}$ (ii) $z^2=5+\mathrm{i}$

10. Find the exact values of $a, b\in\mathbb{R}$ such that $(3+a\mathrm{i})(b-\mathrm{i})=-4\mathrm{i}$. Give your answers in the form $k\sqrt{3}$. *[4 marks]*

11. Find the exact values of $a, b\in\mathbb{R}$ such that:

$$(1+a\mathrm{i})(1+b\mathrm{i})=b+9\mathrm{i}-a. \quad [4\ marks]$$

12. Solve the equation $\mathrm{i}z+2=\mathrm{i}-3z$. *[4 marks]*

13. (a) Find values x and y such that $(x+\mathrm{i}y)(2+\mathrm{i})=-\mathrm{i}$.

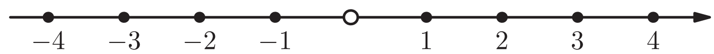
(b) Evaluate $-\dfrac{3\mathrm{i}}{2+\mathrm{i}}$ in the form $x+\mathrm{i}y$. *[5 marks]*

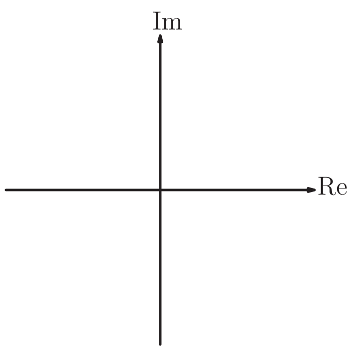
14. (a) Find real numbers a and b such that $(a+bi)^2 = -3-4i$.
(b) Hence solve the equation: $z^2 + i\sqrt{3}z + i = 0$ *[6 marks]*

15B Geometric interpretation

When you first worked with numbers, you may have used a number line.

When you met negative numbers, this number line was extended to the left.





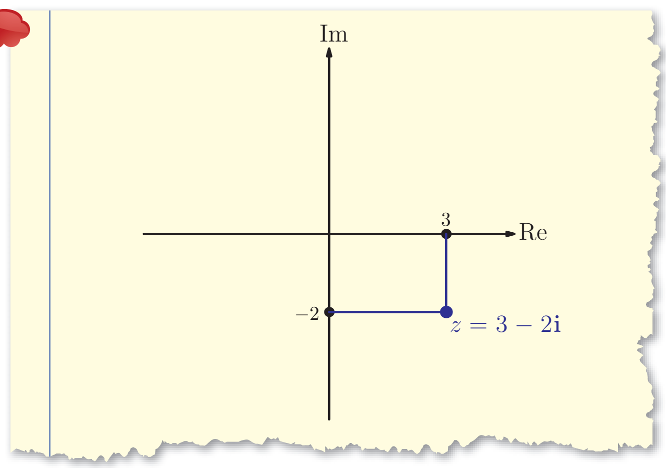
But how can complex numbers be illustrated on this diagram?

The answer was provided by the land surveyor Caspar Wessel who suggested that another axis, perpendicular to the real number line, could represent the imaginary part of a number. However, this idea was popularised by Jean-Robert Argand, and is called an **Argand Diagram**.

Worked example 15.4

Represent $3-2i$ on an Argand Diagram.

3 on the x-axis (real part) and −2 on the y-axis (imaginary part)



Once you see the power of imaginary numbers, it is tempting to go back and see if adding a third axis produces even more powerful tools. However, there is a result called the Fundamental Theorem of Algebra which proves that complex numbers are sufficient to produce all solutions to all polynomial equations. However, that does not mean that there might not be uses found for a third axis and many mathematicians have worked on this.

To describe the position of points in the Argand Diagram we have used their real part and their imaginary part in **Cartesian form**. However, this is not the only way to represent these numbers; they can also be described using the distance away from the origin (which we call the **modulus,** $|z|$ or r) and an angle (which we call the **argument,** arg z or θ). The argument is conventionally measured anticlockwise from the positive real axis. The description of a complex number like this is called the **polar form**.

As you will see in Section E, many operations are much easier to carry out in polar form, but most examination questions are asked using Cartesian form and answers are often also required in Cartesian form. You will therefore have to convert between the two forms. To do this requires trigonometry.

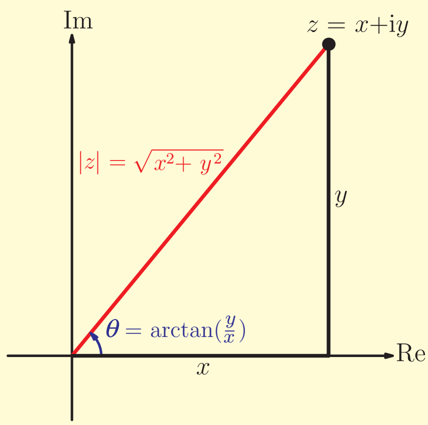
Is this definition of $|z|$ the same as, or different from, $|x|$, where x is real?

KEY POINT 15.3

To convert from Cartesian to polar form:

$$|z| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$



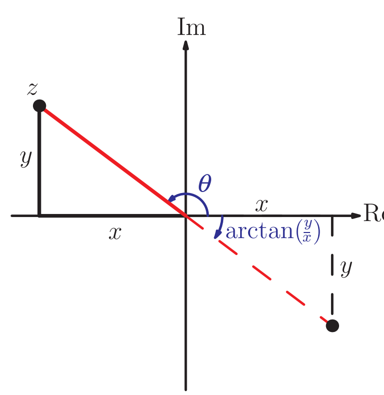
EXAM HINT

Your calculator can convert between Cartesian and polar forms. See Calculator sheet 8.

EXAM HINT

The examination question should specify if the argument is wanted as an angle between 0 and 2π or $-\pi$ and π.

Do not forget that if you know $\tan\theta$ there are two possible values for θ in either of these regions. You can decide which of these is the argument of the complex number you are investigating by considering which quadrant of the Argand diagram it is in. In general:



If $\text{Re}(z) > 0$ then $\theta = \arctan\left(\frac{y}{x}\right)$

If $\text{Re}(z) < 0$ then $\theta = \arctan\left(\frac{y}{x}\right) + \pi$

Worked example 15.5

Convert 2 – 3i to polar form, giving your answer to 3 significant figures.

Find the modulus

$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13} = 3.61$ (3SF)

Find tan θ

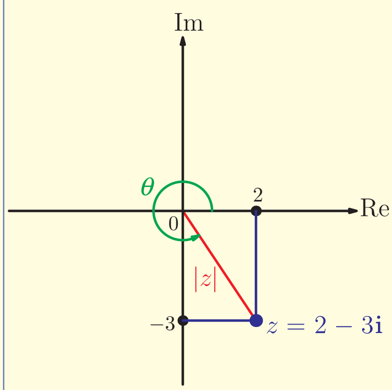
$\tan\theta = \frac{-3}{2}$

Find possible values of θ between 0 and 2π (add π to get the next value)

$\tan^{-1}(-3/2) = -0.983$

$\theta = 2.16$ or 5.30

Where is the number on the Argand diagram?



Deduce the correct value of θ

$\theta > \pi$, so $\theta = 5.30$ (3SF)

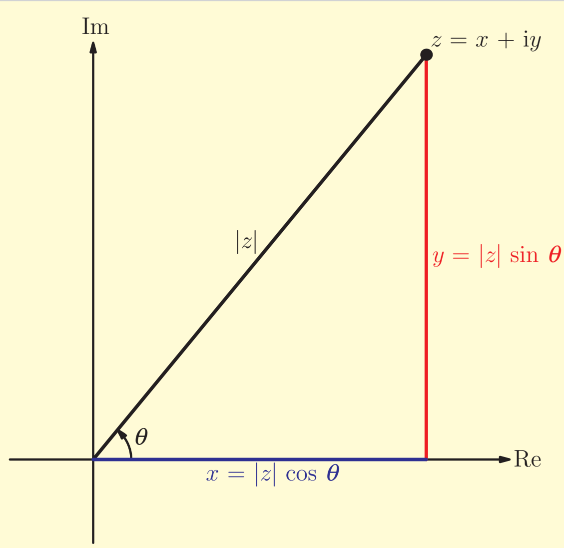
So $|z| = 3.61$ and $\arg z = 5.30$ (3SF)

KEY POINT 15.4

To convert from polar form to Cartesian form:

$x = |z|\cos\theta \qquad y = |z|\sin\theta$

Although the diagram shows a complex number in the first quadrant, these formulae hold for all quadrants.



These formulae are a nice way to express complex numbers which, although strictly in Cartesian form, highlights the polar information:

$$z = x + \mathrm{i}y$$
$$= |z|\cos\theta + \mathrm{i}|z|\sin\theta$$
$$= |z|(\cos\theta + \mathrm{i}\sin\theta)$$
$$= |z|\,\mathrm{cis}\,\theta$$
$$= r\,\mathrm{cis}\,\theta$$

Here $\mathrm{cis}\,\theta$ is just shorthand for $\cos\theta + \mathrm{i}\sin\theta$.

$r\,\mathrm{cis}\,\theta$ is an acceptable way to write a complex number when it is asked for in polar form.

Worked example 15.6

A complex number z has modulus 3 and argument $\frac{\pi}{6}$. Write the number in Cartesian form.

Use the conversion formulae:
$x = |z|\cos\theta$
$y = |z|\sin\theta$

$$x = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

$$y = 3\sin\frac{\pi}{6} = \frac{3}{2}$$

Write the answer in the required form

$$z = \frac{3\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}$$

Exercise 15B

1. Represent the following numbers on an Argand diagram. Use a separate diagram for each part.
   (a) (i) $z = 4 + \mathrm{i}$ and $w = 2\mathrm{i}$
   (ii) $z = -3 + \mathrm{i}$ and $w = -3\mathrm{i}$
   (b) (i) Let $z = 4 + 3\mathrm{i}$. Represent z, $-z$ and $\mathrm{i}z$.
   (ii) Let $z = -2 + 5\mathrm{i}$. Represent z, $-z$ and $\mathrm{i}z$.
   (c) (i) Let $z = 2 + \mathrm{i}$. Represent z, $3z$, $-2z$ and $-\mathrm{i}z$.
   (ii) Let $z = 1 - 3\mathrm{i}$. Represent z, $2z$ and $2\mathrm{i}z$.

2. Find the modulus and the argument of
   (a) (i) 6 (ii) 13
   (b) (i) −3 (ii) −1.6
   (c) (i) 4i (ii) 0.5i

(d) (i) $-2i$ (ii) $-5i$

(e) (i) $1+i$ (ii) $2+\sqrt{3}i$

(f) (i) $-1-\sqrt{3}i$ (ii) $4-4i$

You may need a reminder of some trigonometric identities from chapters 10 and 12.

3. Find the modulus and the argument of

(a) (i) $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (ii) $\sqrt{7}\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)$

(b) (i) $\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)$ (ii) $\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)$

(c) (i) $3\left(\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}\right)$ (ii) $7\left(\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}\right)$

(d) (i) $-10\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (ii) $-2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(e) (i) $6\left(\cos\left(-\frac{\pi}{10}\right)+i\sin\left(-\frac{\pi}{10}\right)\right)$

(ii) $\frac{1}{2}\left(\cos\left(-\frac{\pi}{3}\right)-i\sin\left(-\frac{\pi}{3}\right)\right)$

4. Find the modulus and the argument of:

(a) (i) $4+2i$ (ii) $4-3i$

(b) (i) $i-\sqrt{3}$ (ii) $6i+\sqrt{2}$

(c) (i) $-3-i$ (ii) $-3+2i$

5. Given the modulus and the argument of z, write z in Cartesian form.

(a) (i) $|z|=4, \arg(z)=\frac{\pi}{3}$ (ii) $|z|=\sqrt{2}, \arg(z)=\frac{\pi}{4}$

(b) (i) $|z|=2, \arg(z)=\frac{3\pi}{4}$ (ii) $|z|=2, \arg(z)=\frac{2\pi}{3}$

(c) (i) $|z|=3, \arg(z)=-\frac{\pi}{2}$ (ii) $|z|=4, \arg(z)=-\pi$

6. Write these complex numbers in Cartesian form without using trigonometric functions. Display each one on an Argand diagram.

(a) (i) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ (ii) $\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$

(b) (i) $3\operatorname{cis}\frac{\pi}{2}$ (ii) $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$

(c) (i) $4\operatorname{cis}0$ (ii) $\cos\pi + \mathrm{i}\sin\pi$

7. Write the following complex numbers in the form $|z|\operatorname{cis}\theta$.

(a) (i) $z = 4\mathrm{i}$ (ii) $z = -5$

(b) (i) $z = 2 - 2\sqrt{3}\,\mathrm{i}$ (ii) $z = \frac{(\sqrt{3}+\mathrm{i})}{3}$

8. If x and y are real numbers, find the modulus and the tangent of the argument of z where:

(a) (i) $z = x - \mathrm{i}y$ (ii) $z = -x - \mathrm{i}y$

(b) (i) $z = 4x - 3\mathrm{i}y$ (ii) $z = 9x + 9\mathrm{i}y$

(c) (i) $z = 3\mathrm{i} + x + \mathrm{i}y$ (ii) $z = 2 + \mathrm{i} + x + \mathrm{i}y$

(d) (i) $z = (x + \mathrm{i}y)^2$ (ii) $z = (x - \mathrm{i}y)^2$

9. (a) Write $3\operatorname{cis}\left(\frac{7\pi}{4}\right)$ in Cartesian form in terms of surds only.

(b) Write $4\mathrm{i} - 4$ in the form $|z|\operatorname{cis}\theta$. *[6 marks]*

10. Let $z = 1 + \sqrt{3}\mathrm{i}$ and $w = 3\sqrt{3} - 3\mathrm{i}$.

(a) Find the modulus and the argument of z and w.

(b) Represent z and w on the same Argand diagram.

(c) Find the modulus and the argument of zw. Comment on your answer. *[8 marks]*

11. If z and w are complex numbers such that $|z + w| = |z - w|$, prove that $\arg z - \arg w = \pm\frac{\pi}{2}$. *[6 marks]*

15C Properties of complex conjugates

In Section 15A we saw that quadratic equations may have two complex roots. If you know that one of them is 3 + 2i then you could deduce that the other one is 3 – 2i because the imaginary part arises from the term after the ± sign in the quadratic formula (the square root of the discriminant). These two numbers form what is called a **conjugate pair**. They only differ in the sign of the imaginary part.

With imaginary numbers being 'only' a few hundred years old, there is some variation in the notations used in different cultures. In the United States the complex conjugate is labelled $\bar{z}$, while in engineering the square root of −1 is often called j instead of i.

KEY POINT 15.5

If $z = x + iy$ then the **complex conjugate** of z is given the symbol z^* and it equals $x - iy$.

On an Argand diagram, the conjugate is the reflection of the original number in the real axis.

At first the concept of conjugates may not appear particularly useful, but conjugate pairs have some very powerful properties.

Worked example 15.7

Show that both $z + z^*$ and zz^* are always real.

Write z and z^* in terms of their real and imaginary parts

Let $z = x + iy$, where x and y are real,
so $z^* = x - iy$

Find $z + z^*$

$z + z^* = (x + iy) + (x - iy) = 2x$ which is real

Find zz^*

$zz^* = (x + iy)(x - iy) = x^2 - ixy + iyx - i^2y^2$
$= x^2 + y^2$ which is real

The results in the Worked example above have important links with some concepts you have met earlier: $z + z^*$ is twice the real part of z and zz^* is the square of the modulus of z.

These results will be used when solving polynomial equations in Section 15 D – see Worked example 15.14.

KEY POINT 15.6

$$\text{Re}(z) = \frac{1}{2}(z + z^*)$$

$$\text{Im}(z) = \frac{1}{2i}(z - z^*)$$

$$|z|^2 = zz^*$$

One extremely important application of these results is when we have to divide by complex numbers. We can use the fact that $|z|^2$ is real to turn complex division into complex multiplication by multiplying top and bottom of the fraction by the complex conjugate of the denominator.

Worked example 15.8

Write $\frac{3+2\text{i}}{5-\text{i}}$ in the form $x + \text{i}y$.

Multiply numerator and denominator by the complex conjugate of the denominator

$$\frac{3+2\text{i}}{5-\text{i}} = \frac{3+2\text{i}}{5-\text{i}} \times \frac{5+\text{i}}{5+\text{i}}$$

Use $zz^* = |z|^2$

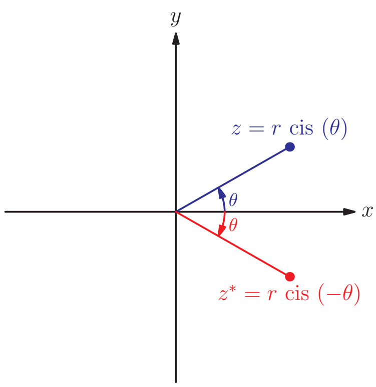
$$= \frac{15+10\text{i}+3\text{i}+2\text{i}^2}{5^2+(-1)^2}$$

$$= \frac{15-2+13\text{i}}{26}$$

$$= \frac{13+13\text{i}}{26}$$

$$= 0.5 + 0.5\text{i}$$

Because the complex conjugate is a reflection of the original number in the real axis we can make a link between the description of conjugate pairs in polar form.



KEY POINT 15.7

If $z = r\,\text{cis}\,\theta$ then $z^* = r\,\text{cis}\,(-\theta)$.

The algebra of complex conjugates is very intuitive.

Worked example 15.9

Prove that $(zw)^* = z^*w^*$.

Define variables in Cartesian form

Let
$z = x + \text{i}y$
$w = u + \text{i}v$

Express complex conjugates using the same variables

$z^* = x - \text{i}y$
$w^* = u - \text{i}v$

Work on the LHS for a bit, until we get stuck

$\text{LHS} = (zw)^*$
$= \left((x+\text{i}y)(u+\text{i}v)\right)^* = \left((xu-yv)+\text{i}(xv+yu)\right)^*$
$= (xu-yv) - \text{i}(xv+yu)$

→

continued . . .

Work on the RHS, trying to meet in the middle

$\text{RHS} = z^* w^*$

$= (x - iy)(u - iv) = xu - ixv - iyu - yv$

$= (xu - yv) - i(xv + yu)$

Conclude

Therefore LHS = RHS

This rule is an example of several similar rules which can all be proved in similar ways.

KEY POINT 15.8

$$(z \pm w)^* = z^* \pm w^*$$

$$(zw)^* = z^* w^*$$

$$\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$$

$$(z^n)^* = (z^*)^n$$

Worked example 15.10

If $z = (1 + iw)^2$ find z^* in terms of w.

$z^* = \left((1 + iw)^2\right)^*$

Use $(z^n)^* = (z^*)^n$

$= \left((1 + iw)^*\right)^2$

Use $(z + w)^* = z^* + w^*$

$= \left(1^* + (iw)^*\right)^2$

Note that $1^* = 1$

$= \left(1 + (iw)^*\right)^2$

Use $(zw)^* = z^* w^*$

$= (1 + i^* w^*)^2$

Note that $i^* = -i$

$= (1 - iw)^2$

The next example shows how the idea of equating real and imaginary parts can be used to solve equations involving complex conjugates.

Worked example 15.11

Find a complex number z such that $3z + 2z^* = 5 + 2i$.

As before, the best way to describe complex conjugates is in Cartesian form

Let $z = x + iy$

Then $3z + 2z^* = 5 + 2i$

$\Rightarrow 3(x + iy) + 2(x - iy) = 5 + 2i$

$\Rightarrow 5x + iy = 5 + 2i$

We can now equate real and imaginary parts

$\Rightarrow \begin{cases} 5x = 5 \\ y = 2 \end{cases}$

$\therefore x = 1, y = 2$

So $z = 1 + 2i$

Exercise 15C

1. Find the complex conjugate of the following numbers:

(a) (i) $2-3i$ (ii) $4+4i$

(b) (i) $i-3$ (ii) $3i+2$

(c) (i) $3i$ (ii) i

(d) (i) -45 (ii) 9

2. Write the following in the form $x + yi$. Check your answers using a calculator.

(a) (i) $\frac{3-2i}{1+2i}$ (ii) $\frac{4i}{3-5i}$

(b) (i) $\frac{4}{i}$ (ii) $-\frac{1}{i}$

(c) (i) $\frac{4+i}{4-i}$ (ii) $\frac{2i+1}{2i-1}$

(d) (i) $\frac{(1+i)^2}{1-i}$ (ii) $\frac{(i-2)^2}{i+2}$

3. Solve the following equations:

(i) $2z-3=4-3(i+z)$ (ii) $2iz+1=4i(z-3)$

4. Solve these simultaneous equations:

(i) $2z - 3iw = 5, \quad (1+i)z + 3w = -4i$

(ii) $(1+i)z + (1-i)w = 1, \quad (1-i)z + 2iw = i$

5. Find the complex number z if:

(i) $2z^* - 1 = 4i$ (ii) $3z^* + 2 = 9i$

6. By writing $z = x + iy$ solve the equations:

(i) $z + 2z^* = 2 - 7i$ (ii) $2z + iz^* = -3 - i$

7. If x and y are real numbers find the complex conjugate z^* when:

(a) (i) $z = 3 + (x + iy)$ (ii) $z = x - (2 - iy)$

(b) (i) $z = (x + 3iy) + (2 - i)$ (ii) $z = (3 + 3i) - (x - iy)$

(c) (i) $x + iy + \frac{1}{x + iy}$ (ii) $x + iy - \frac{1}{x + iy}$

(d) (i) $\frac{x}{x + iy} - \frac{x}{x - iy}$ (ii) $\frac{x}{x + iy} + \frac{x}{x - iy}$

8. Find z^* in terms of w^*:

(a) (i) $z = 1 + 2w$ (ii) $z = 3w - 1$

(b) (i) $z = 1 + 3i + w$ (ii) $z = 2 - i - w$

(c) (i) $z = iw$ (ii) $z = (2 - 3i)w$

(d) (i) $z = w^2$ (ii) $z = \sqrt{w}$

(e) (i) $z = \frac{1}{w}$ (ii) $z = -\frac{3}{w}$

(f) (i) $z = (4 + iw)^2$ (ii) $z = \frac{1}{1 + w}$

9. (a) Let $z = x + yi$. Find in terms of x and y the real and imaginary part of $3iz + 2z^*$.

(b) Find the complex number z such that $3iz + 2z^* = 4 - 4i$.

[5 marks]

10. Find real numbers x and y such that $x + 3iy = z + 4iz^*$ where $z = 2 + i$. *[5 marks]*

11. Prove that $(z^*)^2 = (z^2)^*$. *[4 marks]*

12. Find all solutions to $z + 3z^* = i$. *[5 marks]*

13. Find all solutions to $z + i = 1 - z^*$. *[5 marks]*

14. If $z = r\operatorname{cis}\theta$, write each in terms of r and θ, simplifying your answers as far as possible.

(a) $z + z^*$ (b) zz^* (c) $\dfrac{z}{z^*}$ *[6 marks]*

15. Given that $|z| = \sqrt{3}$, solve the equation $2z^* + \dfrac{3}{iz} = \sqrt{15}$.

[5 marks]

16. Given that $|z| = 3$, solve the equation $z - \dfrac{12i}{z^*} = 5$. *[5 marks]*

17. Prove that if $z + \dfrac{1}{z}$ is real then either $|z|$ or z is real. *[6 marks]*

18. If $z = x + iy$, find the real and the imaginary parts of $\dfrac{z}{z+1}$ in terms of x and y, simplifying your answers as far as possible. *[6 marks]*

19. If $z = \operatorname{cis}\theta$, express the real and imaginary parts of $\dfrac{z-1}{z+1}$ in terms of θ, simplifying your answer as far as possible. *[6 marks]*

15D Complex solutions to polynomial equations

You may want to remind yourself about working with polynomials from chapter 3.

We have already seen that quadratic equations can have complex solutions. We can use the ideas about roots and factors to construct a quadratic equation with given roots.

Worked example 15.12

Find a quadratic equation with roots 7 and 2i.

Make a link between roots and factors	$(x-7)$ and $(x-2i)$ are factors of the quadratic.
Multiply together the factors	$(x-7)(x-2i) = x^2 - 7x - 2ix + 14i$
Write down an appropriate equation	So the quadratic equation is: $x^2 - 7x - 2ix + 14i = 0$

We can now factorise some quadratics which could not previously be factorised using real numbers. In particular there is an extension of the difference of two squares identity.

KEY POINT 15.9

$$a^2 + b^2 = (a + \mathrm{i}b)(a - \mathrm{i}b)$$

We can also factorise other polynomials with complex roots, using the factor theorem: if a polynomial $f(z)$ has a root $z = a$ then $f(z)$ has a factor $(z - a)$.

Worked example 15.13

Solve the equation $x^2 - 4x + 40 = 0$. Hence factorise $x^2 - 4x + 40$.

Use the quadratic equation

$$\begin{aligned} x &= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 40}}{2} \\ &= \frac{4 \pm \sqrt{-144}}{2} \\ &= \frac{4 \pm 12\mathrm{i}}{2} \\ &= 2 \pm 6\mathrm{i} \end{aligned}$$

Make a link between roots and factors

Therefore

$$x^2 - 4x + 40 = \big(x - (2 + 6\mathrm{i})\big)\big(x - (2 - 6\mathrm{i})\big)$$

In Worked example 15.13 we again saw a quadratic equation whose solutions were conjugate pairs. However, Worked example 15.12 constructed a quadratic whose solutions were not in conjugate pairs. Are there any striking differences between these two quadratics? You can see that in Worked example 15.13 all of the coefficients are real. It turns out that this result generalises to any polynomial with real coefficients – called a **real polynomial**. The proof of this is demonstrated on Fill-in proof 14: 'Solutions to real polynomials' on the CD-ROM.

KEY POINT 15.10

The solutions of real polynomials come in conjugate pairs:
If $f(z) = 0$ then $f(z^*) = 0$.

Another important result about solutions of polynomial equations is the **Fundamental Theorem of Algebra**, which tells us how many solutions to expect. We have already met this in the context of real roots, when we used it to decide how many times the graph of a polynomial function can cross the x-axis: the graph of a polynomial of degree n can have at most nx-intercepts.

See section 3C.

If we include complex roots, then the Fundamental Theorem of Algebra gives us an even more precise answer. It applies to both real and complex polynomials.

KEY POINT 15.11

The Fundamental Theorem of Algebra

A polynomial of degree n has exactly n roots, some of which may be repeated.

For example, x^4+9x^2 is a polynomial of order 4 and so it should have 4 roots. Indeed,

$$x^4+9x^2 = x^2(x^2+9) = (x-0)(x-0)(x+3\text{i})(x-3\text{i})$$

where we used the result of Key point 15.9 to factorise the sum of two squares. So the four roots are $0, 0, -3\text{i}$ and 3i. Notice again that, since the polynomial is real, the complex roots form conjugate pairs.

The last two Key points are very useful when solving cubic and quartic equations. We know how many solutions to expect, and if we know one complex solution, z, then we know that its complex conjugate, z^*, is also a solution. This implies that the polynomial has a factor $(x-z)(x-z^*)$. Since we know how many other factors we are looking for, we can use comparing coefficients to find them.

This is related to the results in Key point 15.6.

These are also related to the sums and products of roots we will meet in the next section.

EXAM HINT

The structure $(x-z)(x-z^*)$ occurs so often it is useful to remember the result:

$$(x-z)(x-z^*) = x^2 - 2Re(z) + |z|^2$$

Worked example 15.14

Given that $1-\mathrm{i}\sqrt{6}$ is a root of the function $f(x)=x^3+x^2+x+21$, find the remaining roots.

We know how many roots to expect

By the Fundamental Theorem of Algebra, a polynomial of order three has 3 roots.

Use the fact that the polynomial has real coefficients

Since the polynomial is real the roots occur in conjugate pairs, so $1+\mathrm{i}\sqrt{6}$ is also a root.

Link roots to factors

So $\left(x-\left(1-\mathrm{i}\sqrt{6}\right)\right)$ and $\left(x-\left(1+\mathrm{i}\sqrt{6}\right)\right)$ are factors of f(x).

Therefore $\left(x-\left(1-\mathrm{i}\sqrt{6}\right)\right)\left(x-\left(1+\mathrm{i}\sqrt{6}\right)\right)$ is a factor.

Multiply this out using $z+z^*=2\,\mathrm{Re}\,(z)$ and $zz^*=|z|^2$

$$\left(x-\left(1-\mathrm{i}\sqrt{6}\right)\right)\left(x-\left(1+\mathrm{i}\sqrt{6}\right)\right)$$
$$=x^2-2x+7$$

Compare coefficients to find the third factor

$(x^3+x^2+x+21)=(x+k)(x^2-2x+7)$

By inspection, $k=3$

So the solutions are $x=1\pm\mathrm{i}\sqrt{6}$ and $x=-3$

Exercise 15D

1. Solve the equation $f(x)=0$ and hence factorise $f(x)$.
   (a) (i) $f(x)=x^2-2x+2$ (ii) $f(x)=x^2+6x+25$
   (b) (i) $f(x)=x^2+3x+4$ (ii) $f(x)=x^2+2x+5$
   (c) (i) $f(x)=3x^2-2x+10$ (ii) $f(x)=5x^2+4x+2$

2. Factorise into linear factors:
   (a) (i) z^2+4 (ii) z^2+25
   (b) (i) $4z^2+49$ (ii) $9z^2+64$
   (c) (i) z^4-1 (ii) $16z^4-81$

3. Find the real values of a and b such that the quadratic equation $x^2+ax+b=0$ has roots:
   (a) (i) $5\mathrm{i}$ and $-5\mathrm{i}$ (ii) $-3\mathrm{i}$ and $3\mathrm{i}$
   (b) (i) $-4\mathrm{i}+3$ and $4\mathrm{i}+3$ (ii) $1+2\mathrm{i}$ and $1-2\mathrm{i}$

4. Find a quadratic equation with roots:
   (a) (i) 4 and $3\mathrm{i}$ (ii) 8 and $-\mathrm{i}$
   (b) (i) $-\mathrm{i}$ and $2\mathrm{i}$ (ii) $5\mathrm{i}$ and i

(c) (i) $3i$ only (ii) $-i$ only

(d) (i) $1-2i$ and $5+3i$ (ii) $3+i$ and $2-i$

(e) (i) $2+i$ only (ii) $2-2i$ only

5. Given that $z = 3i$ is one root of the equation $z^3 - 2z^2 + 9z - 18 = 0$, find the remaining roots. *[5 marks]*

6. Given that $z = 1 + 2i$ is one root of the equation $z^3 + z^2 - z + 15 = 0$, find the remaining roots. *[5 marks]*

7. Two roots of the cubic equation $z^3 + bz^2 + cz + d = 0$ $(b, c, d \in \mathbb{R})$ are -2 and $2-3i$.

(a) Write down the third root.

(b) Find the values of b, c and d. *[6 marks]*

8. Find a quartic equation with real coefficients and roots $3i$ and $5-i$. *[6 marks]*

9. Two of the roots of the polynomial $f(x) = ax^5 + bx^4 - cx^2 + d$ are $x = 2i$ and $x = 3 - i$. Explain why this implies that $f(x)$ has exactly one real root.

10. $f(x)$ is a polynomial of order 4 and has two real roots, $x = 1$ and $x = 5$. The graph of $y = f(x)$ is tangent to the x-axis at $x = 5$.

(a) Write in factorial form an expression for $f(x)$, explaining clearly your reasoning.

(b) Hence sketch a possible graph of $y = f(x)$.

11. Let $f(z) = z^4 + z^3 + 5z^2 + 4z + 4$.

(a) Show that $f(2i) = 0$.

(b) Hence find the remaining solutions of the equation $f(z) = 0$.

(c) Write $f(z)$ as a product of two real quadratic factors. *[9 marks]*

15E Sums and products of roots of polynomials

In the last section we saw that we can use factorised form to find a quadratic equation with given roots. For example, one possible quadratic equation with roots 3 and 7 is:

$$(x-3)(x-7)=0$$
$$\Leftrightarrow x^2-10x+21=0$$

See Worked example 15.12

But there are infinitely many other quadratic equations with the same roots, because we can multiply the whole equation by a constant. Another example of an equation with roots 3 and 7 is $3x^2-30x+63=0$.

In general, if a quadratic equation has roots p and q then

$$(x-p)(x-q)=0$$
$$\Leftrightarrow x^2-(p+q)x+pq=0$$

so any quadratic equation with roots p and q has the form

$$ax^2-a(p+q)x+apq=0.$$

If we divide the coefficient of x by the coefficient of x^2 the ratio is $(p+q)$, and the ratio of the constant term and the coefficient of x^2 is pq.

This relationship between the roots and the coefficients of the equation can also be expressed as follows:

KEY POINT 15.12

If the quadratic equation $ax^2+bx+c=0$ has roots p and q then:

$$p+q=-\frac{b}{a} \text{ and } pq=\frac{c}{a}.$$

These observations apply to both real and complex equations. The easiest application is to find the coefficients in the equation when we know the roots.

We saw an example of this with complex conjugate roots in Worked example 15.13

Worked example 15.15

The equation $4x^2 + bx + c = 0$ has roots $-\frac{4}{3}$ and 2i. Find the values of b and c.

The coefficients can be found by using sum and product of the roots.

$$p + q = -\frac{b}{a} \Rightarrow -\frac{4}{3} + 2i = -\frac{b}{4}$$

$$\Rightarrow b = \frac{16}{3} - 8i$$

$$pq = \frac{c}{a} \Rightarrow -\frac{4}{3} \times 2i = \frac{c}{4}$$

$$\Rightarrow c = -\frac{32i}{3}$$

Sometimes we want to find an equation whose roots are related to the roots of a given equation.

Worked example 15.16

The quadratic equation $3x^2 - 4x + 7 = 0$ has roots p and q. Find a quadratic equation with integer coefficients and roots p^2 and q^2.

We don't need to find p and q, just their sum and product

$$p + q = -\frac{-4}{3} = \frac{4}{3}$$

$$pq = \frac{7}{3}$$

The coefficients of the new equation are related to the roots p^2 and q^2

Let the equation be $ax^2 + bx + c = 0$.

Then $p^2 + q^2 = -\frac{b}{a}$ and $p^2q^2 = \frac{c}{a}$

All equations with the required roots are multiples of each other, so we can set $a = 1$

Set $a = 1$. Then

$b = -(p^2 + q^2)$ and $c = p^2q^2$

We need to relate $p^2 + q^2$ and p^2q^2 to $p + q$ and pq. The second one is easier

$$p^2q^2 = (pq)^2 = \left(\frac{7}{3}\right)^2$$

$$\therefore c = \frac{49}{9}$$

continued . . .

We can try squaring $p + q$ to get $p^2 + q^2$

$$(p+q)^2 = p^2 + 2pq + q^2$$

$$\Rightarrow p^2 + q^2 = (p+q)^2 - 2pq$$

$$= \left(\frac{4}{3}\right)^2 - 2\left(\frac{7}{3}\right)$$

$$= -\frac{26}{9}$$

$$\therefore b = \frac{26}{9}$$

The equation is

$$x^2 + \frac{26}{9}x + \frac{49}{9} = 0$$

We want the equation with integer coefficients, so multiply through by 9

$$\Leftrightarrow 9x^2 + 26x + 49 = 0$$

The rules relating sums and products of roots to the coefficients apply to polynomial equations of any degree. They can be derived by considering the factorised form of the equation.

KEY POINT 15.13

For a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$:

- the sum of the roots is $\frac{-a_{n-1}}{a_n}$.
- the product of the roots is $\frac{(-1)^n a_0}{a_n}$.

We can use these rules, combined with our knowledge about complex conjugate roots of real polynomials, to find unknown coefficients if we know the roots of an equation.

Worked example 15.17

A quartic equation $x^4 + ax^3 + 14x^2 - 18x + b = 0$ has real coefficients and two of its roots are $3i$ and $1-2i$. Find the values of a and b.

As we know two of the complex roots, we can find the other two

The four roots are:
$3i, -3i, 1-2i, 1+2i$

We can find the coefficients using sum and product of the roots

$$-\frac{a}{1} = (3i) + (-3i) + (1-2i) + (1+2i) = 2$$

$$\therefore a = -2$$

$$(-1)^4 \frac{b}{1} = (3i)(-3i)(1-2i)(1+2i)$$

$$= 9(1^2 + 2^2) = 45$$

$$\therefore b = 45$$

We can also find relationships between the roots and the other coefficients of the equation. For example, if the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots p, q and r, then by expanding $(x-p)(x-q)(x-r) = 0$ we get that:

$$p + q + r = -\frac{b}{a}$$

$$pq + qr + rp = \frac{c}{a}$$

$$pqr = -\frac{d}{a}$$

EXAM HINT

You can be asked to prove and use these and other relationships. It may be useful to remember the ones for quadratic and cubic equations.

Worked example 15.18

The roots p, q, r of the equation $x^3 - 3x^2 + cx + d = 0$ form an arithmetic progression. Show that $c + d = 2$.

If form an arithmetic progression then $q - p = r - q$

$q - p = r - q$
$\Leftrightarrow p + r = 2q$

We know the three equations relating the roots to the coefficients. The most useful one seems to be $p + q + r = -\frac{b}{a}$

$p + q + r = -\frac{-3}{1} = 3$
$\Rightarrow q + 2q = 3$
$\Rightarrow q = 1$

To involve c and d we need to use the other two relationships as well

$pq + qr + rp = c$
$\Rightarrow p + r + rp = c$

We know that $p + r = 2q = 2$

$\Rightarrow 2 + rp = c$
$\Rightarrow rp = c - 2$
$pqr = -d$
$\Rightarrow pr = -d$
$\Rightarrow c - 2 = -d$
$\Rightarrow c + d = 2$, as required

Exercise 15E

1. Write down the value of the sum and the product of roots of the following equations:
   - (a) (i) $x^3 + 4x^2 - 3x + 1 = 0$ (ii) $x^3 + 6x^2 + 2x - 3 = 0$
   - (b) (i) $4x^3 - x + 3 = 0$ (ii) $5x^3 + x^2 - 8 = 0$
   - (c) (i) $3x^5 + 3x^4 - x + 2 = 0$ (ii) $-2x^5 + 2x^4 - 3 = 0$
   - (d) (i) $-4x^{10} + 5x^9 + 3 = 0$ (ii) $9x^9 - 8x^8 + 3x^5 + 5 = 0$

2. Given the roots of the equations, find the missing coefficients:
   - (a) (i) $3x^3 - ax^2 + b = 0$, roots $1, -1, -2$
     (ii) $2x^3 + ax^2 + 10x + c = 0$, roots $1, 1, 2$
   - (b) (i) $x^3 - ax^2 - 4x - b = 0$, roots $3, 2i, -2i$
     (ii) $x^3 + bx^2 + 9x + d = 0$, roots $2 + i, 2 - i, 1$

3. The quartic equation $x^4 - ax^3 + bx^2 - cx + d = 0$ has real coefficients, and two of its roots are $3i$ and $3 - i$.

(a) Write down the other two roots.

(b) Hence find the values of a and d. *[5 marks]*

4. The equation $4x^3 - 2x^2 + 4x + 1 = 0$ has roots p, q and r. Find the value of $p^2qr + pq^2r + pqr^2$. *[3 marks]*

5. When two resistors of resistances R_1 and R_2 are connected in series in an electric circuit, the total resistance in the circuit is $R = R_1 + R_2$. When they are connected in parallel, the total resistance satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

Two resistors have resistances equal to the two roots of the quadratic equation $3R^2 - 12R + 4 = 0$. Find the total resistance in the circuit if the two resistors are connected:

(i) in series

(ii) in parallel. *[5 marks]*

6. The cubic equation $3x^3 - 5x - 3 = 0$ has roots α, β and γ.

(a) Write down the value of $\alpha\beta\gamma$.

(b) Show that $\alpha + \beta = \frac{1}{\alpha\beta}$. *[4 marks]*

7. The quadratic equation $5x^2 - 3x + 2 = 0$ has roots p and q.

Find a quadratic equation with roots $\frac{1}{p}$ and $\frac{1}{q}$. *[4 marks]*

8. A random number generator can produce four possible values, each of which is equally likely. The four values satisfy the equation:

$$x^4 - 9x^3 + 26x^2 - 29x + 10 = 0$$

Estimate the mean of a large sample of random values.

[3 marks]

9. The equation $x^4 + bx^3 + cx^2 + dx + e = 0$ has roots $p, 2p, 3p$ and $4p$. Show that $125e = 3b^4$. *[4 marks]*

10. (a) Show that:

(i) $p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp)$

(ii) $p^2q^2 + q^2r^2 + r^2p^2 = (pr + qr + rp)^2 - 2pqr(p + q + r)$

(b) Given that the cubic equation $ax^3+bx^2+cx+d=0$ has roots p, q and r:

(i) Write down the values of $p+q+r$ and pqr in terms of a, b, c and d.

(ii) Show that $pq+qr+rp=\frac{c}{a}$.

(c) The equation $2x^3-5x+2=0$ has roots x_1, x_2 and x_3.

(i) Show that $x_1^2+x_2^2+x_3^2=5$.

(ii) Find the values of $x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2$ and $x_1^2x_2^2x_3^2$.

(iii) Hence find a cubic equation with integer coefficients and roots x_1^2, x_2^2 and x_3^2. *[16 marks]*

15F Operations in polar form

In Cartesian form, addition and subtraction are quite easy but multiplication and division are more difficult. Raising to a large power is even harder. These operations are much easier in polar form.

KEY POINT 15.14

> When you multiply two complex numbers in polar form you *multiply* their moduli and *add* their arguments:
>
> $$|zw|=|z||w|$$
>
> $$\arg(zw)=\arg z+\arg w$$

You will need compound angle identities from Section 12B

Although most of the time you will just use these results, you could be asked to prove them. This requires some trigonometry, as shown in the next example.

Worked example 15.19

Prove that when you multiply two complex numbers, you multiply their moduli and add their arguments.

Write two complex numbers in general terms with $\lvert z_1\rvert=r_1$, $\arg z_1=\theta_1$ and $\lvert z_2\rvert=r_2$, $\arg z_2=\theta_2$	Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$

continued . . .

Multiply them together and group real and imaginary parts

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))$$

But the real and imaginary parts look familiar – they are the compound angle formulae. This is in the form $|z|\text{cis}\,\theta$

$$= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

So by comparison we can state the modulus and argument of z_1z_2

$$|z_1z_2| = r_1r_2$$
$$\arg(z_1z_2) = \theta_1 + \theta_2$$

Interpret what we have found

So the modulus of the product is the product of the original two moduli, and the argument of the product is the sum of the original two arguments.

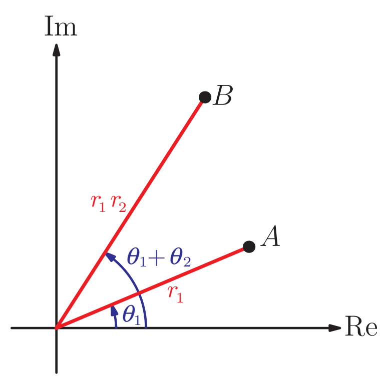
A similar proof gives the following result for the division of complex numbers.

KEY POINT 15.15

When you divide two complex numbers you *divide* their moduli and *subtract* their arguments.

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$\arg\left(\frac{z}{w}\right) = \arg z - \arg w$$



Multiplication of complex numbers has an interesting geometrical interpretation. On an Argand diagram, let A be the point corresponding to the complex number $z_1 = r_1 \text{ cis } \theta_1$, and let B be the point corresponding to the complex number $z_1(r_2 \text{ cis } \theta_2) = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$.

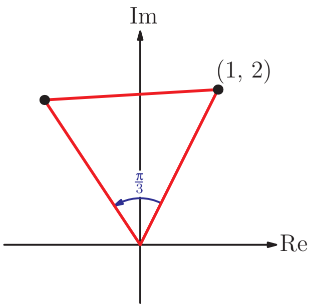
Then $OA = r_1$, $OB = r_1 r_2$, and $A\hat{O}B = (\theta_1 + \theta_2) - \theta_1 = \theta_2$. Hence multiplication by $r_2 \text{ cis } \theta_2$ corresponds to rotation around the origin though angle θ_2 and an enlargement with scale factor r_2.

In 1843 Irish mathematician Sir William Rowan Hamilton created a four dimensional version of complex numbers called quaternions. These have some nice properties for describing rotations in three dimensions. Although they were the precursor of vectors, until recently they were largely just a curiosity. However, they are now used extensively in computer graphics!

This result is very powerful in some situations that have nothing to do with complex numbers.

Worked example 15.20

The equilateral triangle shown in the diagram has one vertex at the origin and another at (1, 2). Find the coordinates of the third vertex.



The third vertex can be obtained by rotation through 60° anti-clockwise around the origin and no enlargement. This corresponds to multiplication by the complex number with modulus 1 and argument $\frac{\pi}{3}$

Point (1, 2) corresponds to the complex number $1 + 2i$

Rotation 60° around the origin corresponds to multiplication by $cis\frac{\pi}{3}$

The complex number corresponding to the third vertex is

$$
\begin{aligned}
(1+2i)\left(cis\frac{\pi}{3}\right) &= (1+2i)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\\
&= (1+2i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\
&= \left(\frac{1}{2}-\sqrt{3}\right)+\left(\frac{\sqrt{3}}{2}+1\right)i
\end{aligned}
$$

This complex number gives the coordinates of the third vertex

So the coordinates are

$$\left(\frac{1}{2}-\sqrt{3}, \frac{\sqrt{3}}{2}+1\right)$$

EXAM HINT

Note that there is another equilateral triangle with vertices (0, 0) and (1, 2); it can be obtained by rotating clockwise through 60°, corresponding to multiplication by $\text{cis}\left(-\frac{\pi}{3}\right)$.

You could try solving this problem using coordinate geometry or trigonometry; the calculations are much more complicated, as you may be able to guess by looking at the answer!

We can apply the result for multiplication to find powers of complex numbers. If a complex number has modulus r and argument θ, then multiplying $z \times z$ gives that z^2 has modulus r^2 and argument 2θ. Repeating this process we find the following result for powers:

KEY POINT 15.16

When you raise a complex number to a power you raise the modulus to the same power and multiply the argument by the power.

$$|z^n| = |z|^n$$

$$\arg(z^n) = n \arg z$$

If the rules of complex arguments look familiar, it may be because they are the same as the rules of logarithms. In fact, it turns out that the logarithm of a complex number is closely related to its argument.

It turns out that the exponent can be any real number. This result is so important that it is given a name.

KEY POINT 15.17

De Moivre's theorem

$$(r(\cos\theta + \mathrm{i}\sin\theta))^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

The IB requires you to be able to prove this result for positive integer exponents by induction. We shall return to this in chapter 24. However, it is important to note that it is true for all real exponents n.

Worked example 15.21

Evaluate $\dfrac{(1+\mathrm{i})^{10}}{4\mathrm{i}}$.

Express (1+i) in polar form

$1+\mathrm{i} = \sqrt{2}\,\mathrm{cis}\left(\dfrac{\pi}{4}\right)$

continued . . .

Use De Moivre to find $(1+i)^{10}$

By De Moivre:

$$\left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^{10} = \left(\sqrt{2}\right)^{10}\operatorname{cis}\left(\frac{10\pi}{4}\right)$$
$$= 2^5\operatorname{cis}\left(\frac{5\pi}{2}\right)$$
$$= 32\operatorname{cis}\left(\frac{\pi}{2}\right), \left(\text{since } \frac{5\pi}{2} = 2\pi + \frac{\pi}{2}\right)$$

Express 4i in polar form

$$4i = 4\operatorname{cis}\left(\frac{\pi}{2}\right)$$

Use the rules of division

$$\frac{32\operatorname{cis}\left(\frac{\pi}{2}\right)}{4\operatorname{cis}\left(\frac{\pi}{2}\right)} = \frac{32}{8}\operatorname{cis}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = 4\operatorname{cis}0 = 4$$

Exercise 15F

1. Evaluate the following expressions, giving your answers in the form $r\operatorname{cis}\theta$.

(a) (i) $3\operatorname{cis}\left(\frac{\pi}{6}\right)\times 7\operatorname{cis}\left(\frac{\pi}{5}\right)$

(ii) $\operatorname{cis}\left(-\frac{\pi}{9}\right)\times 4\operatorname{cis}\left(\frac{\pi}{3}\right)$

(b) (i) $\frac{8\operatorname{cis}(6)}{2\operatorname{cis}(2)}$ (ii) $\frac{15\operatorname{cis}\left(\frac{\pi}{7}\right)}{5\operatorname{cis}\left(\frac{\pi}{2}\right)}$

(c) (i) $\left(2\operatorname{cis}\left(\frac{\pi}{5}\right)\right)^6$ (ii) $\left(3\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^4$

2. Write in the form $\cos\theta + i\sin\theta$:

(a) (i) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

(ii) $\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(b) (i) $\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}\right)$

(ii) $\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\left(\cos\frac{2\pi}{5}-\mathrm{i}\sin\frac{2\pi}{5}\right)$

(c) (i) $\dfrac{\cos\frac{3\pi}{5}+\mathrm{i}\sin\frac{3\pi}{5}}{\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}}$

(ii) $\dfrac{\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}}{\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}}$

(d) (i) $\dfrac{\cos\frac{\pi}{5}+\mathrm{i}\sin\frac{\pi}{5}}{\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}}$

(ii) $\dfrac{\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}}{\cos\frac{2\pi}{5}-\mathrm{i}\sin\frac{2\pi}{5}}$

(e) (i) $\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^2\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)^3$

(ii) $\left(\cos\frac{\pi}{8}+\mathrm{i}\sin\frac{\pi}{8}\right)^4\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)^2$

(f) (i) $\dfrac{\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)^6}{\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^3}$

(ii) $\dfrac{\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)^2}{\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)^6}$

3. Let $z=\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}$.

(a) Write z^2, z^3 and z^4 in the form $r\operatorname{cis}\theta$.

(b) Represent z, z^2, z^3 and z^4 on the same Argand diagram.

[5 marks]

4. Let $z = \text{cis}\left(\frac{2\pi}{3}\right)$.

(a) Write z^2, z^3 and z^4 in polar form.

(b) Represent z, z^2, z^3 and z^4 on the same Argand diagram.

(c) For which natural numbers n is $z^n = z$? *[6 marks]*

5. (a) Find the modulus and argument of $1+\sqrt{3}\text{i}$.

(b) Hence find $\left(1+\sqrt{3}\text{i}\right)^5$ in polar form.

(c) Hence find $\left(1+\sqrt{3}\text{i}\right)^5$ in complex surd form. *[8 marks]*

6. (a) Write $-\sqrt{2}+\sqrt{2}\text{i}$ in the form $r\,\text{cis}\,\theta$.

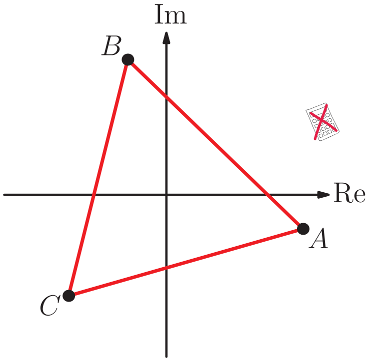
(b) Hence find $\left(-\sqrt{2}+\sqrt{2}\text{i}\right)^6$ in simplified Cartesian form.

[6 marks]

7. Use trigonometric identities to show that:

(a) $\frac{1}{\text{cis}\,\theta} = \text{cis}(-\theta) = \text{cis}(2\pi-\theta)$

(b) $\frac{\text{cis}\,\theta_1}{\text{cis}\,\theta_2} = \text{cis}(\theta_1-\theta_2)$ *[6 marks]*



8. The diagram shows an equilateral triangle with its centre at the origin and one vertex A (4, −1).

(a) Write down the complex number corresponding to the vertex A.

(b) Hence find the coordinates of the other two vertices. *[6 marks]*

15G Complex exponents

The arguments of complex numbers are added when you multiply. This may remind you of rules of exponents; when you multiply two numbers with the same base the powers are added together. By analogy, perhaps $r\,\text{cis}\,\theta = ra^{k\theta}$ for some base a and constant k? But which values of a and k should we use?

Differentiation is a method of finding gradients of curves. You will learn about how to apply it to sin and cos in chapter 16.

You will learn later that differentiating $\cos\theta + \text{i}\sin\theta$ gives $-\sin\theta + \text{i}\cos\theta$ which is the same as $\text{i}(\cos\theta+\text{i}\sin\theta)$, so differentiating has the effect of just multiplying by i. But you will also see that differentiating e^{kx} has the effect of multiplying by k. This suggests the following important result:

KEY POINT 15.18

$r \operatorname{cis} \theta = re^{i\theta}$

The form $re^{i\theta}$ is known as **Euler's form**, named after the Swiss mathematician Leonhard Euler who first established this relationship between trigonometric and exponential functions in the 1740s.

Arguing by analogy is not a valid mathematical method of proof. The above result has not been derived. It is actually a mathematical definition. So can we say whether or not it is true?

Euler famously calculated $e^{i\pi} = \operatorname{cis}\pi = \cos\pi + i\sin\pi = -1 + 0i$. Therefore:

$$e^{i\pi} + 1 = 0.$$

This is considered by many mathematicians to be a very aesthetically pleasing result. Do you agree?

Notice that it contains all the fundamental operations of mathematics – addition, multiplication and raising to a power – and four of the fundamental constants of mathematics: 1 (the fundamental constant of arithmetic), π (the fundamental constant of geometry), e (the fundamental constant of calculus) and i (the fundamental constant of complex numbers). Bringing all these together with the most fundamental mathematical relation – equality – gives zero. Complexity, elegance, truth and surprise – all the elements of great art!

This result allows us to find complex powers of numbers.

Worked example 15.22

Find 3^i.

We know what $e^{i\theta}$ means, so express 3 as a power of e	$3^i = (e^{\ln 3})^i$
Use rules of indices	$= e^{i\ln 3}$
Use the definition of complex exponential	$= \cos(\ln 3) + i\sin(\ln 3)$
We can calculate the answer to 3SF	$= 0.455 + 0.891i$

Exercise 15G

1. Write the following complex numbers in Cartesian form without using trigonometric functions.

(a) (i) $3e^{i\frac{\pi}{6}}$ (ii) $4e^{\frac{i\pi}{4}}$

(b) (i) $4e^{i\pi}$ (ii) $5e^{2\pi i}$

(c) (i) $e^{\frac{2\pi i}{3}}$ (ii) $2e^{\frac{3\pi}{2}i}$

2. Write the following complex numbers in the form $re^{i\theta}$.

(a) (i) $5+5i$ (ii) $2\sqrt{3}-2i$

(b) (i) $-\frac{1}{2}+\frac{1}{2}i$ (ii) $2+3i$

(c) (i) $-4i$ (ii) -5

3. Write the answer to each calculation in the form $re^{i\theta}$.

(a) (i) $4e^{i\frac{\pi}{6}} \times 5e^{i\frac{\pi}{4}}$ (ii) $\dfrac{5e^{i\frac{3\pi}{4}}}{10e^{i\frac{\pi}{4}}}$

(b) (i) $\dfrac{\left(2e^{i\frac{\pi}{4}}\right)^3}{\left(5e^{i\frac{\pi}{3}}\right)^2}$ (ii) $\dfrac{2e^{i\frac{\pi}{3}}}{\left(e^{i\frac{\pi}{6}}\right)^5}$

4. Represent these complex numbers on an Argand diagram:

(a) (i) $e^{i\frac{\pi}{3}}$ (ii) $e^{i\frac{3\pi}{4}}$

(b) (i) $5e^{i\frac{\pi}{2}}$ (ii) $2e^{-i\frac{\pi}{3}}$

5. Show that $\dfrac{e^{iz}+e^{-iz}}{2}=\cos z$. Hence find the value of $\cos(2i)$ correct to 3 significant figures. *[4 marks]*

6. Find 5^i in the form $x+iy$. *[4 marks]*

7. Find 3^{2-i} in the form $x+iy$. *[5 marks]*

8. Given that $\dfrac{e^{iz}+e^{-iz}}{2}=\cos z$ find possible values of complex numbers z for which $\cos z=2$. *[5 marks]*

9. (a) Express i in the form $re^{i\theta}$.

(b) Hence find the exact value of i^i. *[5 marks]*

15H Roots of complex numbers

What is $\sqrt[3]{1}$? On your calculator, try cubing $\dfrac{-1+\sqrt{3}i}{2}$. Does this change your answer to the first question?

You have already seen that a cubic equation has three **complex roots**. Therefore there should be three solutions to the equation $z^3=1$. One of them is $z=1$. The calculation above suggests that another one is $z=\dfrac{-1+\sqrt{3}i}{2}$. Using the result about solutions of real polynomials, the third solution should be $z=\dfrac{-1-\sqrt{3}i}{2}$.

In Example 15.3 we found square roots by writing $z = x + iy$ and equating real and imaginary parts. However, this method becomes too complicated when the equation involves higher powers. Fortunately, De Moivre's theorem gives us a different way to find roots of complex numbers. In complex numbers, when we wish to find an nth root there will be n different solutions.

De Moivre's theorem was given in Key point 15.17

The procedure is best illustrated with an example.

Worked example 15.23

Solve the equation $z^3 = 1$, giving your answers in the form $x + iy$.

Find the modulus and argument of the RHS

$|1| = 1$

$\arg(1) = 0$

Write $z = r\text{cis}(\theta)$ and equate the two sides

$(r\text{cis}(\theta))^3 = 1\text{cis}(0)$

Use De Moivre's Theorem

$r^3\text{cis}(3\theta) = 1\text{cis}(0)$

Compare the moduli

Therefore $r^3 = 1$

Remember that r is a real number

r is the real cube root of 1 so $r = 1$

If $0 < \theta < 2\pi$ then $0 < 3\theta < 6\pi$. Since adding on 2π to the argument returns to the same complex number there are 3 possible values for 3θ between 0 and 6θ

$3\theta = 0,\ 2\pi$ or 4π

$\therefore \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Write down all three solutions in polar form

$z_1 = 1\text{cis}(0)$

$z_2 = 1\text{cis}\left(\frac{2\pi}{3}\right)$

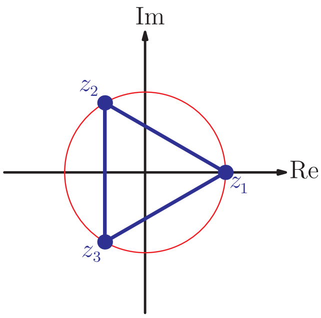
$z_3 = 1\text{cis}\left(\frac{4\pi}{3}\right)$

Convert to Cartesian form using $\text{cis}\ \theta = \cos\theta + i\sin\theta$

$z_1 = 1$

$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$



If we plot the three solutions to Worked example 15.23 on an Argand diagram, we notice an interesting pattern: the three points are vertices of an equilateral triangle.

The three solutions all have the same modulus, so they lie on a circle of radius 1. The arguments of any two of the numbers differ by $\frac{2\pi}{3}$, hence they are equally spaced around the circle. The three solutions of the equation $z^3 = 1$ are often referred to as the third roots of unity.

A similar pattern is seen if we solve the equation $z^n = 1$ for any positive whole number n. There will be n roots, each with modulus 1, represented on the Argand diagram by n points equally spaced around the unit circle. Remembering that one of the roots is always $z = 1$, this leads to the full description of the n roots.

KEY POINT 15.19

The equation $z^n = 1$ has n solutions:

$$1, \quad \text{cis}\left(\frac{2\pi}{n}\right), \quad \text{cis}\left(\frac{4\pi}{n}\right), \quad \ldots, \quad \text{cis}\left(\frac{2(n-1)\pi}{n}\right)$$

The same method extends to finding n^{th} roots of any complex number. The general procedure is:

KEY POINT 15.20

To solve $z^n = w$:

- first write w in polar form
- use De Moivre's theorem to write z^n in polar form
- compare moduli, remembering that moduli are always real
- compare arguments, remembering that adding 2π onto the argument does not change the number
- write n different solutions in polar form
- convert z to Cartesian form if required.

Worked example 15.24

Solve the equation $z^4 = 8 + 8\sqrt{3}\mathrm{i}$.

Find the modulus and argument of the RHS

$$|8+8\sqrt{3}\mathrm{i}| = \sqrt{64+192} = 16$$

$$\tan\theta = \frac{8\sqrt{3}}{8} = \sqrt{3}$$

$$\text{so } \arg(8+8\sqrt{3}\mathrm{i}) =$$

Write $z = r\,cis(\theta)$ and equate the two sides

$$(r\,\mathrm{cis}(\theta))^4 = 16\,\mathrm{cis}\left(\frac{\pi}{3}\right)$$

Use De Moivre's theorem

$$r^4\mathrm{cis}(4\theta) = 16\,\mathrm{cis}\left(\frac{\pi}{3}\right)$$

Compare the moduli and the arguments

Therefore $r^4 = 16$ and $\mathrm{cis}(4\theta) = \mathrm{cis}\left(\frac{\pi}{3}\right)$

Remember that r is a positive real number

$$r = \sqrt[4]{16} = 2$$

We are taking a fourth root so find 4 values for θ. Always add 2π to get the next value

$$4\theta = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \frac{19\pi}{3}$$

$$\therefore \theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$$

Write down all four solutions in polar form

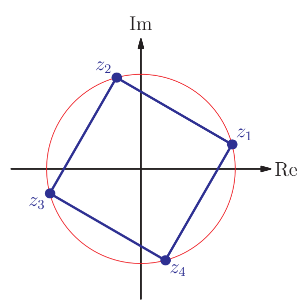
$$z_1 = 2\,\mathrm{cis}\left(\frac{\pi}{12}\right)$$

$$z_2 = 2\,\mathrm{cis}\left(\frac{7\pi}{12}\right)$$

$$z_3 = 2\,\mathrm{cis}\left(\frac{13\pi}{12}\right)$$

$$z_4 = 2\,\mathrm{cis}\left(\frac{19\pi}{12}\right)$$

If you represent these four solutions on the Argand diagram you will see that they form a square with vertices on the circle of radius 2.



Exercise 15H

1. Find all three cube roots of the following numbers:

   (a) (i) 27 (ii) 100

   (b) (i) 8i (ii) i

   (c) (i) 1 + i (ii) $2-\sqrt{3}\mathrm{i}$

2. Find the fourth roots of unity in the form $\operatorname{cis}\theta$. *[4 marks]*

3. Solve the equation $z^3=-8$ for $z\in\mathbb{C}$. Give your answers in the form $x+y\mathrm{i}$. *[4 marks]*

4. Solve the equation $z^3-\sqrt{2}(4-4\mathrm{i})=0$, giving your answers in Cartesian form. *[6 marks]*

5. Find all complex solutions of the equation $z^4+81\mathrm{i}=0$, giving your answers in the form $r\operatorname{cis}\theta$. *[5 marks]*

6. (a) Write $4+4\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

   (b) Hence solve the equation $z^4=4+4\sqrt{3}\mathrm{i}$ giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

   (c) Show your solutions on an Argand diagram. *[7 marks]*

7. (a) Solve the equation $z^4=-16$, giving your answers in Cartesian form.

   (b) Hence express z^4+16 as a product of two real quadratic factors. *[6 marks]*

8. Let $1, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the distinct sixth roots of unity.

   (a) Show that $\omega_n=\omega_1^n$ for $n=2,3,4,5$.

   (b) Hence show that $1+\omega_1+\omega_2+\omega_3+\omega_4+\omega_5=0$. *[5 marks]*

9. Multiply out and simplify $(a+b\omega)(a+b\omega^2)$ where $\omega=\mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}$. *[5 marks]*

10. (a) Find the cube roots of −1.

    (b) Expand $(x+2)^3$.

    (c) Hence or otherwise solve the equation $z^3+6z^2+12z+9=0$. *[7 marks]*

15I Using complex numbers to derive trigonometric identities

The polar form of a complex number provides a link between complex numbers and trigonometry. This turns out to be a powerful tool for deriving trigonometric identities.

We have two ways of raising a complex number to a power: we can either use the Cartesian form and multiply out the brackets, or write it in polar form and use De Moivre's theorem. Equating the two answers allows us to derive formulae for trigonometric ratios of multiple angles. We already know some such identities, for example $\cos 2\theta = 2\cos^2\theta - 1$.

Worked example 15.25

Derive a formula for $\cos(4\theta)$ in terms of $\cos\theta$.

Start with an expression for a complex number involving $\cos\theta$, and find two different expression for z	Let $z = \cos\theta + i\sin\theta$ Then $z^4 = (\cos\theta + i\sin\theta)^4$
First use the binomial theorem	$z^4 = \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$
Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$	$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$
Now use De Moivre's theorem	$z^4 = \cos(4\theta) + i\sin(4\theta)$
The two expressions for z^4 must have equal real and imaginary parts	Equating real parts: $\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
We want the answer in terms of $\cos\theta$ only, so use $\sin^2\theta = 1 - \cos^2\theta$	$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$
Simplify the final expression	Hence $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$

These trigonometric identities can be proved without using complex numbers but this will mean many lines of tedious working. The fact that complex numbers proved correct results in a much shorter way was a major factor in convincing mathematicians that they should be accepted.

Note that by equating imaginary parts of the two expressions we can obtain a similar expression for $\sin(4\theta)$.

Another important link with trigonometry comes from considering the exponential description of complex numbers:

$$e^{i\theta}\cos\theta = +i\sin\theta$$

Equally:

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}$$

In the last line we use symmetry properties of the sine and cosine functions. From adding and subtracting these two equations we can establish two very useful identities:

KEY POINT 15.21

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This result can be generalised further.

Worked example 15.26

Let $z = \cos\theta + i\sin\theta$. Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Use De Moivre's theorem for positive and negative integers, remembering that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin x$

$$z^n = \cos n\theta + i\sin n\theta$$

$$\frac{1}{z^n} = z^{-n} = \cos n\theta - i\sin n\theta$$

Add the two equations, eliminating the $i\sin n\theta$ terms

Adding:

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

Subtract the two equations, eliminating the $i\cos n\theta$ terms

Subtracting:

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

The results proved above can be used to derive another class of trigonometric identities; expressing powers of trigonometric functions in terms of functions of multiple angles.

One example of such an identity is $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$.

Worked example 15.27

Express $\sin^5\theta$ in terms of sines of multiples of θ.

We want to use the results from the previous example, so start with a complex number $z = \text{cis}\,\theta$

Let $z = \cos\theta + i\sin\theta$

Expand $\left(z - \frac{1}{z}\right)^5$ using the binomial expansion.

Then

$$\left(z-\frac{1}{z}\right)^5 = z^5 + 5z\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

Simplify the fractions, taking care with negative signs

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

Group the terms to get expressions of the form $z^n - \frac{1}{z^n}$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

Use the result from the previous example

Using $z^n - \frac{1}{z^n} = 2i\sin n\theta$ on both sides of the equation:

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\therefore \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

These trigonometric identities are one example of how complex numbers can be used to establish facts about real numbers and functions. Other such applications include a formula for cubic equations, calculations involving alternating current, and analysing the motion of waves. Does the fact that complex numbers have real world applications make them more 'real'?

Trigonometric identities such as these will be very useful when we integrate trigonometric functions – See section 19C.

Exercise 15I

1. Use the binomial expansion to find real and imaginary parts of $(\cos\theta + \mathrm{i}\sin\theta)^3$. Hence find an expression for $\sin 3\theta$ in terms of $\sin\theta$. *[6 marks]*

2. (a) Find the real part of $(\cos\theta + \mathrm{i}\sin\theta)^4$.

(b) Hence express $\cos 4\theta$ in terms of $\cos\theta$. *[6 marks]*

3. Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Hence show that $32\cos^5\theta = A\cos 5\theta + B\cos 3\theta + C\cos\theta$ where A, B and C are constants to be found. *[6 marks]*

4. Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$.

(b) Expand $\left(z + z^{-1}\right)^6$ and $\left(z - z^{-1}\right)^6$.

(c) Hence show that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$. *[8 marks]*

See Section 16B if you are unfamiliar with the lim notation.

5. (a) Use the binomial expansion to find the real and imaginary parts of $(\cos\theta + \mathrm{i}\sin\theta)^5$.

(b) Hence show that $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$.

(c) Find $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin\theta}$. *[8 marks]*

Summary

- In this chapter we defined a new type of number, the **imaginary number**, based on i, where $\mathrm{i}^2 = -1$.
- The arithmetic of these complex numbers is the same as real numbers.
- It is useful to interpret complex numbers geometrically using an **Argand diagram**.
- There are two natural ways of describing numbers on the Argand diagram:
  Cartesian form $(x + \mathrm{i}y)$ where x is the real part and $\mathrm{i}y$ the imaginary part
  Polar form $(r\,\mathrm{cis}\,\theta)$ where r is the modulus and θ the argument.
- The polar and Cartesian forms are linked by:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right) \text{ (Remember, if } x < 0 \text{ to add } \pi.)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

- The reflection of a complex number $z = x + iy$ in the real axis gives its complex conjugate $z^* = x - iy$. This satisfies a very important identity:

$$zz^* = |z|^2$$

- A complex number and its reflection in the real axis are known as **conjugate pairs**.
- A surprising result concerning complex conjugates is that the roots of real polynomials occur in conjugate pairs. This fact can be used to factorise higher order polynomials.
- The **Fundamental Theorem of Algebra** states that a polynomial of order n has exactly n roots (real or complex), some of which may be repeated.
- For a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$,

  the sum of the roots is $\frac{-a_{n-1}}{a_n}$ and the product of the roots is $\frac{(-1)^n a_0}{a_n}$.
- Calculating powers of complex numbers is quite tricky in Cartesian form. It turned out that in polar form multiplication represents rotations and stretches, a fact which can be summarised by:

$$\arg(zw) = \arg z + \arg w$$
$$|zw| = |z||w|$$

- Applying this idea to powers of a complex number results in **De Moivre's theorem**:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

- This theorem allows us to find complex roots which lie on the vertices of regular polygons. It also demonstrates that complex numbers act very much like exponentials, leading to the definition:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

  This is called the **Euler's form** of a complex number.
- These results along with the **Binomial Theorem** allowed us to establish several trigonometric identities – results that can be shown to be true outside of the imaginary world!

Introductory problem revisited

Express sin 5*x* in terms of sin *x*.

Consider a complex number with imaginary parts in 5*x*	$\sin 5x = \mathrm{Im}(\cos 5x + i\sin 5x)$
Use De Moivre's theorem	$= \mathrm{Im}(\cos x + i\sin x)^5$

Use the binomial expansion

$= Im(\cos^5 x + 5i\cos^4 x \sin x - 10\cos^3 x \sin^2 x - 10i\cos^2 x \sin^3 x + 5\cos x \sin^4 x + i\sin^5 x)$

Extract the imaginary part

$= 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$

Use $\sin^2 x + \cos^2 x = 1$ to get rid of cosines

$= 5(1-\sin^2 x)^2 \sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x$

Tidy up

$= 16\sin^5 x - 20\sin^3 x + 5\sin x$

This result can also be found by repeated use of the compound angle formula. Feel free to try this, but be warned, it will take a long time!

Mixed examination practice 15

Short questions

1. Express $z = 3\text{i} - \dfrac{2}{\text{i}+\sqrt{3}}$ in the form $x + \text{i}y$. *[5 marks]*

2. If z and w are complex numbers, solve the simultaneous equations:

$$3z + w = 9 + 11\text{i}$$
$$\text{i}w - z = -8 - 2\text{i}$$
[5 marks]

3. $f(z) = z^3 + az^2 + bz + c$ where a, b and c are real constants.
Two roots of $f(z) = 0$ are $z = 1$ and $z = 1 + 2\text{i}$. Find a, b and c. *[6 marks]*

4. Find the complex number z such that $3z - 5z^* = 4 - 3\text{i}$. *[4 marks]*

5. Find the exact value of $\dfrac{1}{\left(\sqrt{3}+\text{i}\right)^6}$. *[6 marks]*

6. The polynomial $z^3 + az^2 + bz - 65$ has a factor of $(z - 2 - 3\text{i})$. Find the values of the real constants a and b. *[6 marks]*

7. If $w = 1 + \sqrt{3}\text{i}$ and $z = 1 + \text{i}$ show that $\text{Re}\left(\dfrac{w+\sqrt{2}z}{w-\sqrt{2}z}\right) = 0$. *[6 marks]*

8. If $z = 4\left(\cos\dfrac{\pi}{4} + \text{i}\sin\dfrac{\pi}{4}\right)$ and $w = 2\left(\cos\dfrac{\pi}{6} + \text{i}\sin\dfrac{\pi}{6}\right)$ evaluate $\left(\dfrac{z}{w}\right)^6$.
leaving your answer in a simplified form. *[6 marks]*

9. If $\arg\left((a+\text{i})^3\right) = \pi$ and a is real and positive, find the exact value of a. *[6 marks]*

10. Let z and w be complex numbers satisfying $\dfrac{w+\text{i}}{w-\text{i}} = \dfrac{z+1}{z-1}$.
    (a) Express w in terms of z.
    (b) Show that if $\text{Im}(z) = 0$ then $\text{Re}(w) = 0$. *[6 marks]*

11. If $|z + 2\text{i}| = |z - 6\text{i}$ find the imaginary part of z. *[6 marks]*

12. If $|z + 25| = 5|z + 1$ find $|z|$. *[6 marks]*

13. (a) The equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots $\dfrac{1}{3}, \dfrac{2}{3}, 1, 1$ and 3. Find the value of a.
    (b) Let $1, \omega_1, \omega_2, \omega_3, \omega_4$ be the roots of the equation $z^5 = 1$. Find the value of $\omega_1 + \omega_2 + \omega_3 + \omega_4$. *[5 marks]*

14. (a) Show that $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$.

(b) Let α and β be the roots of the quadratic equation $x^2 + 7x + 2 = 0$. Find a quadratic equation with roots α^3 and β^3. *[7 marks]*

15. By considering the product $(2+\text{i})(3+\text{i})$ show that

$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$. *[6 marks]*

16. If $0 < \theta < \frac{\pi}{2}$ and $z = (\sin\theta + \text{i}(1-\cos\theta))^2$ find in its simplest form $\arg z$. *[6 marks]*

17. Let z and w be complex numbers such that $w = \frac{1}{1-z}$ and $|z|^2 = 1$. Find the real part of w. *[6 marks]*

18. If $z = \text{cis}\,\theta$ prove that $\frac{z^2-1}{z^2+1} = \text{i}\tan\theta$. *[6 marks]*

19. $w = \frac{kz}{z^2+1}$ where $z^2 \neq -1$. If $\text{Im}(w) = \text{Im}(k) = 0$ and $\text{Im}(z) \neq 0$

prove that $|z| = 1$. *[6 marks]*

Long questions

1. Let $z_1 = \frac{\sqrt{6} - \text{i}\sqrt{2}}{2}$, and $z_2 = 1 - \text{i}$.

(a) Write z_1 and z_2 in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b) Show that $\frac{z_1}{z_2} = \cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}$.

(c) Find the value of $\frac{z_1}{z_2}$ in the form $a + b\text{i}$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise find the exact values of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$. *[8 marks]*

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2. (a) Express $\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$.

(b) Hence show that $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right)^9 = c\text{i}$ where c is a real number to be found.

(c) Find one pair of possible values of positive integers m and n such that:

$\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right)^m = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\text{i}\right)^n$ *[8 marks]*

3. Let $z = \cos\theta + i\sin\theta$, for $-\frac{\pi}{4} < \theta \leq \frac{\pi}{4}$.

(a) (i) Find z^3 using the binomial theorem.

(ii) Use De Moivre's theorem to show that:

$\cos 3\theta = 4\cos 3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin 3\theta$.

(b) Hence prove that $\frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

(c) Given that $\sin\theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$.

[10 marks]

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4. If ω is a complex third root of unity and x and y are real numbers prove that:

(a) $1 + \omega + \omega^2 = 0$.

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = x^2 - xy + y^2$. *[7 marks]*

5. (a) A cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots x_1, x_2, x_3.

(i) Write down the values of $x_1 + x_2 + x_3$ and $x_1 x_2 x_3$ in terms of a, b, c and d.

(ii) Show that $x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{c}{a}$.

(b) The roots α, β and γ of the equation $2x^3 + bx^2 + cx + 16 = 0$ form a geometric progression.

(i) Show that $\beta = -2$.

(ii) Show that $c = 2b$. *[14 marks]*

6. Let $z = \cos\theta + i\sin\theta$.

(a) Show that $2\cos\theta = z + \frac{1}{z}$.

(b) Show that $2\cos n\theta = z^n + \frac{1}{z^n}$.

(c) Consider the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$.

(i) Show that the equation can be written as $6\cos 2\theta - 2\cos\theta + 2 = 0$.

(ii) Find all four complex roots of the original equation. *[7 marks]*

7. Let $\omega = e^{\frac{2i\pi}{5}}$.

(a) Write ω^2, ω^3 and ω^4 in the form $e^{i\theta}$.

(b) Explain why $\omega^1 + \omega^2 + \omega^3 + \omega^4 = -1$.

(c) Show that $\omega + \omega^4 = 2\cos\left(\frac{2\pi}{5}\right)$ and $\omega^2 + \omega^3 = 2\cos\left(\frac{4\pi}{5}\right)$.

(d) Form a quadratic equation in $\cos\left(\frac{2\pi}{5}\right)$ and hence show that
$\cos\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}$. *[10 marks]*

8. (a) By considering $(\cos\theta + \mathrm{i}\sin\theta)^3$ find the expressions for $\cos(3\theta)$ and $\sin(3\theta)$.

(b) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.

(c) Hence show that $\tan\left(\frac{\pi}{12}\right)$ is a root of the equation $x^3-3x^2-3x+1=0$.

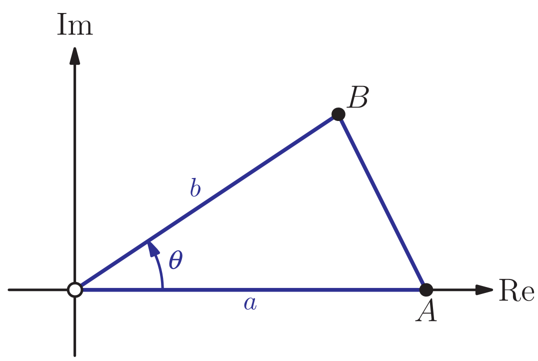
(d) Show that $(x-1)$ is a factor of x^3-3x^2-3x+1 and hence find the exact solutions of the equation $x^3-3x^2-3x+1=0$.

(e) By considering $\tan\left(\frac{\pi}{4}\right)$ explain why $\tan\left(\frac{\pi}{12}\right)<1$.

(f) Hence state the exact value of $\tan\left(\frac{\pi}{12}\right)$. *[14 marks]*

9. (a) Points P and Q in the Argand diagram correspond to complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$. Show that $PQ = |z_1 - z_2|$.

(b) The diagram shows a triangle with one vertex at the origin, one at the point A $(a, 0)$ and one at the point B such that $OB = b$ and $A\hat{O}B = \theta$.



(i) Write down the complex number corresponding to point A.

(ii) Write down the number corresponding to point B in polar form.

(iii) Write down an expression for the length of AB in terms of a, b and θ.

(iv) Hence prove the cosine rule for the triangle AOB:
$|AB|^2 = |OA|^2 + |OB|^2 + 2|OA||OB|\cos\theta$ *[13 marks]*

16 Basic differentiation and its applications

In this chapter you will learn:

- how to find the gradients of curves from first principles, a process called differentiation
- how to differentiate x^n
- how to differentiate $\sin x$, $\cos x$ and $\tan x$
- how to differentiate e^x and $\ln x$
- to find the equations of tangents and normals to curves at given points
- to find maximum and minimum points on curves.

Introductory problem

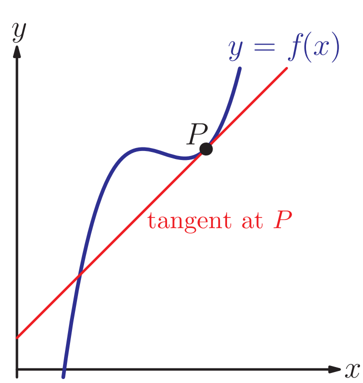
The cost of petrol used in a car, in £ per hour, is $\frac{12+v^2}{100}$ where v is measured in miles per hour and $v > 0$.

If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

In real life, things change. Planets move, babies grow and prices rise. Calculus is the study of things that change, and one of its important tools is differentiation; the ability to find the rate at which the y-coordinate of a curve is changing when the x-coordinate changes. For a straight-line graph this is determined by the **gradient**, but it requires more work to apply the same idea to curves, where the gradient is different at different points.

16A Sketching derivatives

Our first task is to establish exactly what is meant by the gradient of a curve. We are clear on what is meant by the gradient of a straight line and we can use this idea to make a more general definition: the gradient of a curve at a point P is the gradient of the tangent to the curve at that point.



A **tangent** is a straight line which touches the curve without crossing it.

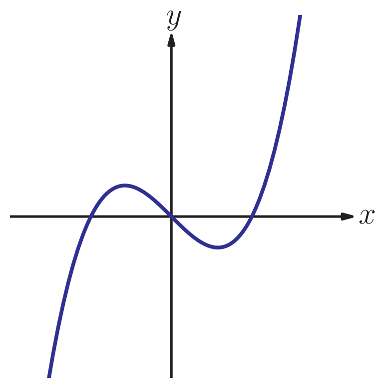
We have already met tangents in chapter 3.

Note that when we say that the tangent at P does not cross the curve we mean that this is only the case locally (close to the point P). The tangent might also intersect a different part of the curve.

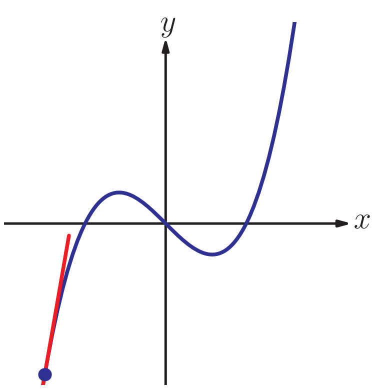
The **derivative** of a function, $f(x)$, is another function that gives the gradient of $y = f(x)$ at any point in the x domain.
It is often useful to be able to roughly sketch the derivative.

Worked example 16.1

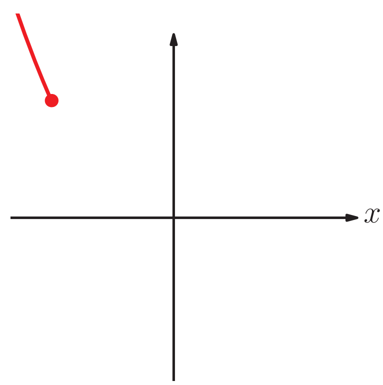
Sketch the derivative of this function.



Imagine we are tracking a point moving along the curve from left to right; we will track the tangent to the curve at the moving point and form the graph of its gradient

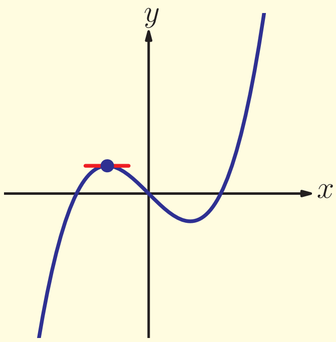


The curve is increasing from left to right, but more and more slowly...

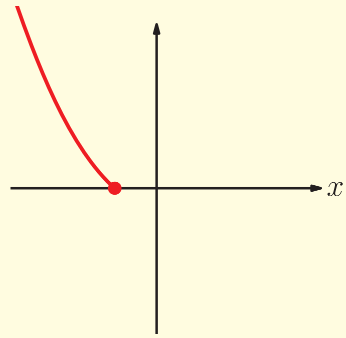


... so the gradient is positive and falling

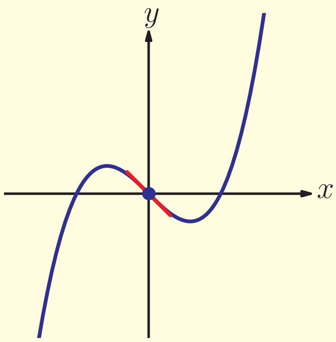
continued . . .



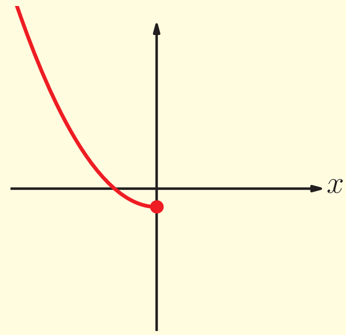
The tangent is horizontal...



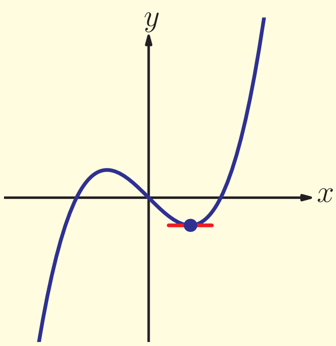
... so the gradient is zero



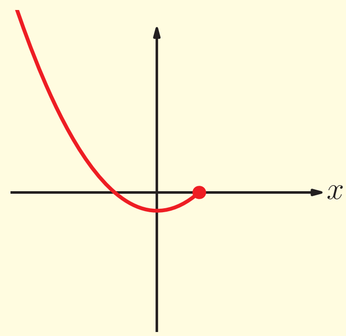
The curve is now decreasing. . . .



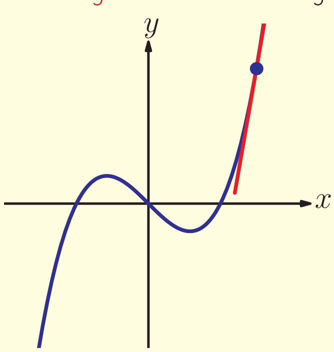
... so the gradient is negative



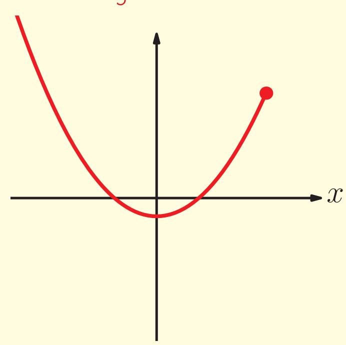
The tangent is horizontal again...



... so the gradient is zero



The curve is increasing, and does so faster and faster...

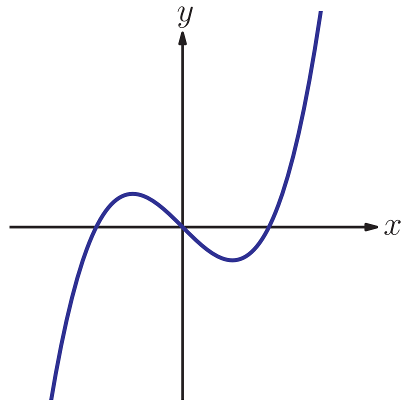


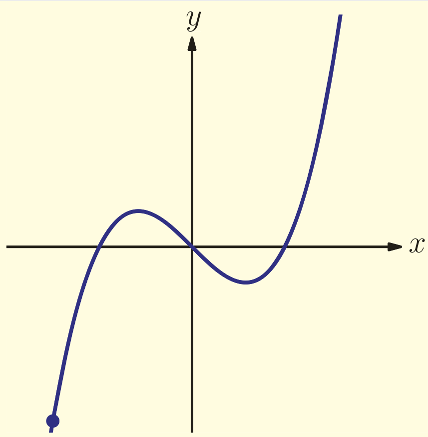
... so the gradient is positive and getting larger

We can also apply the same reasoning backwards.

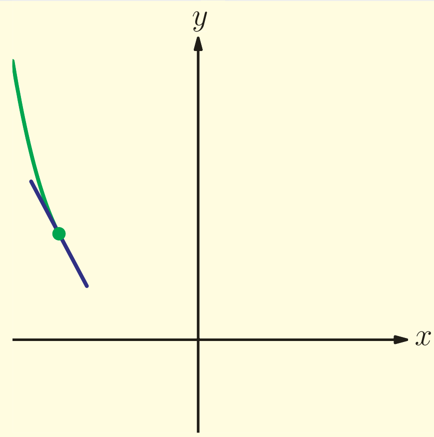
Worked example 16.2

You are given the derivative of a function. Sketch a possible graph of the original function.

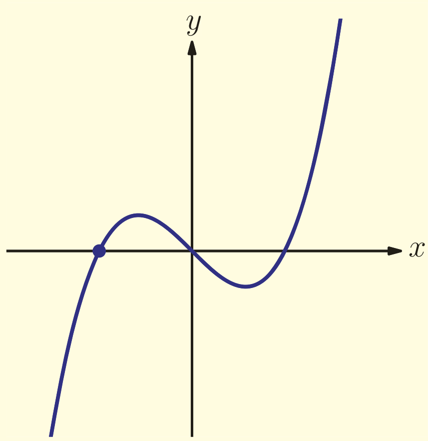




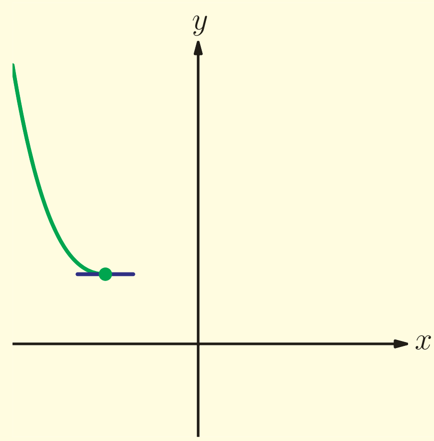
The gradient is negative...



... so the curve is decreasing.

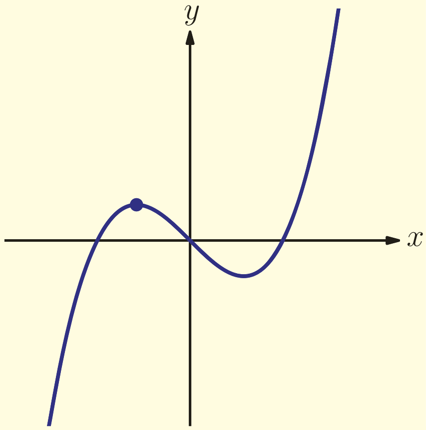


The gradient is zero...

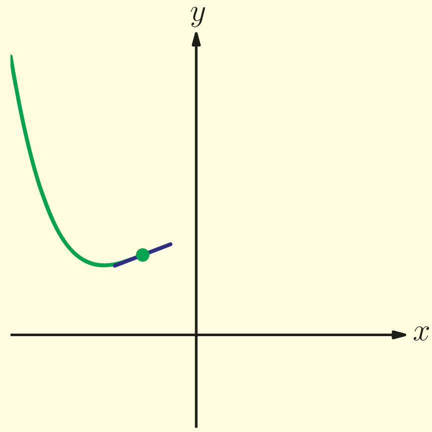


... so the tangent is horizontal.

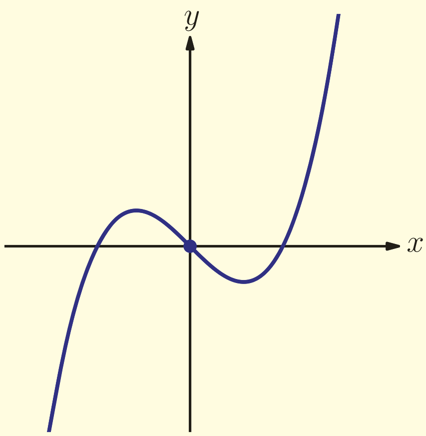
continued . . .



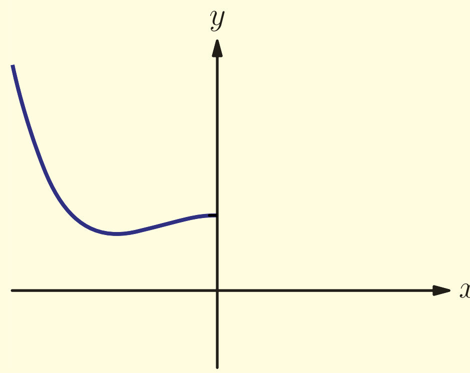
The gradient is positive...



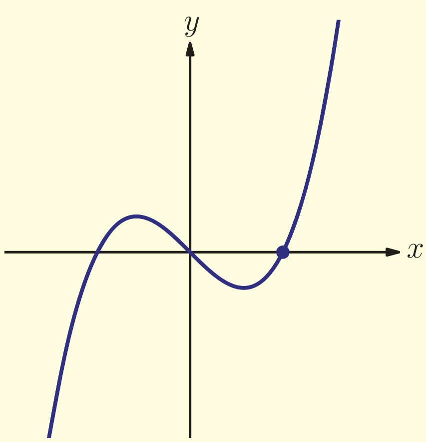
... so the curve is increasing.



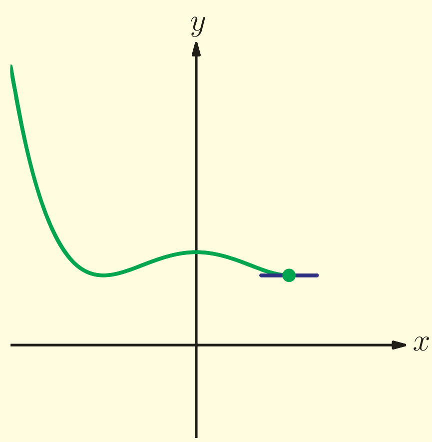
The gradient is zero...



.... so the tangent is horizontal.

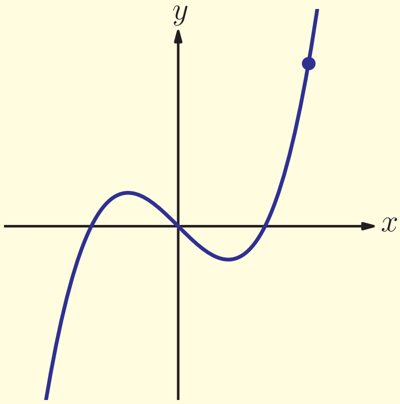


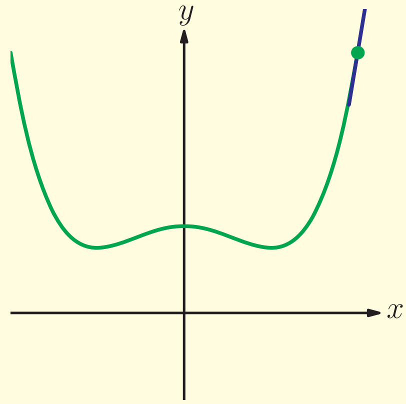
The gradient is zero...



... so the tangent is horizontal.

continued . . .





Notice in this example that there was more than one possible graph we could have drawn, depending on where we started the sketch. In chapter 17 you will learn more about this ambiguity when you 'undo' differentiation.

The relationship between a graph and its derivative can be summarised as follows:

KEY POINT 16.1

When the curve is increasing the gradient is positive.

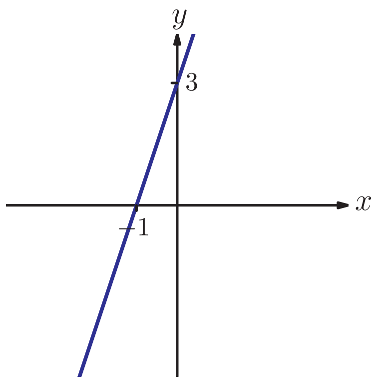
When the curve is decreasing the gradient is negative.

When the tangent is horizontal the gradient is zero; a point on the curve where this happens is called a **stationary point** or **turning point**.

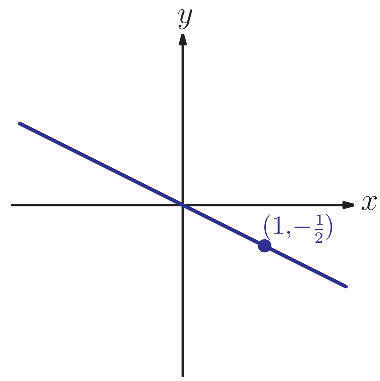
Exercise 16A

1. Sketch the derivatives of the following showing intercept with the x-axis:

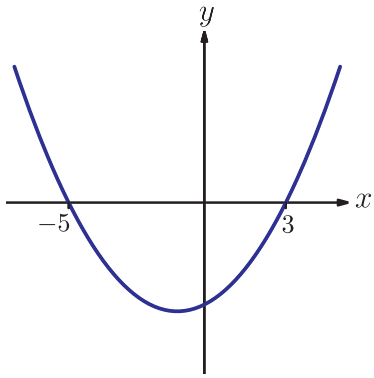
(a) (i)



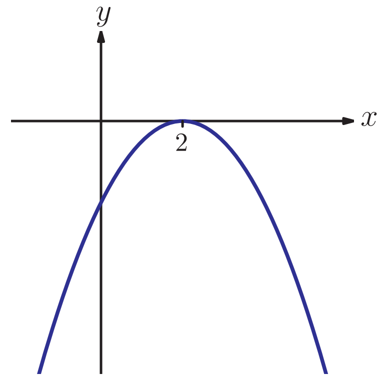
(ii)



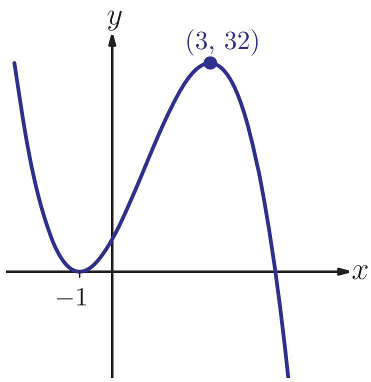
(b) (i)



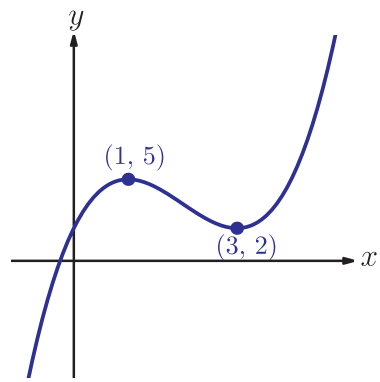
(ii)



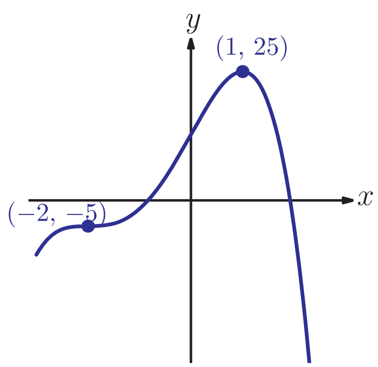
(c) (i)



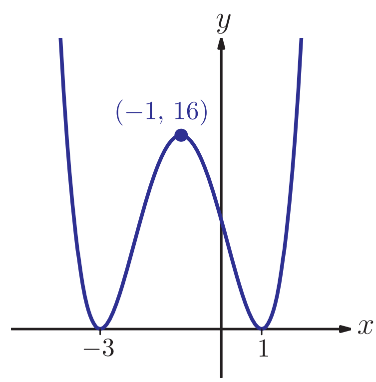
(ii)



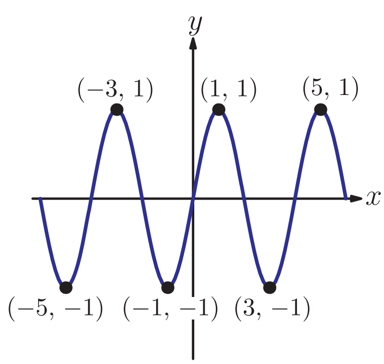
(d) (i)



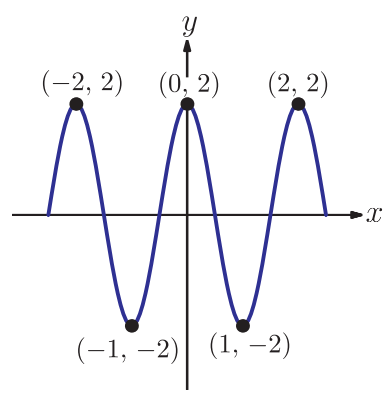
(ii)



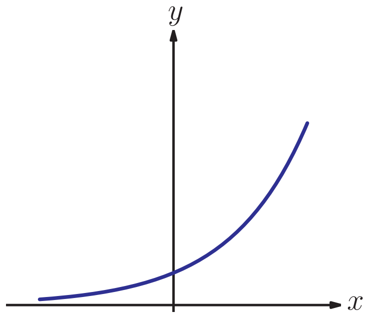
(e) (i)



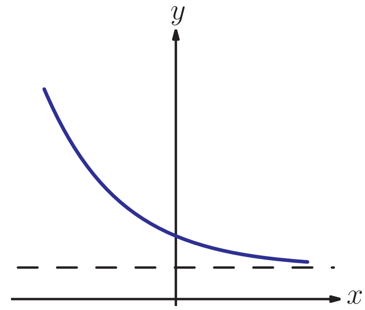
(ii)



(f) (i)

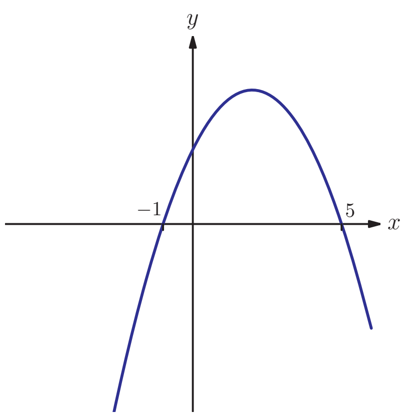


(ii)



2. Each of the following represents a graph of a function's derivative. Sketch a possible graph for the original function, indicating any stationary points.

(a)



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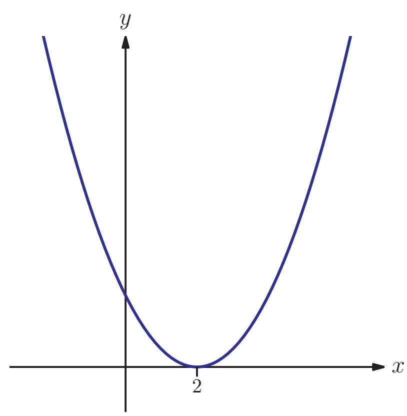

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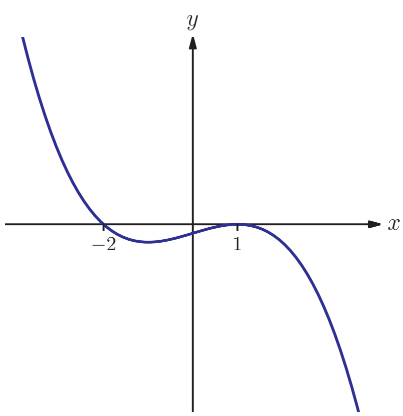

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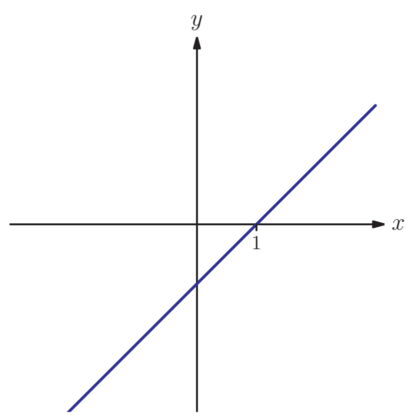


3. For each of the following statements decide if they are always true, sometimes true or always false.

(a) At a point where the derivative is positive, the original function is positive.

(b) If the original function is negative then the derivative is also negative.

(c) The derivative crossing the axis corresponds to a stationary point on the graph.

(d) When the derivative is zero, the graph is at a local maximum or minimum point.

(e) If the derivative function is always positive then part of the original function is above the x-axis.

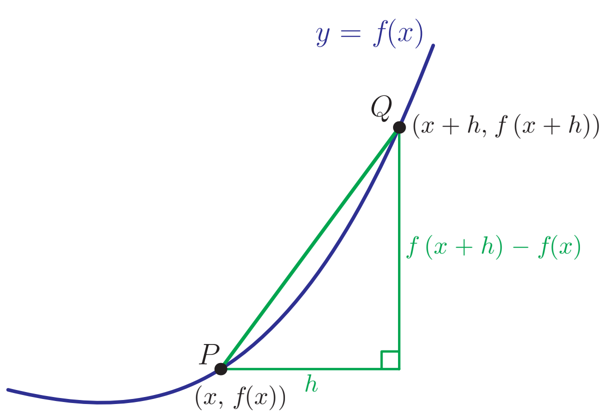
(f) At the lowest value of the original function, the derivative is zero.

16B Differentiation from first principles

You will probably find that drawing a tangent to a graph is very difficult to do accurately, and that your line actually crosses the curve at two points. The line segment between these two intersection points is called a **chord**. If the two points are close together, the gradient of the chord is very close to the gradient of the tangent. We can use this to establish a method for calculating the derivative for a given function.

Self-discovery worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM leads you through several examples of this method. Here we summarise the general procedure.

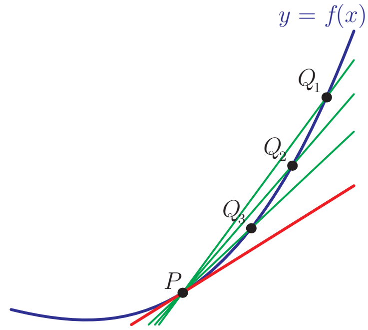
Consider a point $P(x, f(x))$ on the graph of the function $y = f(x)$ and move a distance h away from x to the point $Q(x+h, f(x+h))$.



We can find an expression for the gradient of the chord PQ:

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

As the point Q becomes closer and closer to P, the gradient of the chord PQ becomes a closer and closer approximation to the gradient of the tangent at P.



To denote this idea of the distance h approaching zero, we use $\lim_{h\to 0}$, which reads as 'the limit as h tends to 0'. This idea of a limit is very much like that encountered for asymptotes on graphs in chapters 2 and 4, where the graph tends to the asymptote (the limit) as x tends to ∞.

The process of finding $\lim_{h\to 0}$ of the gradient of the chord PQ is called **differentiation from first principles** and with this notation, we have the following definition:

KEY POINT 16.2

Differentiation from first principles

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is the **derivative of** $f(x)$. It can also be written as f', y' or $\frac{dy}{dx}$ where $y = f(x)$. The process of finding the derivative is called **differentiation**.

EXAM HINT

Differentiation from first principles means finding the derivative using this definition, rather than any of the rules we will meet in the later sections.

We can use this definition to find the derivative of simple polynomial functions.

Worked example 16.3

For the function $y = x^2$, find $\frac{dy}{dx}$ from first principles.

Use the formula

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

We do not want to let the denominator tend to zero so first simplify the numerator and hope the h in the denominator will cancel

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

Divide top and bottom by h

$$= \lim_{h \to 0} (2x + h)$$

Finally let $h \to 0$

$$= 2x$$

We can use the same method with other functions too, but it may require more complicated algebraic manipulation.

Worked example 16.4

Differentiate $f(x) = \sqrt{x}$ from first principles.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

We do not want to let the denominator tend to zero so manipulate the numerator to get a factor of h

We can get rid of the square roots by multiplying top and bottom of the fraction by $\sqrt{x+h} + \sqrt{x}$ and using the difference of two squares

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

We can now divide top and bottom by h…

$$= \frac{1}{\sqrt{h+x} + \sqrt{x}}$$

→

... and let $h \to 0$

$$\therefore f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{h+x}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

Exercise 16B

1. Find the derivatives of the following functions from first principles:

   (a) (i) $f(x) = x^3$ (ii) $f(x) = x^4$

   (b) (i) $f(x) = -4x$ (ii) $f(x) = 3x^2$

   (c) (i) $f(x) = x^2 - 6$ (ii) $f(x) = x^2 - 3x + 4$

2. Prove from first principles that the derivative of $x^2 + 1$ is $2x$.

   [4 marks]

3. Prove from first principles that the derivative of 8 is zero.

   [4 marks]

4. Prove from first principles that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$.

   [4 marks]

5. If k is a constant prove that the derivative of $kf(x)$ is $kf'(x)$.

   [4 marks]

6. Prove from first principles that the derivative of $\frac{1}{\sqrt{x}}$ is $-\frac{1}{2x\sqrt{x}}$.

   [5 marks]

16C Rules of differentiation

From Exercise 16B, and the results of Self-discovery worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM, some properties of differentiation are suggested:

KEY POINT 16.3

- If $y = x^n$ then:

$$\frac{dy}{dx} = nx^{n-1}$$

KEY POINT 16.3 continued...

- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- Differentiation of the sum of various terms can proceed term by term.

Fill-in proof sheet 15 'Differentiating polynomials' on the CD-ROM proves these results for positive integer values; however, this result holds for all rational powers.

A special case is when $n = 0$. Since $x^0 = 1$, we can say that $\frac{dy}{dx} = 0x^{-1} = 0$. This is because the gradient of the graph $y = 1$ is zero everywhere; it is a horizontal line. In fact, the derivative of any constant is zero.

If you need to review rules of exponents, see chapter 2.

You often have to simplify an expression before differentiating, using the laws of algebra, in particular the laws of exponents.

Worked example 16.5

Find the derivative of the following functions:

(a) $f(x) = x^2\sqrt{x}$ (b) $g(x) = \frac{1}{\sqrt[3]{x}}$

First rewrite the function in the form x^n using the laws of exponents

(a) $f(x) = x^2\sqrt{x} = x^2 x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$

Differentiate

$f'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{\frac{3}{2}}$

Cube root can be written as a power

(b) $g(x) = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$

$g'(x) = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}$

EXAM HINT

Note that you cannot differentiate products by differentiating each of the factors and multiplying them together – we will see in chapter 18 that there is a more complicated rule for dealing with products.

Worked example 16.6

Find the derivative of the following functions:

(a) $f(x) = 5x^3$

(b) $g(x) = x^4 - \frac{3}{2}x^2 + 5x - 4$

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

Differentiate x^3 then multiply by 5

(a) $f'(x) = 5 \times 3x^2 = 15x^2$

Differentiate each term separately

(b) $g'(x) = 4x^3 - \frac{3}{2} \times 2x + 5 = 4x^3 - 3x + 5$

We need to write this as a sum of terms of the form x^n

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

$= \frac{4x - 14}{x^{\frac{1}{2}}}$

$= 4x^{1-\frac{1}{2}} - 14x^{-\frac{1}{2}}$

$= 4x^{\frac{1}{2}} - 14x^{-\frac{1}{2}}$

Now differentiate each term separately

$h'(x) = 4 \times \frac{1}{2}x^{\frac{1}{2}-1} - 14\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$= 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

Exercise 16C

1. Differentiate the following:

(a)	(i) $y = x^4$	(ii) $y = x$
(b)	(i) $y = 3x^7$	(ii) $y = -4x^5$
(c)	(i) $y = 10$	(ii) $y = -3$
(d)	(i) $y = 4x^3 - 5x^2 + 2x - 8$	(ii) $y = 2x^4 + 3x^3 - x$
(e)	(i) $y = \frac{1}{3}x^6$	(ii) $y = -\frac{3}{4}x^2$
(f)	(i) $y = 7x - \frac{1}{2}x^3$	(ii) $y = 2 - 5x^4 + \frac{1}{5}x^5$

(g) (i) $y = x^{\frac{3}{2}}$ (ii) $y = x^{\frac{2}{3}}$

(h) (i) $y = 6x^{\frac{4}{3}}$ (ii) $y = \frac{3}{5}x^{\frac{5}{6}}$

(i) (i) $y = 3x^4 - x^2 + 15x^{\frac{2}{5}} - 2$ (ii) $y = x^3 - \frac{3}{5}x^{\frac{5}{3}} + \frac{4}{3}x^{\frac{1}{2}}$

(j) (i) $y = x^{-1}$ (ii) $y = -x^{-3}$

(k) (i) $y = x^{-\frac{1}{2}}$ (ii) $y = -8x^{-\frac{3}{4}}$

(l) (i) $y = 5x - \frac{8}{15}x^{-\frac{5}{2}}$ (ii) $y = -\frac{7}{3}x^{-\frac{3}{7}} + \frac{4}{3}x^{-6}$

2. Find $\frac{dy}{dx}$ for the following:

(a) (i) $y = \sqrt[3]{x}$ (ii) $y = \sqrt[5]{x^4}$

(b) (i) $y = \frac{3}{x^2}$ (ii) $y = -\frac{2}{5x^{10}}$

(c) (i) $y = \frac{1}{\sqrt{x}}$ (ii) $y = \frac{8}{3\sqrt[4]{x^3}}$

(d) (i) $y = x^2(3x - 4)$ (ii) $y = \sqrt{x}(x^3 - 2x + 8)$

(e) (i) $y = (x+2)(\sqrt[3]{x} - 1)$ (ii) $y = \left(x + \frac{2}{x}\right)^2$

(f) (i) $y = \frac{3x^5 - 2x}{x^2}$ (ii) $y = \frac{9x^2 + 3}{2\sqrt[3]{x}}$

3. Find $\frac{dy}{dx}$ if:

(a) (i) $x + y = 8$ (ii) $3x - 2y = 7$

(b) (i) $y + x + x^2 = 0$ (ii) $y - x^4 = 2x$

16D Interpreting derivatives and second derivatives

$\frac{dy}{dx}$ has two related interpretations:

- It is the gradient of the graph of y against x.
- It measures how fast y changes when x is changed – this is called the **rate of change** of y with respect to x.

Remember that $\frac{dy}{dx}$ is itself a function – its value changes with x. For example, if $y = x^2$ then $\frac{dy}{dx}$ is equal to 6 when $x = 3$, and it is equal to –2 when $x = -1$. This corresponds to the fact that the gradient of the graph of $y = x^2$ changes with x, or that the rate of change of y varies with x.

EXAM HINT

We can also write this using function notation:

If $f(x) = x^2$ then $f'(3) = 6$ and $f'(-1) = -2$

To calculate the gradient (or the rate of change) at any particular point, we simply substitute the value of x into the equation for the derivative.

Worked example 16.7

Find the gradient of the graph $y = 4x^3$ at the point where $x = 2$.

The gradient is given by the derivative, so find $\frac{dy}{dx}$	$\frac{dy}{dx} = 12x^2$
Substitute the value for x	When $x = 2$: $\frac{dy}{dx} = 12 \times 2^2 = 48$ So the gradient is 48

EXAM HINT

Your calculator can find the gradient at a given point, but it cannot find the expression for the derivative. See Calculator sheet 8 on the CD-ROM.

If we know the gradient of a graph at a particular point, we can find the value of x at that point. This involves solving an equation.

The sign of the gradient tells us whether the function is increasing or decreasing.

Worked example 16.8

Find the values of x for which the graph of $y = x^3 - 7x + 1$ has gradient 5.

The gradient is given by the derivative	$\frac{dy}{dx} = 3x^2 - 7$
We know the value of $\frac{dy}{dx}$ so we can form an equation	$3x^2 - 7 = 5$ $\Rightarrow 3x^2 = 12$ $\Rightarrow x^2 = 4$ $\Rightarrow x = 2$ or -2

KEY POINT 16.4

If $\frac{dy}{dx}$ is positive the function is increasing – as x gets larger so does y.

If $\frac{dy}{dx}$ is negative the function is decreasing – as x gets larger y gets smaller.

In Section 16H we will discuss what happens when $\frac{dy}{dx} = 0$.

Worked example 16.9

Find the range of values of x for which the function $f(x) = 2x^3 - 6x$ is decreasing.

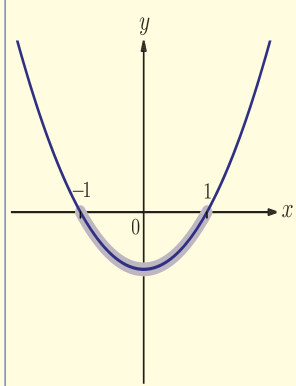
A decreasing function has negative gradient

$f'(x) < 0$

$\Rightarrow 6x^2 - 6 < 0$

$\Rightarrow x^2 - 1 < 0$

This is a quadratic inequality, so we need to look at the graph of $x^2 - 1$



$-1 < x < 1$

There is nothing special about the variables y and x. We can just as easily say that $\frac{dB}{dQ}$ is the gradient of the graph of B against Q or that $\frac{d(bananas)}{d(monkeys)}$ measures how fast bananas change when you change the variable monkeys. To emphasise which variables we are using, we call $\frac{dy}{dx}$ the **derivative of y with respect to x**.

You may wonder why it is so important to emphasise that we are differentiating with respect to x (or Q or *monkeys*). In this course we are only considering functions of one variable, but it is possible to generalise calculus to include functions which depend on several variables. This has many applications, particularly in physics and engineering.

Worked example 16.10

Given that $a = \sqrt{S}$, find the rate of change of a when $S = 9$.

The rate of change is given by the derivative

$a = S^{\frac{1}{2}}$

$\frac{da}{dS} = \frac{1}{2}S^{-\frac{1}{2}} = \frac{1}{2\sqrt{S}}$

Substitute the value for S

When $S = 9$: $\frac{da}{dS} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$\frac{d}{dx}$ is called an operator – it acts on functions to turn them into other functions. So when we differentiate $y = 3x^2$ what we are really doing is applying the $\frac{d}{dx}$ operator to both sides of the identity:

$$\frac{d}{dx}(y) = \frac{d}{dx}(3x^2)$$

$$\Rightarrow \frac{dy}{dx} = 6x$$

So $\frac{dy}{dx}$ is just $\frac{d}{dx}$ applied to y.

The $\frac{d}{dx}$ operator can also be applied to things which have already been differentiated. This is then called the **second derivative**.

KEY POINT 16.5

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is given the symbol $\frac{d^2y}{dx^2}$ or $f''(x)$ and it refers to the rate of change of the gradient.

We can differentiate again to find the third derivative $\left(\frac{d^3y}{dx^3} \text{ or } f'''(x)\right)$, fourth derivative $\left(\frac{d^4y}{dx^4} \text{ or } f^{(4)}(x)\right)$, and so on.

Worked example 16.11

Given that $f(x) = 5x^3 - 4x$:

(a) Find $f''(x)$.

(b) Find the rate of change of the gradient of the graph of $y = f(x)$ at the point where $x = -1$.

Differentiate $f(x)$ and then differentiate the result

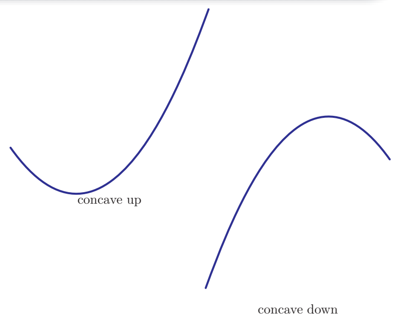
(a) $f'(x) = 15x^2 - 4$

$f''(x) = 30x$

The rate of change of the gradient means the second derivative

(b) $f''(-1) = -30$

We can use the second derivative to find out more about the shape of the graph. Remember that the second derivative is the rate of change of the gradient. So when the second derivative is positive, the gradient is increasing. This means that the graph is curving upwards; we say that it is **concave up**. When the second derivative is negative, the gradient is decreasing so the graph curves downwards; we say that it is **concave down**.



Exercise 16D

1. Write the following rates of change as derivatives:
   (a) The rate of change of z as t changes.
   (b) The rate of change of Q with respect to P.
   (c) How fast R changes when m is changed.
   (d) How quickly balloon volume (V) changes over time (t).
   (e) The rate of increase of the cost of apples (y) as the weight of the apple (x) increases.
   (f) The rate of change of the rate of change of z as y changes.
   (g) The second derivative of H with respect to m.

2. (a) (i) If $f = 5x^{\frac{1}{3}}$ what is the derivative of f with respect to x?
   (ii) If $p = 3q^5$ what is the derivative of p with respect to q?
   (b) (i) Differentiate $d = 3t + 7t^{-1}$ with respect to t.
   (ii) Differentiate $r = c + \frac{1}{c}$ with respect to c.
   (c) (i) Find the second derivative of $y = 9x^2 + x^3$ with respect to x.
   (ii) Find the second derivative of $z = \frac{3}{t}$ with respect to t.

3. (a) (i) If $y = 5x^2$, find $\frac{dy}{dx}$ when $x = 3$.

(ii) If $y = x^3 + \frac{1}{x}$, find $\frac{dy}{dx}$ when $x = 1.5$.

(b) (i) If $A = 7b + 3$, find $\frac{dA}{db}$ when $b = -1$.

(ii) If $f = \theta^2 + \theta^{-3}$, find $\frac{df}{d\theta}$ when $\theta = 0.1$.

(c) (i) Find the gradient of the graph of $A = x^3$ when $x = 2$.

(ii) Find the gradient of the tangent to the graph of $z = 2a + a^2$ when $a = -6$.

(d) (i) How quickly does $f = 4T^2$ change as T changes when $T = 3$?

(ii) How quickly does $g = y^4$ change as y changes when $y = 2$?

(e) (i) What is the rate of increase of W with respect to p when p is -3 if $W = -p^2$?

(ii) What is the rate of change of L with respect to c when $c = 6$ if $L = 7\sqrt{c} - 8$?

You may think that it is contradictory to talk about the rate of change of *y* as *x* changes if we are fixing *x* to have a certain value. Think about x passing through this point.

4. (a) (i) If $y = ax^2 + (1-a)x$ where a is a constant, find $\frac{dy}{dx}$.

(ii) If $y = x^3 + b^2$ where b is a constant, find $\frac{dy}{dx}$.

(b) (i) If $Q = \sqrt{ab} + \sqrt{b}$ where b is a constant, find $\frac{dQ}{da}$.

(ii) If $D = 3(av)^2$ where a is a constant, find $\frac{dD}{dv}$.

5. (a) (i) If $y = x^3 - 5x$, find $\frac{d^2y}{dx^2}$ when $x = 9$.

(ii) If $y = 8 + 2x^4$, find $\frac{d^2y}{dx^2}$ when $x = 4$.

(b) (i) If $S = 3A^2 + \frac{1}{A}$, find $\frac{d^2S}{dA^2}$ when $A = 1$.

(ii) If $J = v - \sqrt{v}$, find $\frac{d^2J}{dv^2}$ when $v = 9$.

(c) (i) Find the second derivative of B with respect to n if $B = 8n$ and $n = 2$.

(ii) Find the second derivative of g with respect to r if $g = r^7$ and $r = 1$.

6. (a) (i) If $y = 3x^3$ and $\frac{dy}{dx} = 36$, find x.

(ii) If $y = x^4 + 2x$ and $\frac{dy}{dx} = 5$, find x.

(b) (i) If $y = 2x + \frac{8}{x}$ and $\frac{dy}{dx} = -31$, find y.

(ii) If $y = \sqrt{x} + 3$ and $\frac{dy}{dx} = \frac{1}{6}$ find y.

7. (a) (i) Find the interval in which $x^3 - 4x$ is an increasing function.

(ii) Find the interval in which $x^3 - 3x^2$ is a decreasing function.

(b) (i) Find the interval in which $3x + \frac{2}{x}$ is a decreasing function.

(ii) Find the interval in which $x - \sqrt{x}$ is an increasing function.

(c) (i) Find the interval in which the graph of $y = x^3 - 4x + 3$ is concave up.

(ii) Find the interval in which the graph of $y = x^3 + 6x^2 - 1$ is concave up.

(d) (i) Find the set of values of x for which the graph of $f(x) = x^4 - 6x^3 + 12x^2$ is concave down.

(ii) Find the set of values of x for which the graph of $f(x) = x^4 - 54x^2$ is concave down.

8. Find all points of the graph of $y = x^3 - 2x^2 + 1$ where the gradient equals the y-coordinate. *[5 marks]*

9. In what interval on the graph of $y = 7x - x^2 - x^3$ is the gradient decreasing? *[5 marks]*

10. In what interval on the graph of $y = \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 3x + 6$ is the gradient increasing? *[6 marks]*

11. Find an alternative expression for $\frac{d^n}{dx^n}(x^n)$.

16E Trigonometric functions

Using the techniques from Section 16A we can sketch the derivative of the graph of $y = \sin x$. The result is a graph that looks just like $y = \cos x$. On Fill-in proof sheet 17 'Differentiating trigonometric functions' on the CD-ROM you can see why this is the case. Results for $y = \cos x$ and $y = \tan x$ can be established in a similar manner giving these results:

In Section 18C we will prove the derivative of tan x using the quotient rule.

KEY POINT 16.6

Differentiating trigonometric functions gives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

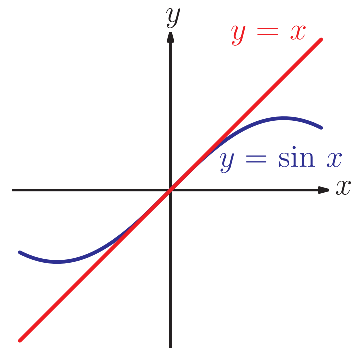
Reciprocal trigonometric functions were covered in Section 12D.

EXAM HINT

Whenever you are doing calculus you MUST work in radians.

It is possible to do calculus using degrees, or any other unit for measuring angles, but using radians gives the simplest rules, which is why they are the unit of choice for almost all mathematicians.

These rules only work if x is measured in radians since they are based upon the result that $\sin x \approx x$ for very small values of x. You can check on your calculator that $\sin x \approx x$ for radians but not for degrees. The result can also be seen on the graph and is proved on Fill-in proof sheet 16 'The small angle approximations' on the CD-ROM.



All rules of differentiation from Section 16C still apply.

Worked example 16.12

Differentiate $y = 3\tan x - 2\cos x$.

Differentiate using the rules in key point 16.6. Note that $\sec^2 x$ can also be written as $\frac{1}{\cos^2 x}$

$$\frac{dy}{dx} = 3(\sec^2 x) - 2(-\sin x)$$

$$= 3\sec^2 x + 2\sin x$$

Exercise 16E

1. Differentiate the following:
   (a) (i) $y = 3\sin x$ (ii) $y = 2\cos x$
   (b) (i) $y = 2x - 5\cos x$ (ii) $y = \tan x + 5$
   (c) (i) $y = \frac{\sin x + 2\cos x}{5}$ (ii) $y = \frac{1}{2}\tan x - \frac{1}{3}\sin x$

2. Find the gradient of $f(x) = \sin x + x^2$ at the point $x = \frac{\pi}{2}$. *[5 marks]*

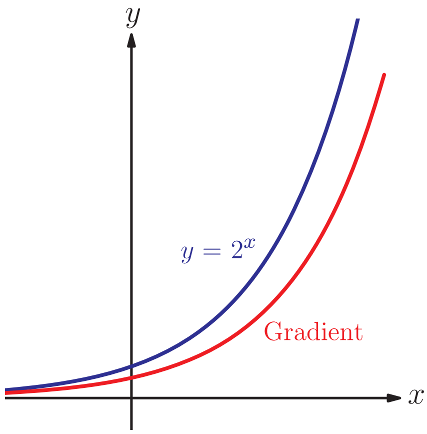
3. Find the gradient of $g(x) = \frac{1}{4}\tan x - 3\cos x - x^3$ at the point $x = \frac{\pi}{6}$. *[5 marks]*

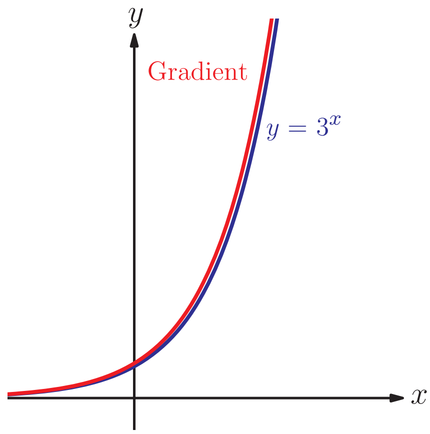
4. Given $h(x) = \sin x + \cos x \quad 0 \le x < 2\pi$, find the values of x for which $h'(x) = 0$. *[6 marks]*

5. Given $y = \frac{1}{4}\tan x + \frac{1}{x^2} \quad 0 < x \le 2\pi$ solve the equation $\frac{dy}{dx} = 1 - \frac{2}{x^3}$. *[6 marks]*

16F The exponential and natural logarithm functions

Use your calculator to plot the graphs of $y = 2^x$ and $y = 3^x$ and their derivatives. The results looks like another exponential function.





It appears that there is a number somewhere between two and three where the derivative of the graph would be exactly the same as the original exponential. It turns out that this is the graph of $y = e^x$ where $e = 2.718\ldots$ It is the same as the base of the natural logarithm defined in Section 2E.

We will see how to differentiate exponential functions with bases other than e in Section 20D.

KEY POINT 16.7

$$\frac{d}{dx}(e^x) = e^x$$

The natural logarithm function $y = \ln x$ behaves in a surprising way, having a derivative of a completely different form.

KEY POINT 16.8

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

This result is proved on Fill-in proof sheet 18 'Differentiating logarithmic functions graphically' on the CD-ROM.

Exercise 16F

1. Differentiate the following:

   (a) (i) $y = 3e^x$ (ii) $y = \dfrac{2e^x}{5}$

   (b) (i) $y = -2\ln x$ (ii) $y = \dfrac{1}{3}\ln x$

   (c) (i) $y = \dfrac{\ln x}{5} - 3x + 4e^x$ (ii) $y = 4 - \dfrac{e^x}{2} + 3\ln x$

2. (i) Find the exact value of the gradient of the graph of $f(x) = \dfrac{1}{2}e^x - 7\ln x$ at the point $x = \ln 4$.

   (ii) Find the exact value of the gradient of the graph $f(x) = e^x - \dfrac{\ln x}{2}$ when $x = \ln 3$. *[4 marks]*

3. Find the value of x where the gradient of $f(x) = 5 - 2e^x$ is -6. *[4 marks]*

4. Find the value of x where the gradient of $g(x) = x^2 - 12\ln x$ is 2. *[4 marks]*

There is an easier way to do some parts of Question 5 using a method from Section 18A. For now, you will have to use your algebra skills!

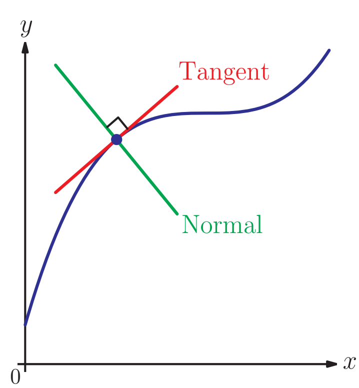
5. Differentiate:

   (a) (i) $y = \ln x^3$ (ii) $y = \ln 5x$

   (b) (i) $y = e^{x+3}$ (ii) $y = e^{x-3}$

   (c) (i) $y = e^{2\ln x}$ (ii) $y = e^{3\ln x+2}$

   (d) (i) $y = \log_3 x$ (ii) $y = 4\log_6 x$



16G Tangents and normals

The tangent to a curve at a given point is a straight line which touches the curve and has the same gradient at that point. Finding the equation of the tangent at a point relies on knowing the gradient of the function at that point. This can be found by differentiating the function. We then have both the gradient of the line and a point on it and we can use the standard procedure for finding the equation of a straight line.

Normals are lines which pass through the graph and are perpendicular to the tangent. They have many uses, such as finding centres of curvature of graphs and working out how light is reflected from curved mirrors. However, in the International Baccalaureate® you are only likely to be asked to calculate their equations. To do this you use the fact that if two lines with gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$.

See Prior learning section W on the CD-ROM.

Worked example 16.13

(a) Find the equation of the tangent to the graph of the function $f(x) = \cos x + e^x$ at the point $x = 0$.

(b) Find the equation of the normal to the graph of the function $g(x) = x^3 - 5x^2 - x^{\frac{3}{2}} + 22$ at $(4, -2)$.

In each case give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

We need the gradient, which is $f'(0)$

(a) $f'(x) = -\sin x + e^x$

$\therefore f'(0) = -\sin 0 + e^0 = 1$

To find the equation of a straight line we also need coordinates of one point
The tangent passes through the point on the graph where $x = 0$. Its y-coordinate is $f(0)$

When $x = 0$,

$$y = f(0) = \cos 0 + e^0 = 1 + 1 = 2$$

Put all the information into the equation of a line

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 1(x - 0)$$
$$\Rightarrow y = x + 2$$
$$\Rightarrow y - x - 2 = 0$$

The normal is perpendicular to the tangent, so we need the gradient of the tangent first

(b) $f'(x) = 3x^2 - 10x - \frac{3}{2}x^{\frac{1}{2}}$

Find the gradient at $x = 4$

$$\therefore f'(4) = 3(4)^2 - 10(4) - \frac{3}{2}(4)^{\frac{1}{2}} = 48 - 40 - 3 = 5$$

For perpendicular lines, $mm_1 = -1$

Therefore gradient of normal,

$$m = \frac{-1}{5}$$

We are given both x and y-coordinates of the point, so put all the information into the equation of a line

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = \frac{-1}{5}(x - 4)$$
$$\Rightarrow 5y + 10 = -x + 4$$
$$\Rightarrow x + 5y + 6 = 0$$

The procedure for finding the equations of tangents and normals can be summarised as follows:

EXAM HINT

Your calculator may be able to find the equation of a tangent at a given point.

KEY POINT 16.9

For the point on the curve $y = f(x)$ with $x = a$:

- the gradient of the tangent is $f'(a)$
- the gradient of the normal is $-\dfrac{1}{f'(a)}$
- the coordinates of the point are $x_1 = a,\ y_1 = f(a)$.

To find the equation of the tangent or the normal use $y - y_1 = m(x - x_1)$ with the appropriate gradient.

You may not be given the coordinates of the point where the tangent touches the curve, but asked to find them given another point.

Worked example 16.14

The tangent at point P on the curve $y = x^2 + 1$ passes through the origin. Find the possible coordinates of P.

We want to find the equation of the tangent at P, so use unknowns for its coordinates

Let P have coordinates (p, q)

As P lies on the curve, (p, q) satisfies $y = x^2 + 1$

Then $q = p^2 + 1$

The gradient of the tangent is given by $\dfrac{dy}{dx}$ when $x = p$

$\dfrac{dy}{dx} = 2x$

When $x = p: \dfrac{dy}{dx} = 2p$

$\therefore m = 2p$

Write the equation of the tangent, remembering it passes through (p, q)

Equation of the tangent:

$y - q = 2p(x - p)$

$\Rightarrow y - (p^2 + 1) = 2p(x - p)$

→

continued . . .

Tangent passes through the origin, so set $x=0, y=0$

Passes through $(0,0)$:
$0-(p^2+1)=2p(0-p)$
$\Rightarrow -p^2-1=-2p^2$
$\Rightarrow p^2=1$
$\Rightarrow p=1$ or -1

We can now find q

When $p=1,\ q=2$
When $p=-1,\ q=2$
So the coordinates of P are (1, 2) or (–1, 2)

Exercise 16G

1. Find the equations of the tangent and normal to the following:
   (a) $y=\dfrac{x^2+4}{\sqrt{x}}$ at $x=4$
   (b) $y=3\tan x-2\sqrt{2}\sin x$ at $x=\dfrac{\pi}{4}$
   (c) $y=3-\dfrac{1}{5}e^x$ at $x=2\ln 5$

2. Find the coordinates of the point on the curve $y=\sqrt{x}+3x$ where the gradient is 5. *[4 marks]*

3. Find the equation of the tangent to the curve $y=e^x+x$ which is parallel to $y=3x$. *[4 marks]*

4. Find the x-coordinates of the points on the curve $y=x^3-3x^2$ where the tangent is parallel to the normal of the point at (1, –1). *[6 marks]*

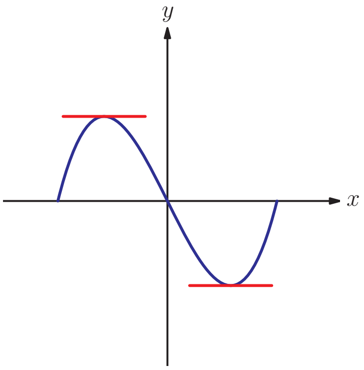
5. Find the coordinates of the point where the tangent to the curve $y=x^3-3x^2$ at $x=2$ meets the curve again. *[6 marks]*

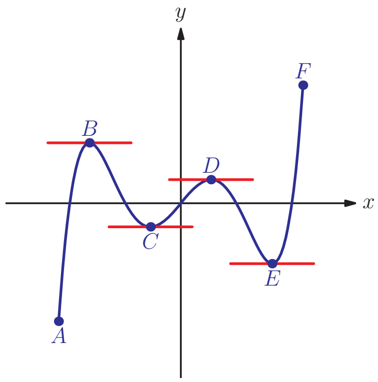
6. Find the coordinates of the point on the curve $y=(x-1)^2$ where the normal passes through the origin. *[5 marks]*

7. Points P and Q lie on the graph of $f(x)=2\sin x$ and have x-coordinates $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$.
   (a) Evaluate $f'\left(\dfrac{\pi}{6}\right)$.
   (b) Find the angle between the tangent at P and the chord PQ, giving your answer to the nearest tenth of a degree. *[11 marks]*

8. A tangent is drawn on the graph $y=\dfrac{k}{x}$ at the point where $x=a$, $(a>0)$. The tangent intersects the y-axis at P and the x-axis at Q. If O is the origin show that the area of the triangle OPQ is independent of a. *[8 marks]*

9. Show that the tangent to the curve $y = x^3 - x$ at the point with x-coordinate a meets the curve again at a point with x-coordinate $-2a$. *[6 marks]*





16H Stationary points

In real life people are interested in maximising their profits, or minimising the drag on a car. We can use calculus to describe such things mathematically as points on a graph.

The gradient at both the maximum and minimum point on the above graph is zero and therefore:

KEY POINT 16.10

> To find local maximum and local minimum points, we solve the equation $\frac{dy}{dx} = 0$.

We use the phrase **local maximum** and **local minimum** because it is possible that the largest or smallest value of the whole function occurs at the endpoint of the graph, or that there are other points which also have gradient of zero. The points that we have found are just the largest or smallest values of y in that part of the graph.

Points which have a gradient of zero are called **stationary points**.

Worked example 16.15

Find the coordinates of the stationary points of $y = 2x^3 - 15x^2 + 24x + 8$.

Stationary points have $\frac{dy}{dx} = 0$ so we need to differentiate

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

Then form an equation

For stationary points $\frac{dy}{dx} = 0$:

$6x^2 - 30x + 24 = 0$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0$

$\Rightarrow x = 1$ or $x = 4$

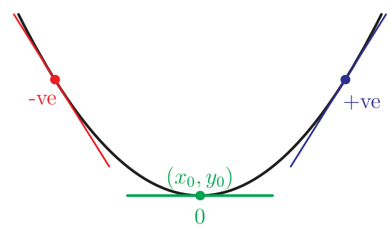
continued . . .

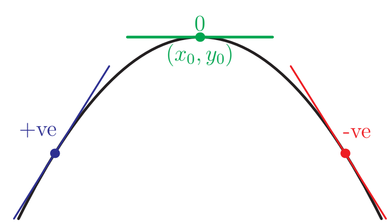
Remember to find the y-coordinate for each point

When $x = 1$:
$y = 2(1)^3 - 15(1)^2 + 24(1) + 8 = 19$
When $x = 4$:
$y = 2(4)^3 - 15(4)^2 + 24(4) + 8 = -8$
Therefore,
stationary points are (1, 19) and (4, −8)

The calculation in Worked example 16.15 does not tell us whether the stationary points we found are maximum or minimum points.

It can be seen from the diagrams that one way of testing for the nature of a stationary point is to look at the gradient either side of the point. You can do this by substituting nearby x-values into the expression for $\frac{dx}{dy}$. For a minimum point the gradient moves from negative to positive. For a maximum point the gradient moves from positive to negative.
We can also interpret these conditions by looking at the sign of the second derivative. Around a minimum point the curve is concave up, so $\frac{d^2y}{dx^2}$ is positive. Around a maximum point the curve is concave down and $\frac{d^2y}{dx^2}$ is negative.





See the end of Section 16D

This leads to the following test.

KEY POINT 16.11

Given a stationary point (x_0, y_0) of a function $y = f(x)$, if:

- $\frac{d^2y}{dx^2} < 0$, at x_0, then (x_0, y_0) is a *maximum*
- $\frac{d^2y}{dx^2} > 0$, at x_0, then (x_0, y_0) is a *minimum*
- $\frac{d^2y}{dx^2} = 0$, at x_0, then no conclusion can be drawn, so test the gradient either side of (x_0, y_0)

Worked example 16.16

Classify the stationary points of the function $y = 2x^3 - 15x^2 + 24x + 8$ from Worked example 16.15.

We have already found the stationary points

Stationary points are (1, 19) and (4, −8)

The nature of stationary points is determined by the value of the second derivative

$$\frac{d^2y}{dx^2} = 12x - 30$$

At $x = 1$:

$$\frac{d^2y}{dx^2} = 12(1) - 30 = -18 < 0$$

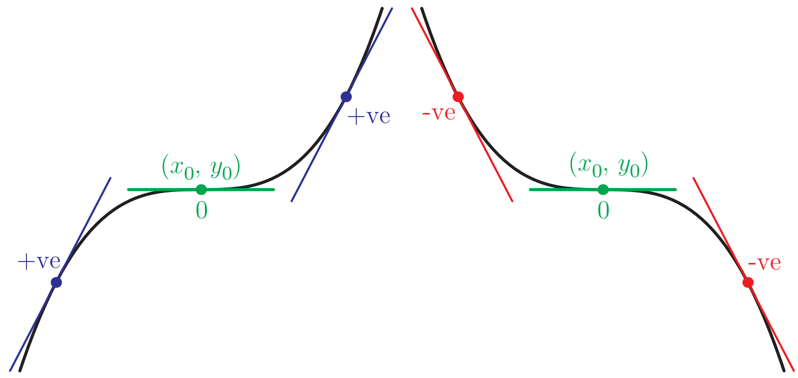
$\therefore (1, 19)$ is a maximum

At $x = 4$:

$$\frac{d^2y}{dx^2} = 12(4) - 30 = 18 > 0$$

$\therefore (4, -8)$ is a minimum

All local maximum points and local minimum points have $\frac{dy}{dx} = 0$, but the reverse is not true: A point with $\frac{dy}{dx} = 0$ does not have to be a maximum or a minimum point. There are two other possibilities:



These possibilities are called **points of inflexion**, and are labelled (x_0, y_0) on the above diagrams. Note that at those points the line with zero gradient actually crosses the curve. The gradient is either positive on both sides of a point of inflexion (positive point of inflexion), or negative on both sides (a negative point of inflexion).

In UK English, 'inflexion' may be spelt 'inflection'.

Worked example 16.17

Find the coordinates and nature of the stationary points of $y = 3 + 4x^3 - x^4$.

Stationary points have $\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 12x^2 - 4x^3$ For stationary points $\frac{dy}{dx} = 0$: $12x^2 - 4x^3 = 0$ $\Rightarrow 4x^2(3-x) = 0$ $\Rightarrow x = 0$ or $x = 3$
Find y-coordinates	When $x = 0$: $y = 3 + 4(0)^3 - (0)^4 = 3$ When $x = 3$: $y = 3 + 4(3)^3 - (3)^4 = 30$ Therefore, stationary points are: (0, 3) and (3, 30)
The nature of the stationary points is determined by the second derivative	Find the nature of these points: $\frac{d^2y}{dx^2} = 24x - 12x^2$ At $x = 0$: $\frac{d^2y}{dx^2} = 24(0) - 12(0)^2 = 0$
As $\frac{d^2y}{dx^2} = 0$ we need to check the gradient either side of the stationary point	Therefore, examine $\frac{dy}{dx}$: At $x = -1$: $\frac{dy}{dx} = 12(-1)^2 - 4(-1)^3 = 16 > 0$ At $x = 1$: $\frac{dy}{dx} = 12(1)^2 - 4(1)^3 = 8 > 0$ $\therefore (0, 3)$ is a positive point of inflexion.

continued . . .

At $x = 3$:

$$\frac{d^2y}{dx^2} = 24(3) - 12(3)^2 = -36 < 0$$

$\therefore (3, 30)$ is a maximum

When $\frac{d^2y}{dx^2} = 0$, the stationary point is NOT always a point of inflexion.

Worked example 16.18

Find the coordinates and nature of the stationary points of $f(x) = x^4$:

$f'(x) = 4x^3$

Stationary points have $f'(x) = 0$

For stationary points $f'(x) = 0$

$4x^3 = 0$

$\Rightarrow x = 0$

Find the y-coordinate

$f(0) = 0$

Therefore, stationary point is:

$(0, 0)$

The nature is determined by $f''(x)$

Find the nature of this point:

$f''(x) = 12x^2$

$f''(0) = 0$

As $f''(0) = 0$, we need to check the gradient on either side

Therefore, examine $f'(x)$:

$f'(-1) = 4(-1)^3 = -4 < 0$

$f'(1) = 4(1)^3 = 4 > 0$

Therefore $(0, 0)$ is a minimum.

Exercise 16H

1. Find and classify the stationary points on the following curves:

(a) (i) $y = x^3 - 5x^2$ (ii) $y = x^4 - 8x^2$

(b) (i) $y = \sin x + \frac{x}{2}$, $-\pi \le x \le \pi$

(ii) $y = 2\cos x + 1$, $0 \le x < 2\pi$

(c) (i) $y = \ln x - \sqrt{x}$ (ii) $y = 2e^x - 5x$

2. Give an example to illustrate that the following statement is incorrect:

'If $y = f(x)$ has exactly two stationary points, at x_1 and x_2, and $f(x_1) > f(x_2)$ then $(x_1, f(x_1))$ must be a local maximum.'

Under what conditions is the statement true?

3. Find and classify the stationary points on the curve $y = x^3 + 3x^2 - 24x + 12$. *[6 marks]*

4. Find and classify the stationary points on the curve $y = x - \sqrt{x}$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \sin x + 4\cos x$ in the interval $0 < x < 2\pi$. *[6 marks]*

6. Show that the function $f(x) = \ln x + \frac{1}{x^k}$ has a stationary point with y-coordinate $\frac{\ln k + 1}{k}$. *[6 marks]*

7. Find the range of the function $f : x \mapsto 3x^4 - 16x^3 + 18x^2 + 6$. *[5 marks]*

8. Find the range of the function $f : x \mapsto e^x - 4x + 2$. *[5 marks]*

9. Find and classify in terms of k the stationary points on the curve $y = kx^3 + 6x^2$. *[6 marks]*

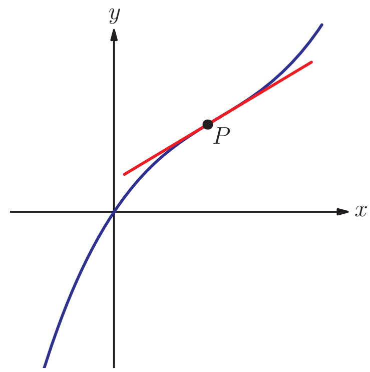
16I General points of inflexion

In the previous section we met stationary points of inflexion, but the idea of a point of inflexion is more general than this.

One definition is that the tangent to the curve at a point of inflexion crosses the curve at the same point. This does not require the point to be a stationary point.

EXAM HINT

Although the red line actually crosses the graph at *P*, it is still referred to as the tangent, because it has the same gradient as the curve at *P*.



Geometrically, this can be interpreted as an 'S-bend', a curve which goes from decreasing gradient to increasing gradient (or vice versa). This means that the curve is concave down on one side of the point of inflexion and concave up on the other. We know that this corresponds to the second derivative changing from negative to positive (or vice versa).

See the end of Section 16D.

KEY POINT 16.12

At a point of inflexion $\frac{d^2y}{dx^2} = 0$.

Unfortunately, as in Worked example 16.18, just because a point has $\frac{d^2y}{dx^2} = 0$ it is not necessarily a point of inflexion. We have to determine the gradient on either side to be sure.

EXAM HINT

If a question states that a curve does have a point of inflexion and there is only one solution to the equation $\frac{d^2y}{dx^2} = 0$, you can then assume you have found the point of inflexion.

Worked example 16.19

Find the coordinates of the point of inflexion on the curve $y = x^3 - 3x^2 + 5x - 1$.

Find $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 3x^2 - 6x + 5$

$\frac{d^2y}{dx^2} = 6x - 6$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$,

$6x - 6 = 0$

$x = 1$

Remember the other coordinate!

When $x = 1$, $y = 1 - 3 + 5 - 1 = 2$

So point of inflexion is at (1, 2)

Exercise 16I

1. Find the coordinates of the point of inflexion on the curve $y = e^x - x^2$. *[5 marks]*

2. The curve $y = x^4 - 6x^2 + 7x + 2$ has two points of inflexion. Find their coordinates. *[5 marks]*

3. Show that all points of inflexion on the curve $y = \sin x$ lie on the x-axis. *[6 marks]*

4. Find the coordinates of the points of inflexion on the curve $y = 2\cos x + x$ for $0 \le x \le 2\pi$. Justify carefully that these points are points of inflexion. *[5 marks]*

5. The point of inflexion on the curve $y = x^3 - ax^2 - bx + c$ is a stationary point of inflexion. Show that $b = 8a^2$. *[6 marks]*

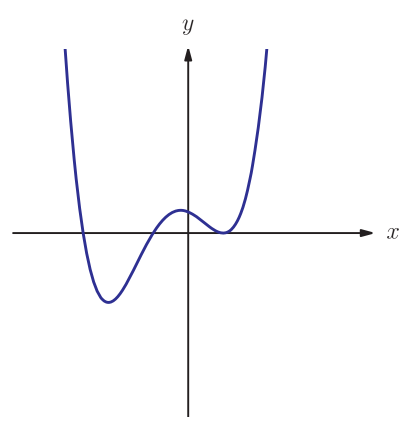
6. The graph shows $y = f'(x)$.

   On a copy of the diagram:

   (a) mark any point corresponding to a local minimum of $f(x)$ with an A

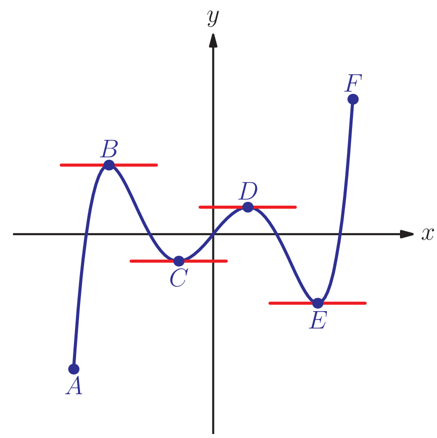
   (b) mark any point corresponding to a local maximum of $f(x)$ with a B

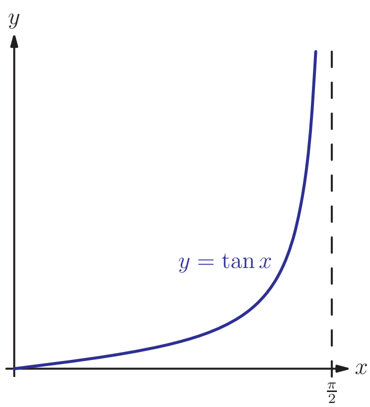
   (c) mark any point corresponding to a point of inflexion of $f(x)$ with a C. *[4 marks]*



16J Optimisation

We can now start to use differentiation to maximise or minimise quantities. In Section 16H we saw how to find stationary points (the points with zero gradient) and how to decide whether they are local maximum or local minimum points. We also noted that a stationary point does not necessarily give the largest or smallest value of the function over the whole domain. For example, on the diagram, points B and D are local maximum points, but the largest value of the function occurs at point F, which is an end point of the domain.





Some functions do not have maximum or minimum values at all. This can happen when the graph has an asymptote. We say that the function is not continuous throughout its domain. For example, the value of $\tan x$ increases without a limit as x increases towards $\frac{\pi}{2}$, so $\tan x$ does not have a maximum value.

If we wish to minimise or maximise A by changing B we do so in four stages:

1. Find the relationship between A and B.
2. Solve the equation $\frac{dA}{dB} = 0$.
3. Decide whether it is a maximum, minimum or point of inflexion by considering $\frac{d^2A}{dB^2}$.
4. Check whether the end points of the domain are actually global maximum or minimum points, and check that there are no vertical asymptotes.

Often the first stage of this process is the most difficult and there are many questions where we have to use a geometric context to make this link. Thankfully in many questions this relationship is given to you.

Worked example 16.20

The height of a swing (h) in metres at a time t seconds is given by $h = 2 - 1.5\sin t$ for $0 < t < 3$. Find the minimum and maximum height of the swing.

Find stationary points

$\frac{dh}{dt} = -1.5\cos t = 0$ at a stationary point

$\Rightarrow \cos t = 0$

$0 < t < 3 \therefore t = \frac{\pi}{2}$ (only one solution)

Classify stationary points

$\frac{d^2h}{dt^2} = 1.5\sin t$

When $t = \frac{\pi}{2}$, $\frac{d^2h}{dt^2} = 1.5 > 0$ so $t = \frac{\pi}{2}$ is a local minimum. This minimum height is

$h = 2 - 1.5\sin\frac{\pi}{2} = 0.5$ metres

Check end points

Check there are no vertical asymptotes

When $t = 0$, $h = 2$m

When $t = 3$, $h = 1.79$m (3SF)

So maximum height is 2 m.

Exercise 16J

1. What are the minimum and maximum values of the expression e^x for $0 \le x \le 1$? *[4 marks]*

2. A rectangle has width x metres and length $30 - x$ metres.

(a) Find the maximum area of the rectangle.

(b) Show that as x changes the perimeter stays constant and find the value of this perimeter. *[5 marks]*

3. Find the maximum and minimum values of the function $y = x^3 - 9x$ if $-2 \le x \le 5$. *[5 marks]*

4. What are the maximum and minimum values of $f(x) = e^x - 3x$ if $0 \le x \le 2$? *[5 marks]*

5. What are the minimum and maximum values of $y = \sin x + 2x$ for $0 \le x \le 2\pi$? *[5 marks]*

6. Find the minimum value of the sum of a positive real number and its reciprocal. *[5 marks]*

7. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{\sqrt{w}}$ ms^{-1} for a time of $\frac{5}{w}$ s. What weight will achieve the maximum distance travelled? *[6 marks]*

8. The time in minutes (t) taken to melt 100 g of butter depends upon the percentage of the butter which is made of saturated fats (p) as in the following function:

$$t = \frac{p^2}{10\,000} + \frac{p}{100} + 2$$

Find the maximum and minimum times to melt 100 g of butter. *[6 marks]*

9. The volume of water in millions of litres (V) in a new tidal lake is modelled by $V = 60\cos t + 100$ where t is the time in days after being completed.

(a) What is the smallest volume of the lake?

(b) A hydroelectric plant produces an amount of electricity proportional to the rate of flow of water. In the first 6 days when is the plant producing maximum electricity? *[6 marks]*

10. The owner of a fast-food shop finds that there is a relationship between the amount of salt s (g/tray) added to the fries and his weekly sales of fries F (100s of portions):

$$F(s) = 4s + 1 - s^2, \quad 0 \le s \le 4.2$$

(a) Find the amount of salt he should put on his fries to maximise his sales.
The total cost C ($ per tray) associated with the sales of fries is given by:

$$C(s) = 0.3 + 0.2F(s) + 0.1s$$

(b) Find the amount of salt he should put on his fries to minimise his costs.

(c) The profit made on his fries is given by the difference between the sales and the costs.
How much salt should he add to maximise his profit? *[8 marks]*

11. A car tank is being filled with petrol such that the volume in the tank in litres (V) over time in minutes (t) is given by

$$V = 300(t^2 - t^3) + 4 \quad \text{for } 0 < t < 0.5$$

(a) How much petrol was initially in the tank?

(b) After 30 seconds the tank was full. What is the capacity of the tank?

(c) At what time is petrol flowing in at the greatest rate? *[8 marks]*

12. x is the surface area of leaves on a tree in m^2. Because leaves may be shaded by other leaves, the amount of energy produced by the tree is given by $2-\frac{x}{10}$ kJ per square metre of leaves.

(a) Find an expression for the total energy produced by the tree.

(b) What area of leaves provides the maximum energy for the tree?

(c) Leaves also use energy. The total energy requirement is given by $0.01x^3$. The net energy produced is the difference between the energy produced by the leaves and the energy used by the leaves. For what range of x do the leaves produce more energy than they use?

(d) Show that the maximum net energy is produced when the tree has leaves with a surface area of $\frac{10(\sqrt{7}-1)}{3}$. *[12 marks]*

Summary

- The **gradient** of a function at the point P is the gradient of the **tangent** to the function's graph at that point.
- To find the gradient of a function we can **differentiate** from first principles:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ (also denoted by } \frac{\mathrm{d}}{\mathrm{d}x}f(x))$$

- For the point on the curve $y=f(x)$ with $x=a$:

  – the gradient of the tangent is $f'(a)$

  – the gradient of the **normal** is $-\frac{1}{f'(a)}$
- If $f(x)=x^n$, then $f'(x)=nx^{n-1}$.
- The **derivative** of a sum is the sum of the derivatives of each term.
- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- The **derivatives** of the trigonometric functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tan x)=\sec^2 x$$

- The derivatives of the exponential and natural logarithm functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x)=\mathrm{e}^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x}$$

- **Stationary points** of a function are points where the gradient is zero, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

- Stationary points can be one of four types:
  - local maximum
  - local minimum
  - positive point of inflexion
  - negative point of inflexion.
- The **second derivative** can be used to test which of these occurs. At a stationary point (x_0, y_0), if
  - $\frac{d^2y}{dx^2} < 0$ at x_0 then (x_0, y_0) is a maximum
  - $\frac{d^2y}{dx^2} > 0$ at x_0 then (x_0, y_0) is a minimum
  - $\frac{d^2y}{dx^2} = 0$ at x_0 then no conclusion can be drawn, so check the sign of the gradient either side of (x_0, y_0).
- **Points of inflexion** can also have a non-zero gradient.
- At a point of inflexion $\frac{d^2y}{dx^2} = 0$.
- Global maximum or minimum points may also occur at the endpoint of a graph.

Introductory problem revisited

The cost of petrol used in a car, in £ per hour, is $\frac{12+v^2}{100}$ where v is measured in miles per hour and $v > 0$. If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

If we have the cost per hour and we want the total cost we must find the total time. But the time taken is $\frac{50}{v}$ hours, so the total cost is $C = \frac{50}{v}\left(12 + \frac{v^2}{100}\right) = \frac{600}{v} + \frac{v}{2}$.

If we wish to find a minimum value of C by changing v we can do this by setting $\frac{dC}{dv} = 0$:

$$-\frac{600}{v^2} + \frac{1}{2} = 0$$
$$\Rightarrow v^2 = 1200$$

$v = 34.6$ mph (3SF) (Taking the positive root since $v > 0$)

To see if we have found a minimum we find $\frac{d^2C}{dv^2} = 1200v^{-3}$ which is positive for any positive v, so the point is a local minimum.

Next, to see if it is global minimum we must consider the end points. Although v is never actually zero as it gets close to it, the $\frac{600}{v}$ term gets very large. When v gets very large the $\frac{v}{2}$ term gets very large. Therefore we have found the global minimum.

Mixed examination practice 16

Short questions

1. Find the equation of the tangent to the curve $y = e^x + 2\sin x$ at the point where $x = \frac{\pi}{2}$. *[5 marks]*

2. Find the equation of the normal to the curve $y = (x-2)^3$ when $x = 2$. *[5 marks]*

3. $f(x)$ is a quadratic function taking the form $x^2 + bx + c$. If $f(1) = 2$ and $f'(2) = 12$ find the values of b and c. *[5 marks]*

4. Find the coordinates of the point of inflexion on the graph of $y = \frac{x^3}{6} - x^2 + x$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \tan x - \frac{4x}{3}$. *[6 marks]*

6. Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$. *[2 marks]*

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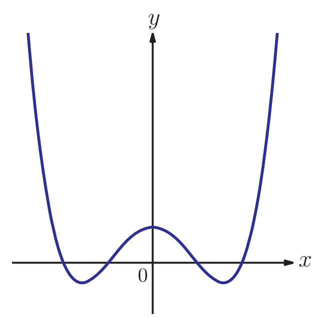
7. The graph shows $y = f'(x)$:



On a sketch of this graph:

(a) Mark points corresponding to a local minimum of $f(x)$ with an A.

(b) Mark points corresponding to a local maximum of $f(x)$ with a B.

(c) Mark points corresponding to a point of inflexion of $f(x)$ with a C. *[6 marks]*

8. On the curve $y = x^3$ a tangent is drawn from the point (a, a^3), $a > 0$ and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find the value of a. *[6 marks]*

Long questions

1. The line $y = 24(x-1)$ is tangent to the curve $y = ax^3 + bx^2 + 4$ at $x = 2$.

(a) Use the fact that the tangent meets the curve to show that $2a + b = 5$.

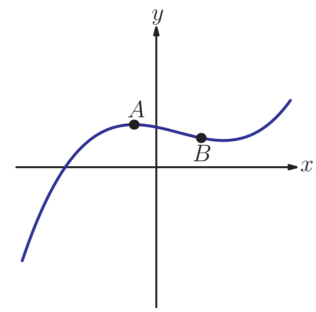
(b) Use the fact that the tangent has the same gradient as the curve to find another relationship between a and b.

(c) Hence find the values of a and b.

(d) The line meets the curve again. Find the coordinates of the other point of intersection. *[12 marks]*

2. The graph shows part of $y = x^3 - x^2 - x + 3$.

The point A is a local maximum and the point B is a point of inflexion.



(a) (i) Find the coordinates of A.

(ii) Find the coordinates of B.

(b) (i) Find the equation of the line containing both A and B.

(ii) Find the x coordinate of the points on the curve at which the tangent is parallel to this line. *[10 marks]*

3. (a) Sketch and label the curves $y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$.

(b) Find the x-coordinate of P, the point of intersection of the two curves.

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR.

(d) Prove that the two tangents at the points where $x = a, a > 0$, on each curve are always perpendicular.

[14 marks]

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4. The population of bacteria (P) in thousands at a time t in hours is modelled by:

$$P = 10 + e^t - 3t,\ t \ge 0$$

(a) (i) Find the initial population of bacteria.

(ii) At what time does the number of bacteria reach 14 million?

(b) (i) Find $\frac{dP}{dt}$.

(ii) Find the time at which the bacteria are growing at a rate of 6 million per hour.

(c) (i) Find $\frac{d^2P}{dt^2}$ and explain the physical significance of this quantity.

(ii) Find the minimum number of bacteria, justifying that it is a minimum.

[12 marks]

17 Basic integration and its applications

In this chapter you will learn:

- to reverse the process of differentiation (this process is called integration)
- to find the equation of a curve given its derivative and a point on the curve
- to integrate sin x, cos x and tan x
- to integrate e^x and $\frac{1}{x}$
- to find the area between a curve and the x- or y-axis
- to find the area enclosed between two curves.

Introductory problem

The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). When it contains alternating current the relationship between I and t is given by $I = \sin t$. When it contains direct current the relationship between I and t is given by $I = k$. What value of k means that the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ is the same whether alternating or direct current is used?

As in many areas of mathematics, as soon as we learn a new process we must then learn how to undo it. However, it turns out that undoing the process of differentiation opens up the possibility of answering a seemingly unconnected problem: what is the area under a curve?

17A Reversing differentiation

We saw in the last chapter how differentiation gives us the gradient of a curve or the rate of change of one quantity with another. What then if we already know the function describing a curve's gradient, or the expression for a rate of change, and wish to find the original function? Our only way of proceeding is to 'undo' the differentiation that has already taken place and this process of reverse differentiation is known as **integration**.

Let us look at two particular cases to get a feel for this process. Each time we will be given $\frac{dy}{dx}$ and need to answer the question 'What was differentiated to give this?'

If $\frac{dy}{dx} = 2x$ then the original function y must have contained x^2 as we know that differentiation decreases the power by 1. Differentiating x^2 gives exactly $2x$, so we have found that if $\frac{dy}{dx} = 2x$ then $y = x^2$.

If $\frac{dy}{dx} = x^{\frac{1}{2}}$ then the original function y must have contained $x^{\frac{3}{2}}$. Differentiating $x^{\frac{3}{2}}$ will give $y = \frac{3}{2}x^{\frac{1}{2}}$ and we do not want the $\frac{3}{2}$. However, if we multiply the $x^{\frac{3}{2}}$ by $\frac{2}{3}$ then when we differentiate the coefficient cancels to 1, so if $\frac{dy}{dx} = x^{\frac{1}{2}}$ then $y = \frac{2}{3}x^{\frac{3}{2}}$.

The integration symbol comes from the old English way of writing the letter 'S'. Originally it stood for the word 'Sum' (or rather, ∫um). As you will see in later sections, the integral does indeed represent a sum of infinitesimally small quantities.

Writing out 'if $\frac{dy}{dx} = x^{\frac{1}{2}}$ then $y = \frac{2}{3}x^{\frac{3}{2}}$, is descriptive but rather laborious and so the notation used for integration is:

$$\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

Here, the dx simply states that the integration is taking place with respect to the variable x in exactly the same way that in $\frac{dy}{dx}$ it states that the differentiation is taking place with respect to x. We could equally well write, for example, $\int t^{\frac{1}{2}}\,dt = \frac{2}{3}t^{\frac{3}{2}}$.

Exercise 17A

You may have heard of the term 'differential equation'. These are the simplest types of differential equation.

1. Find a possible expression for y in terms of x:

(a) (i) $\frac{dy}{dx} = 3x^2$ (ii) $\frac{dy}{dx} = 5x^4$

(b) (i) $\frac{dy}{dx} = -\frac{1}{x^2}$ (ii) $\frac{dy}{dx} = -\frac{4}{x^5}$

(c) (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ (ii) $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$

(d) (i) $\frac{dy}{dx} = 10x^4$ (ii) $\frac{dy}{dx} = 12x^2$

17B Constant of integration

We have seen how to integrate some functions of the form x^n by reversing the process of differentiation but the process as carried out above was not complete.

Let us consider again the first example where we stated that: $\int 2x\,dx = x^2$.

Were there any other possible answers here?

We could have given $\int 2x\,dx = x^2 + 1$ or $\int 2x\,dx = x^2 - \frac{3}{5}$.

Both of these are just as valid as our original answer; we know that when we differentiate the constant ($+1$ or $-\frac{3}{5}$) we just get 0. We could therefore have given any constant; without further information we cannot know what this constant on the original function was before it was differentiated.

Hence our complete answers to the integrals considered in Section 17A are:

$$\int 2x\,dx = x^2 + c$$

$$\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

where the c is an unknown **constant of integration**.

We will see later that, given further information, we can find this constant.

We will see how to find the constant of integration in Section 17F.

Exercise 17B

1. Give three possible functions which when differentiated with respect to x give the following:

   (a) $3x^3$

   (b) 0

2. Find the integrals:

(a) (i) $\int 7x^4 \, dx$ (ii) $\int \frac{1}{3}x^2 \, dx$

(b) (i) $\int \frac{1}{2t^2} \, dt$ (ii) $\int \frac{8}{y^3} \, dy$

17C Rules of integration

To find integrals so far we have used the idea of reversing differentiation for each specific function. Let us now think about applying the reverse process to the general rule of differentiation.

We know that for $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$ or in words:

To differentiate x^n multiply by the old power then decrease the power by 1.

We can express the reverse of this process as follows.

To integrate x^n increase the power by 1 then divide by the new power.

Using integral notation:

We will see how to integrate x^{-1} in Section 17D.

KEY POINT 17.1

The general rule for integrating x^n for any rational power $n \neq -1$ is:

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$$

EXAM HINT

The + c is a part of the answer, and you must write it every time.

Note the condition n $\neq -1$ which ensures that we are not dividing by zero.

It is worth remembering the formula below for integrating a constant: $\int k \, dx = kx + c$, which is a special case of the rule in Key point 17.1

$$\int k \, dx = \int kx^0 \, dx = \frac{k}{1}x^1 + c$$

In Key point 16.3, we saw that if we differentiate $kf(x)$ we get $kf'(x)$; we can reverse this logic to show that:

KEY POINT 17.2

To integrate multiples of functions:

$$\int kf(x) \, dx = k\int f(x) \, dx$$

EXAM HINT

This rule only works if k is a constant.

Since we can differentiate term by term (also in Key point 16.4) then we can also split up integrals of sums.

EXAM HINT

Be warned! You cannot integrate products or quotients by integrating each part separately.

KEY POINT 17.3

For the sum of integrals:

$$\int f(x)+g(x)\,dx=\int f(x)\,dx+\int g(x)\,dx$$

By combining Key points 17.2 and 17.3 with $k=-1$, we can also show that the integral of a difference is the difference of the integrals of the separate parts.

These ideas are demonstrated in the following examples.

Worked example 17.1

Find (a) $\int 6x^{-3}\,dx$ (b) $\int (3x^4-8x^{-\frac{4}{3}}+2)\,dx$

Add one to the power and divide by this new power

(a) $\int 6x^{-3}\,dx=\frac{6}{-3+1}x^{-3+1}+c$

Tidy up

$=\frac{6}{-2}x^{-2}+c$

$=-3x^{-2}+c$

Go through term by term adding one to the power of x and dividing by this new power

Remember the rule for integrating a constant

(b) $\int 3x^4-8x^{-\frac{4}{3}}+2dx=\frac{3}{4+1}x^{4+1}-\frac{8}{-\frac{4}{3}+1}x^{-\frac{4}{3}+1}+2x+c$

Tidy up

$=\frac{3}{5}x^5-\frac{8}{-\frac{1}{3}}x^{-\frac{1}{3}}+2x+c$

$=\frac{3}{5}x^5+24x^{-\frac{1}{3}}+2x+c$

Just as for differentiation, it may be necessary to change terms into the form kx^n before integrating.

Worked example 17.2

Find (a) $\int 5x^2\sqrt[3]{x}\,dx$ (b) $\int \frac{(x-3)^2}{\sqrt{x}}\,dx$

Write the cube root as a power and use rules of exponents

(a) $\int 5x^2\sqrt[3]{x}dx = \int 5x^2x^{1/3}dx = \int 5x^{7/3}dx$

Dividing by $\frac{10}{3}\left(\frac{7}{3}+1\right)$ is the same as multiplying by $\frac{3}{10}$

$= 5\times\frac{3}{10}\times x^{\frac{10}{3}} + c = \frac{3}{2}x^{\frac{10}{3}} + c$

Expand the brackets first, then use rules of exponents

(b) $\int \frac{(x-3)^2}{\sqrt{x}}dx = \int \frac{x^2-6x+9}{x^{\left(\frac{1}{2}\right)}}dx$

$= \int x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}\,dx$

Dividing by a fraction is the same as multiplying by its reciprocal

$= \frac{2}{5}x^{\frac{5}{2}} - 6\times\frac{2}{3}x^{\frac{3}{2}} + 9\times 2x^{\frac{1}{2}} + c$

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$

Exercise 17C

EXAM HINT

In the integral do not forget the dx or the equivalent. We will make more use of it later! The function you are integrating is actually being multiplied by dx so you could write question 1(f)(ii) as $\int \frac{2dx}{x^3}$.

1. Find the following integrals:

(a) (i) $\int 9x^8\,dx$ (ii) $\int 12x^{11}\,dx$

(b) (i) $\int x\,dx$ (ii) $\int x^3\,dx$

(c) (i) $\int 9\,dx$ (ii) $\int \frac{1}{2}\,dx$

(d) (i) $\int 3x^5\,dx$ (ii) $\int 9x^4\,dx$

(e) (i) $\int 3\sqrt{x}\,dx$ (ii) $\int 3\sqrt[3]{x}\,dx$

(f) (i) $\int \frac{5}{x^2}\,dx$ (ii) $\int \frac{2}{x^3}\,dx$

2. Find the following integrals:

(a) (i) $\int 3\,dt$ (ii) $\int 7\,dz$

(b) (i) $\int q^5\,dq$ (ii) $\int r^{10}\,dr$

(c) (i) $\int 12g^{\frac{3}{5}}\,\mathrm{d}g$ (ii) $\int 5y^{\frac{7}{2}}\,\mathrm{d}y$

(d) (i) $\int 4\frac{\mathrm{d}h}{h^2}$ (ii) $\int \frac{\mathrm{d}p}{p^4}$

3. Find the following integrals:

(a) (i) $\int x^2 - x^3 + 2\,\mathrm{d}x$ (ii) $\int x^4 - 2x + 5\,\mathrm{d}x$

(b) (i) $\int \frac{1}{3t^3} + \frac{1}{4t^4}\,\mathrm{d}t$ (ii) $\int 5\times\frac{1}{v^2} - 4\times\frac{1}{v^5}\,\mathrm{d}v$

(c) (i) $\int x\sqrt{x}\,\mathrm{d}x$ (ii) $\int \frac{3\sqrt{x}}{\sqrt[3]{x}}\,\mathrm{d}x$

(d) (i) $\int (x+1)^3\,\mathrm{d}x$ (ii) $\int x(x+2)^2\,\mathrm{d}x$

4. Find $\int \frac{1+x}{\sqrt{x}}\,\mathrm{d}x$. *[4 marks]*

17D Integrating x^{-1} and e^x

When integrating $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$, we were careful to exclude the case $n = -1$.

In Key point 16.8 we saw that $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$. Reversing this gives:

KEY POINT 17.4

$$\int x^{-1}\,\mathrm{d}x = \ln x + c$$

We will modify this rule in Section 17H.

In Key point 16.7, we saw that $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$. We can use this to integrate the exponential function:

KEY POINT 17.5

$$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c$$

Exercise 17D

1. Find the following integrals:

(a) (i) $\int \frac{2}{x}\,dx$ (ii) $\int \frac{3}{x}\,dx$

(b) (i) $\int \frac{1}{2x}\,dx$ (ii) $\int \frac{1}{3x}\,dx$

(c) (i) $\int \frac{x^2-1}{x}\,dx$ (ii) $\int \frac{x^3+5}{x}\,dx$

(d) (i) $\int \frac{3x+2}{x^2}\,dx$ (ii) $\int \frac{x-\sqrt{x}}{x^2}\,dx$

2. Find the following integrals:

(a) (i) $\int 5e^x\,dx$ (ii) $\int 9e^x\,dx$

(b) (i) $\int \frac{2e^x}{5}\,dx$ (ii) $\int \frac{7e^x}{11}\,dx$

(c) (i) $\int \frac{(e^x+3x)}{2}\,dx$ (ii) $\int \frac{(e^x+x^3)}{5}\,dx$

17E Integrating trigonometric functions

We can expand the set of functions that we can integrate by continuing to refer back to work covered in chapter 16.

We saw in Key point 16.6 that $\frac{d}{dx}(\sin x)=\cos x$ which means that $\int \cos x\,dx=\sin x+c$.

Similarly, as $\frac{d}{dx}(\cos x)=-\sin x$, then $\int \sin x\,dx=-\cos x+c$.

KEY POINT 17.6

The integrals of trigonometric functions:

$$\int \sin x\,dx=-\cos x+c$$

$$\int \cos x\,dx=\sin x+c$$

See Exercise 19B for establishing this result.

We do not have a function whose derivative is $\tan x$ and so have no way (yet) of finding $\int \tan x$. We will meet a method that enables us to establish this in chapter 19, but for completeness the result is given here:

KEY POINT 17.6a

$$\int \tan x=\ln|\sec x|$$

EXAM HINT

The integral of tan x is not given in the Formula booklet and is worth remembering

Exercise 17E

1. Find the following integrals:

   (a) (i) $\int \sin x - \cos x \, dx$ (ii) $\int 3\cos x + 4\sin x \, dx$

   (b) (i) $\int 1 + \tan x \, dx$ (ii) $\int \frac{\sin x}{2} + \frac{\tan x}{3} \, dx$

   (c) (i) $\int \frac{x + \sin x}{7} dx$ (ii) $\int \frac{\sqrt{x} + \cos x}{6} dx$

   (d) (i) $\int 1 - (\cos x + \sin x) dx$ (ii) $\int \cos x - 2(\cos x - \sin x) dx$

2. Find $\int \frac{\sin x + \cos x}{2\cos x} dx$. *[5 marks]*

3. Find $\int \frac{\cos 2x}{\cos x - \sin x} dx$. *[5 marks]*

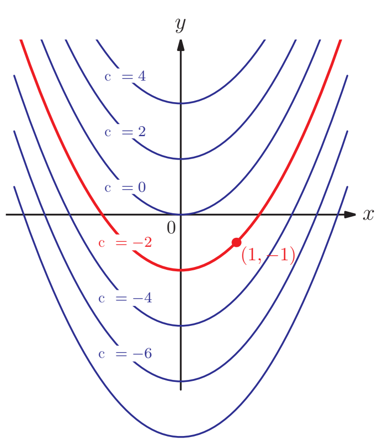
17F Finding the equation of a curve

We have seen how we can integrate the function $\frac{dy}{dx}$ to find the equation of the original curve, except for the unknown constant of integration. This is because the gradient, $\frac{dy}{dx}$, determines the shape of the curve, but not exactly where it is. However, if we are also given the coordinates of a point on the curve we can then determine the constant and hence specify the original function precisely.

Look back to Worked example 16.2 where, given the gradient, we could draw many different curves by changing the starting point.

If we again consider $\frac{dy}{dx} = 2x$ which we met at the start of this chapter, we know that the original function must have equation $y = x^2 + c$ for some constant value c.

If we are also told that the curve passes through the point $(1, -1)$, we can find c and specify which of the family of curves our function must be.



Worked example 17.3

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$ and the curve passes through the point (1, –4). Find the equation of the curve.

To find y from $\frac{dy}{dx}$ we need to integrate

Don't forget + c

$y = \int 3x^2 - 8x + 5dx$

$= x^3 - 4x^2 + 5x + c$

The coordinates of the given point must satisfy this equation, so we can find c.

When $x = 1$, $y = -4$, so

$-4 = (1)^3 - 4(1)^2 + 5(1) + c$

$\Rightarrow -4 = 1 - 4 + 5 + c \Rightarrow c = -6$

$\therefore y = x^3 - 4x^2 + 5x - 6$

The above example illustrates the general procedure for finding the equation of a curve from its gradient function.

KEY POINT 17.7

To find the equation for y given the gradient $\frac{dy}{dx}$ and one point (p, q) on the curve:

1. Integrate $\frac{dy}{dx}$, remembering $+c$.
2. Find the constant of integration by substituting $x = p, y = q$.

Exercise 17F

1. Find the equation of the original curve if:

(a) (i) $\frac{dy}{dx} = x$ and the curve passes through (–2, 7)

(ii) $\frac{dy}{dx} = 6x^2$ and the curve passes through (0, 5)

(b) (i) $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ and the curve passes through (4, 8)

(ii) $\frac{dy}{dx} = \frac{1}{x^2}$ and the curve passes through (1, 3)

(c) (i) $\frac{dy}{dx} = 2e^x + 2$ and the curve passes through (1, 1)

(ii) $\frac{dy}{dx} = e^x$ and the curve passes through (ln5, 0)

(d) (i) $\frac{dy}{dx} = \frac{x+1}{x}$ and the curve passes through (e, e)

(ii) $\frac{dy}{dx} = \frac{1}{2x}$ and the curve passes through (e^2, 5)

(e) (i) $\frac{dy}{dx} = \cos x + \sin x$ and the curve passes through (π, 1)

(ii) $\frac{dy}{dx} = 3\tan x$ and the curve passes through (0, 4)

2. The derivative of the curve $y = f(x)$ is $\frac{1}{2x}$.

(a) Find an expression for all possible functions *f(x)*.

(b) If the curve passes through the point (2, 7), find the equation of the curve. *[5 marks]*

3. The gradient of a curve is found to be $\frac{dy}{dx} = x^2 - 4$.

(a) Find the *x*-coordinate of the maximum point, justifying that it is a maximum.

(b) Given that the curve passes through the point (0, 2), show that the *y*-coordinate of the maximum point is $-7\frac{1}{3}$. *[5 marks]*

4. The gradient of the normal to a curve at any point is equal to the *x*-coordinate at that point. If the curve passes through the point (e^2, 3) find the equation of the curve in the form $y = \ln(g(x))$ where $g(x)$ is a rational function, $x > 0$. *[6 marks]*

17G Definite integration

Until now we have been carrying out a process known as **indefinite integration**: indefinite in the sense that we have an unknown constant each time, for example $\int x^2 \, dx = \frac{1}{3}x^3 + c$.

However, there is also a process called **definite integration** which yields a numerical answer without the involvement of the constant of integration, for example

$$\int_a^b x^2 dx = \left[\frac{1}{3}x^3\right]_a^b = \left(\frac{1}{3}b^3\right) - \left(\frac{1}{3}a^3\right).$$

Here *a* and *b* are known as the **limits** of integration: *a* is the lower limit and *b* the upper limit.

EXAM HINT

Make sure you know how to evaluate definite integrals on your calculator, as explained on Calculator skills sheet 10 on the CD-ROM. It can save you time, and you can evaluate integrals you don't know how to do algebraically. Even when you are asked to find the exact value of the integral, you can check your answer on the calculator.

The square bracket notation means that the integration has taken place but the limits have not yet been applied. To do this we simply evaluate the integrated expression at the upper limit and subtract the integrated expression evaluated at the lower limit.

You may be wondering where the constant of integration has gone. We could write it in as before but we quickly realise that this is unnecessary as it will just cancel out at the upper and lower limit each time:

$$\int_a^b x^2 \mathrm{d}x = \left[\frac{1}{3}x^3 + c\right]_a^b$$
$$= \left(\frac{1}{3}b^3 + c\right) - \left(\frac{1}{3}a^3 + c\right)$$
$$= \frac{1}{3}b^3 - \frac{1}{3}a^3$$

The value of x is a dummy variable, it does not come into the answer. But both a and b can vary and affect the result. Changing x to a different variable does not change the answer. For example:

$$\int_a^b u^2 \,\mathrm{d}u = \frac{1}{3}b^3 - \frac{1}{3}a^3 = \int_a^b x^2 \,\mathrm{d}x$$

Worked example 17.4

Find the exact value of $\int_1^e \frac{1}{x} + 4\,\mathrm{d}x$.

Integrate and write in square brackets

$$\int_1^e \frac{1}{x} + 4\mathrm{d}x = [\ln x + 4x]_1^e$$

Evaluate the integrated expression at the upper and lower limits and subtract the lower from the upper

$$= (\ln(e) + 4(e)) - (\ln(1) + 4(1))$$
$$= (1 + 4e) - (0 + 4) = 4e - 3$$

Exercise 17G

1. Evaluate the following definite integrals, giving exact answers.

   (a) (i) $\int_{2}^{6} x^3 \, dx$ (ii) $\int_{1}^{4} x^2 + x \, dx$

   (b) (i) $\int_{0}^{\pi/2} \cos x \, dx$ (ii) $\int_{\pi}^{2\pi} \sin x \, dx$

   (c) (i) $\int_{0}^{1} e^x \, dx$ (ii) $\int_{-1}^{1} 3e^x \, dx$

2. Evaluate correct to three significant figures:

   (a) (i) $\int_{0.3}^{1.4} \sqrt{x} \, dx$ (ii) $\int_{9}^{9.1} \frac{3}{\sqrt{x}} \, dx$

   (b) (i) $\int_{0}^{1} e^{x^2} \, dx$ (ii) $\int_{1}^{e} \ln x \, dx$

3. Find the exact value of the integral $\int_{0}^{\pi} e^x + \sin x + 1 \, dx$ *[5 marks]*

4. Show that the value of the integral $\int_{k}^{2k} \frac{1}{x} \, dx$ is independent of k. *[4 marks]*

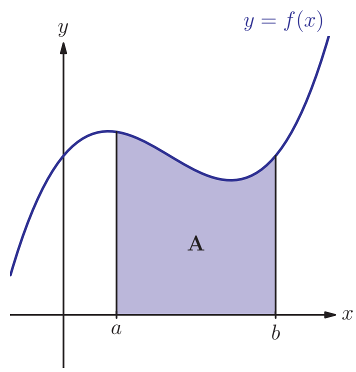
5. If $\int_{3}^{9} f(x) \, dx = 7$, evaluate $\int_{3}^{9} 2f(x) + 1 \, dx$. *[4 marks]*

6. Solve the equation $\int_{1}^{a} \sqrt{t} \, dt = 42$. *[5 marks]*

17H Geometrical significance of definite integration

Now we have a method that gives a numerical value for an integral, the natural question to ask is: what does this number mean?

On Fill-in proof sheet 20 on the CD-ROM, The fundamental theorem of calculus, we show that, as long as $f(x)$ is positive, the definite integral of $f(x)$ between the limits a and b is the area enclosed between the curve, the x-axis and the lines $x = a$ and $x = b$.



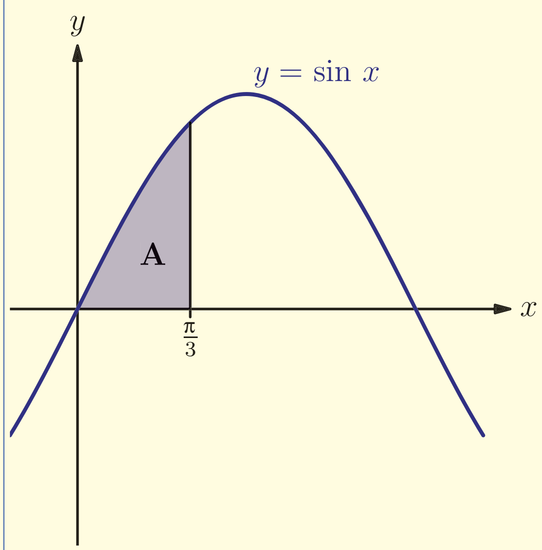
KEY POINT 17.8

$$\text{Area} = \int_{a}^{b} f(x) \, dx$$

Worked example 17.5

Find the exact area enclosed between the x-axis, the curve $y = \sin x$ and the lines $x = 0$ and $x = \frac{\pi}{3}$.

Sketch the graph and identify the area required



Integrate and write in square brackets

$$A = \int_0^{\pi/3} \sin x \, dx = [-\cos x]_0^{\pi/3}$$

Evaluate the integrated expression at the upper and lower limit and subtract the lower from the upper

$$= \left(-\cos\frac{\pi}{3}\right) - (-\cos 0)$$

$$= \left(-\frac{1}{2}\right) - (-1) \quad = \frac{1}{2}$$

If you are sketching the graph on the calculator you can get it to shade and evaluate the required area: see Calculator skills sheet 10 on the CD-ROM. You need to show the sketch as a part of your working if it is not already shown in the question.

In the 17th Century, integration was defined as the area under a curve. The area was broken down into small rectangles, each with a height $f(x)$ and a width of a small bit of x, called Δx. The total area was approximately the sum of all of these rectangles:

$$\sum_{x=a}^{x=b} f(x)\Delta x$$

Isaac Newton, one of the pioneers of calculus, was also a big fan of writing in English rather than Greek. So sigma became the English letter 'S' and delta became the English letter d so that when the limit is taken as the width of the rectangles become vanishingly small then the expression becomes:

$$\int_a^b f(x)\,dx$$

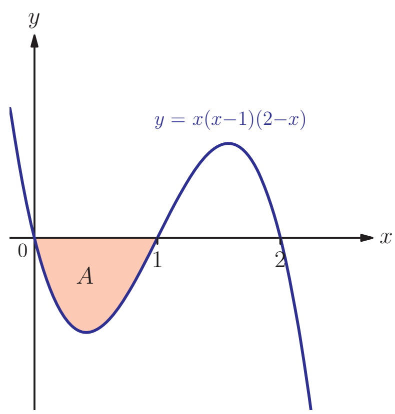
This illustrates another very important interpretation of integration – the infinite sum of infinitesimally small parts.

The Ancient Greeks had developed ideas of limiting processes similar to those used in calculus but it took nearly 2000 years for these ideas to be formalised. This was done almost simultaneously by Isaac Newton and Gottried Leibniz in the 17th Century. Is this a coincidence or is it often the case that a long period of slow progress is often necessary to get to the stage of major breakthroughs? Supplementary sheet 10 looks at some other people who can claim to have invented calculus.

When the curve is entirely below the x-axis the integral will give a negative value. The modulus of this value is the area.

Worked example 17.6

Find the area A in this graph.



Write down the integral to be evaluated, then use calculator

$$\int_0^1 x(x-1)(2-x)\,dx = -0.25 \text{ (by GDC)}$$

The area must be positive

$$\therefore A = 0.25$$

Unfortunately, the relationship between integrals and areas is not so simple when there are parts of the curve above and below the axis. Those bits above the axis contribute positively to the area, but bits below the axis contribute negatively to the area. We must separate out the sections above the axis and below the axis.

Worked example 17.7

(a) Find $\int_1^4 x^2 - 4x + 3\,dx$

(b) Find the area enclosed between the x-axis, the curve $y = x^2 - 4x + 3$ and the lines $x = 1$ and $x = 4$.

Apply standard integration

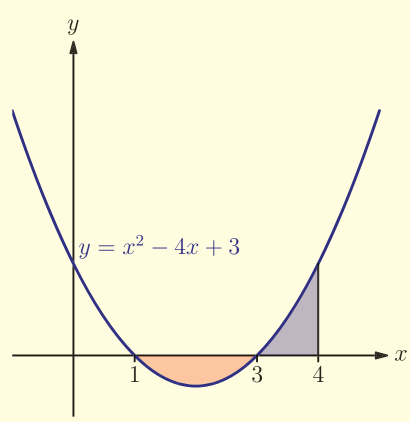
(a) $\int_1^4 x^2 - 4x + 3dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^4$

$= \left(\frac{1}{3}(4)^3 - 2(4)^2 + 3(4)\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1)\right)$

$= \left(\frac{4}{3}\right) - \left(\frac{4}{3}\right) = 0$

The value found above can't be the correct area for (b). Sketch the curve to see exactly which area we are being asked to find

(b)



continued . . .

The area is made up of two parts, so evaluate each of them separately

$$\int_1^3 x^2 - 4x + 3 dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$

$$= (0) - \left(\frac{4}{3}\right) = -\frac{4}{3}$$

$\therefore$ Area below the axis is $\frac{4}{3}$

$$\int_3^4 x^2 - 4x + 3 dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{4}{3}\right) - (0) = \frac{4}{3}$$

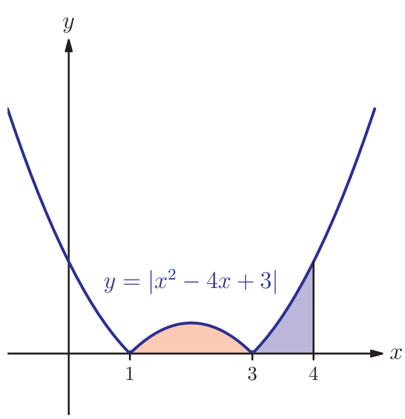
$\therefore$ Area above the axis is $\frac{4}{3}$

Total area $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

The fact that the integral was zero in Worked example 17.7 part (a) means that the area above the axis is exactly cancelled by the area below the axis.

This example warns us that when asked to find an area we must always sketch the graph and identify exactly where each part of the area is. If we are evaluating the area on the calculator we can use the modulus function to ensure that the entire graph is above the x-axis. Using the function from Worked example 17.7:

$$\int_1^4 |x^2 - 4x + 3| \, dx = \frac{8}{3}$$



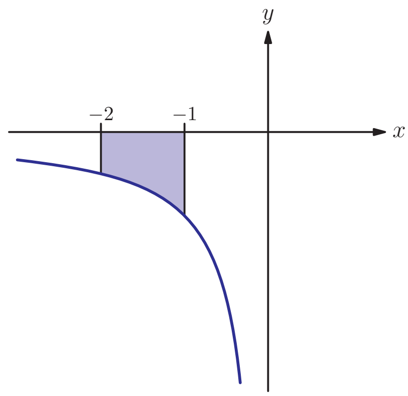
Transformations of graphs using the modulus function were covered in chapter 7.

KEY POINT 17.9

The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b |f(x)|\,dx$.
When working without a calculator, if the curve crosses the x-axis between a and b we need to split the area into several parts and find each one separately.

The interpretation of integrals as areas causes one inconsistency with our previous work. Consider the integral $\int_{-2}^{-1} \frac{1}{x}\,dx$.

Graphically we can see that this area should exist.



However, if we do the integration we find that:

$$\int_{-2}^{-1} \frac{1}{x}\,dx = [\ln x]_{-2}^{-1}$$

$$= \ln(-1) - \ln(-2)$$

$$= \ln\left(\frac{-1}{-2}\right)$$

$$= \ln\left(\frac{1}{2}\right)$$

$$= -\ln 2$$

This is the correct answer (which we could have found using the symmetry of the curve) but it goes through a stage where we had to take logarithms of negative numbers, and this is something we are not allowed to do. We avoid this by redefining the integral of $\frac{1}{x}$ as:

KEY POINT 17.4 AGAIN

$$\int x^{-1}\,dx = \ln|x| + c$$

> You may rightly be a little uncomfortable with inserting a modulus function 'just because it works'. In mathematics, do the ends justify the means?

With this definition we can integrate $y = \frac{1}{x}$ over negative numbers, and the integral above becomes

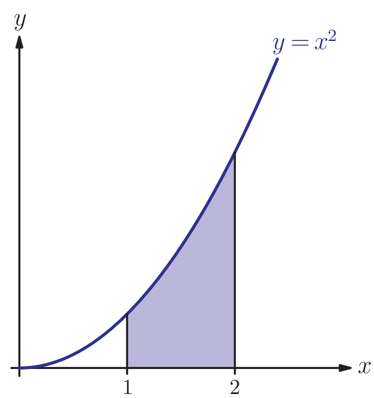
$$\begin{aligned}\int_{-2}^{-1}\frac{1}{x}\,dx &= \left[\ln|x|\right]_{-2}^{-1}\\ &= \ln 1 - \ln 2\\ &= -\ln 2 \text{ as before}\end{aligned}$$

Notice that the answer is negative, since the required area is below the x-axis. We can still not integrate $\frac{1}{x}$ with a negative lower and positive upper limit, since the graph has an asymptote at $x = 0$.

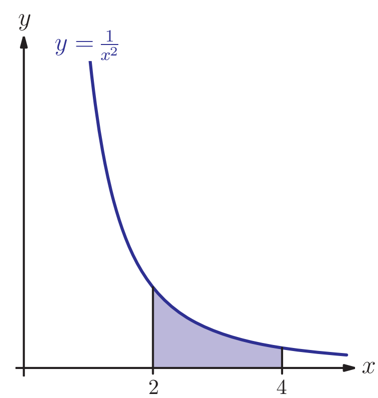
Exercise 17H

1. Find the shaded areas:

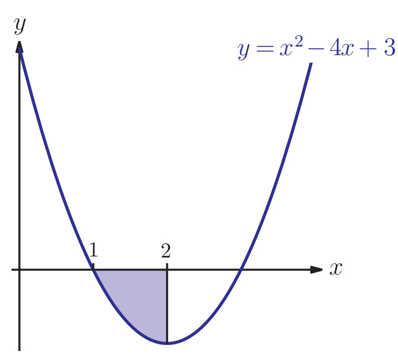
(a) (i)



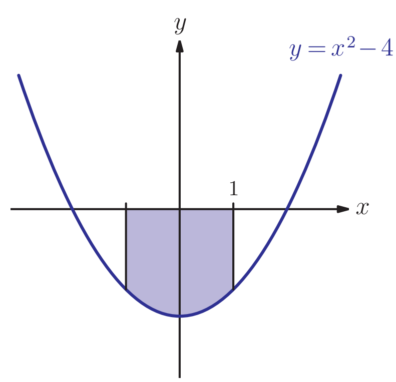
(ii)



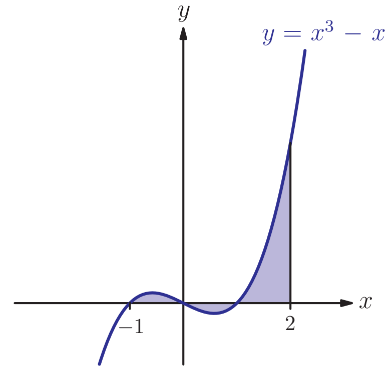
(b) (i)



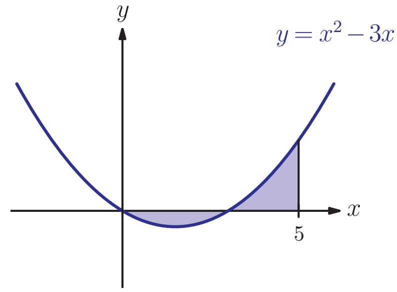
(ii)



(c) (i)



(ii)



EXAM HINT

'Find the area enclosed' means first find a closed region bounded by the curves mentioned, then find its area. A sketch is a very useful tool.

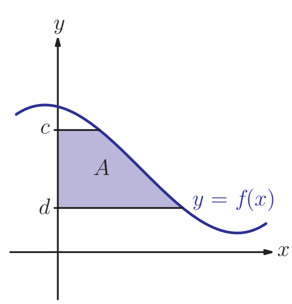
2. The area enclosed by the x-axis, the curve $y = \sqrt{x}$ and the line $x = k$ is 18. Find the value of k. *[6 marks]*

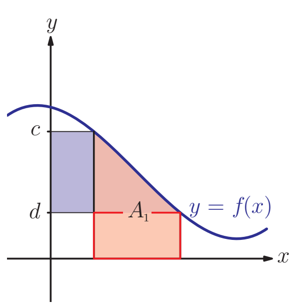
3. (a) Find $\int_0^3 x^2 - 1 \, dx$

 (b) Find the area between the curve $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 3$. *[5 marks]*

4. Between $x = 0$ and $x = 3$, the area of the graph $y = x^2 - kx$ below the x-axis equals the area above the x-axis. Find the value of k. *[6 marks]*

5. Find the area enclosed by the curve $y = 7x - x^2 - 10$ and the x-axis. *[7 marks]*



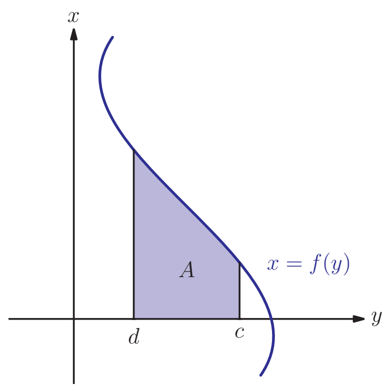


17I The area between a curve and the y-axis

Consider the diagram alongside. How can we find the shaded area A?

One possible strategy is to construct a box around the graph to divide up the regions of interest. You can integrate to find the area labelled A_1 and then, by adding and subtracting the areas of the blue and red rectangles shown, calculate A.

Happily, there is a quicker way: we can treat x as a function of y, effectively reflecting the whole diagram in the line $y = x$, and then use the same method as in the previous section.



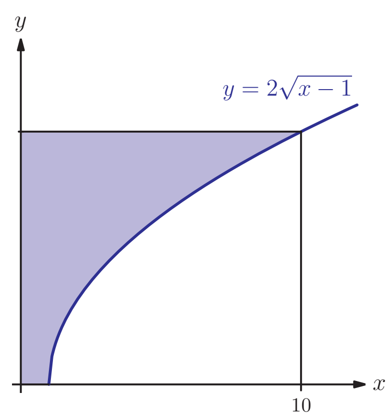
KEY POINT 17.10

The area bounded by the curve $y = f(x)$, the y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d g(y)\,dy$, where $g(y)$ is the expression for x in terms of y.

You may have realised that this is related to inverse functions from Section 5E.

Worked example 17.8

The curve shown has equation $y = 2\sqrt{x-1}$. Find the shaded area.



Express x in terms of y

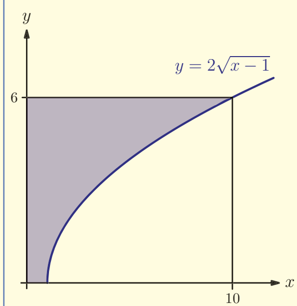
$$x - 1 = \left(\frac{y}{2}\right)^2$$

$$\Rightarrow x = \frac{y^2}{4} + 1$$

continued . . .

Find the limits on the y-axis
It may help to label them on the graph

When $x = 1,\ y = 2\sqrt{1-1} = 0$
When $x = 10,\ y = 2\sqrt{10-1} = 6$



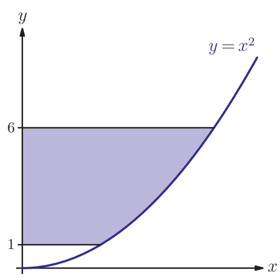
Write down the integral and evaluate using calculator

$$\text{Area} = \int_0^6 \left(\frac{y^2}{4} + 1\right) dy = 24 \text{ (from GDC)}$$

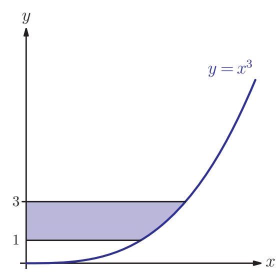
Exercise 17I

1. Find the shaded areas:

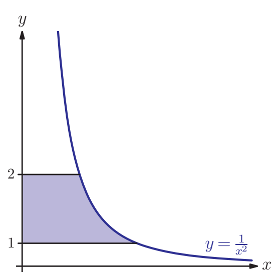
(a) (i)



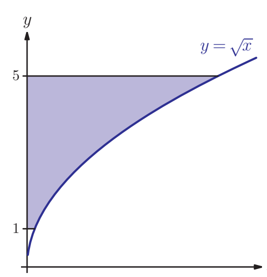
(ii)



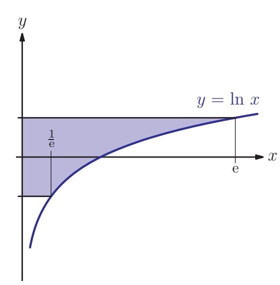
(b) (i)



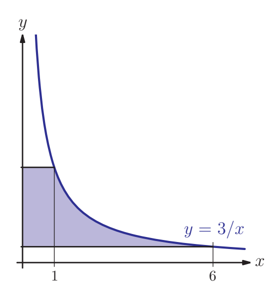
(ii)



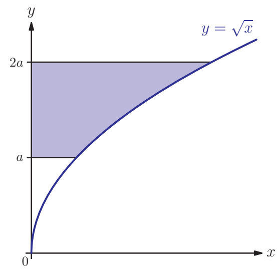
(c) (i)



(ii)

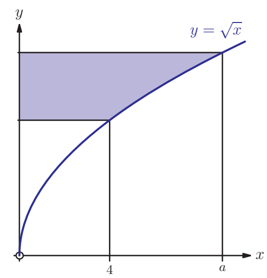


2. The diagram shows the curve $y=\sqrt{x}$. If the shaded area is 504 find the value of a. *[6 marks]*

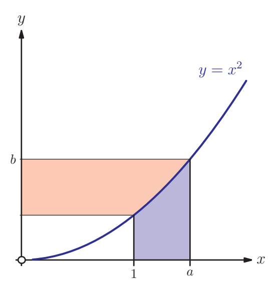


3. Find the exact value of the area enclosed by the graph of $y=\ln(x+1)$, the line $y = 2$ and the y-axis. *[6 marks]*

4. The diagram shows the graph of $y=\sqrt{x}$.
The shaded area is 39 units. Find the value of a. *[7 marks]*



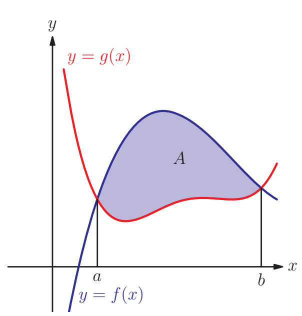
5. The diagram shows the graph of $y=x^2$, where $a\in]1,\infty[$.
The area of the pink region is equal to the area of the blue region. Give two equations for a in terms of b, and hence give a in exact form and determine the size of the blue area. *[8 marks]*



17J The area between two curves

So far we have only looked at areas bounded by a curve and one of the coordinate axes, but we can also find areas bounded by two curves.
The area A in the diagram can be found by taking the area bounded by $f(x)$ and the x-axis and subtracting the area bounded by $g(x)$ and the x-axis, that is:

$$A=\int_a^b f(x)\,\mathrm{d}x-\int_a^b g(x)\,\mathrm{d}x$$



We can do the subtraction before integrating so that we only have to integrate one expression instead of two. This gives an alternative formula for the area.

KEY POINT 17.11

The area A between two curves, $f(x)$ and $g(x)$, is:

$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

where a and b are the x-coordinates of the intersection points of the two curves.

Worked example 17.9

Find the area A enclosed between $y = 2x + 1$ and $y = x^2 - 3x + 5$.

First find the x-coordinates of intersection

For intersection:

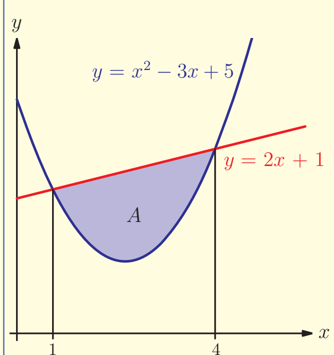
$x^2 - 3x + 5 = 2x + 1$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0$

$\Rightarrow x = 1, 4$

Make a rough sketch to see the relative positions of the two curves



Subtract the lower curve from the higher before integrating

$A = \int_1^4 (2x+1) - (x^2 - 3x + 5) dx$

$= \int_1^4 -x^2 + 5x - 4 \, dx$

$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4$

$= \frac{8}{3} \left(-\frac{11}{6} \right) = \frac{9}{2}$

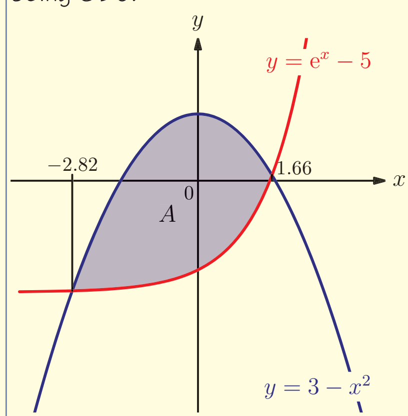
Subtracting the two equations before integrating is particularly useful when one of the curves is partly below the x-axis. If $f(x)$ is always above $g(x)$ then the expression we are integrating, $f(x)-g(x)$, is always positive, so we do not have to worry about the signs of $f(x)$ and $g(x)$ themselves.

Worked example 17.10

Find the area bounded by the curves $y = e^x - 5$ and $y = 3 - x^2$.

Sketch the graph to see the relative position of two curves

Using GDC:



Find the intersection points – use calculator

intersections: $x = -2.818$ and 1.658

Write down the integral representing the area

$$\text{Area} = \int_{-2.818}^{1.658} (3 - x^2) - (e^x - 5)\,dx$$

$$= \int_{-2.818}^{1.658} (8 - x^2 - e^x)\,dx$$

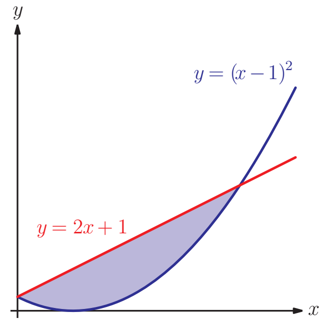
Evaluate the integral using calculator

$$= 21.6 \quad (3SF)$$

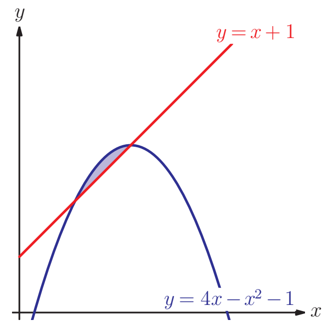
Exercise 17J

1. Find the shaded areas.

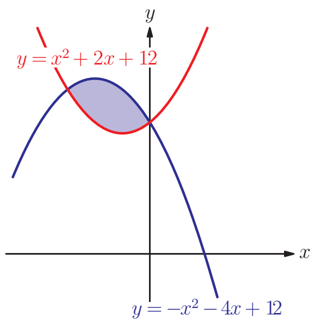
(a) (i)



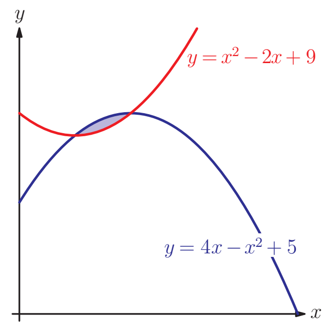
(ii)



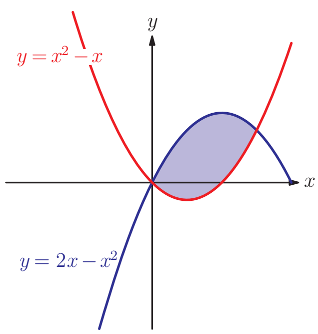
(b) (i)



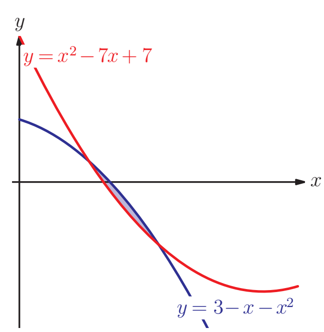
(ii)



(c) (i)



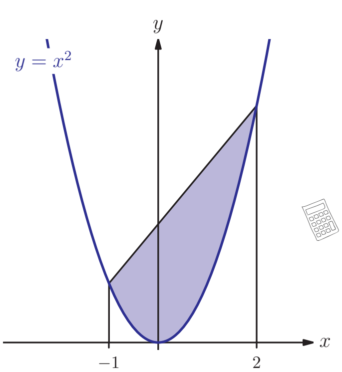
(ii)



2. Find the area enclosed between the graphs of $y = x^2 + x - 2$ and $y = x + 2$. *[6 marks]*

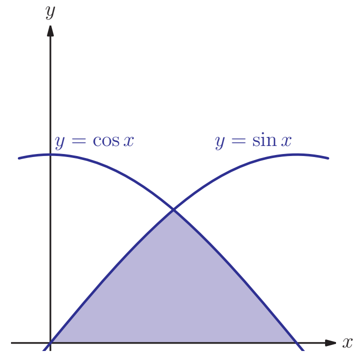
3. Find the area enclosed by the curve $y = e^x$, $y = x^2$, the y-axis and the line $x = 2$. *[6 marks]*

4. Find the area between the curves $y = \frac{1}{x}$ and $y = \sin x$ in the region $0 < x < \pi$. *[6 marks]*



5. Show that the area of the shaded region alongside is $\frac{9}{2}$. *[6 marks]*

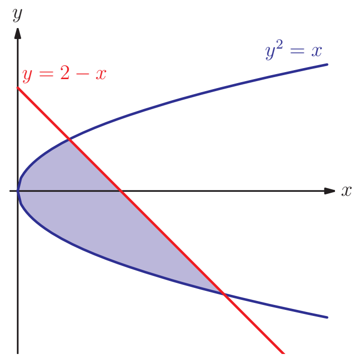
6. The diagram alongside shows the graphs of $y = \sin x$ and $y = \cos x$. Find the shaded area. *[6 marks]*



7. Find the total area enclosed between the graphs of $y = x(x-4)^2$ and $y = x^2 - 7x + 15$. *[6 marks]*

8. The area enclosed between the curve $y = x^2$ and the line $y = mx$ is $10\frac{2}{3}$. Find the value of m if $m > 0$. *[7 marks]*

9. Show that the shaded area in the diagram below is $\frac{9}{2}$. *[8 marks]*



Summary

- **Integration** is the reverse process of differentiation.
- Any integral without limits (**indefinite**) will generate a **constant of integration**.
- For all rational $n \neq -1$ $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$.
- If $n = -1$, we get the natural logarithm function: $\int x^{-1} \, dx = \ln x + c$.
- The integral of the exponential function is: $\int e^x \, dx = e^x + c$.
- The integrals of the trigonometric functions are:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x \, dx = \ln|\sec x| + c$$

- The **definite integral** has limits. $\int_a^b f(x) \, dx$ is found by evaluating the integrated expression at b and then subtracting the integrated expression evaluated at a.

- The area between the curve $y = f(x)$, the x-axis and lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b f(x)\,dx$$

If the curve goes below the x-axis, the value of this integral will be negative.

- On the calculator, we can use the modulus function to ensure we are always integrating a positive function.
- The area between the curve, the y-axis and lines $y = c$ and $y = d$ is given by: $A_1 = \int_c^d g(y)\,dy$.
- The area between two curves is given by:

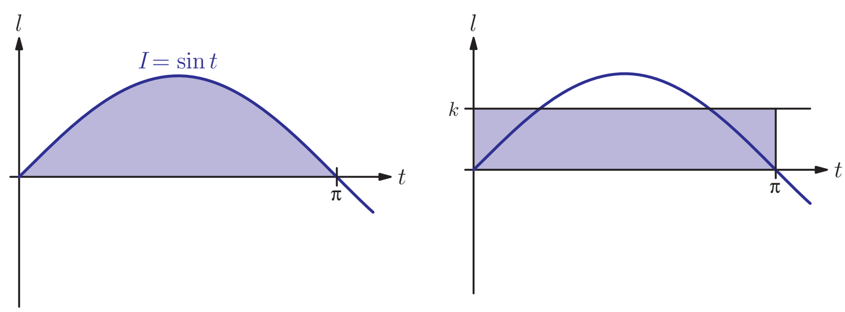
$$A = \int_a^b \left|f(x) - g(x)\right|\,dx$$

where $x = a$ and $x = b$ are the intersection points.

Introductory problem revisited

The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). When there is alternating current the relationship between I and t is given by $I = \sin t$. When it contains direct current the relationship between I and t is given by $I = k$. What value of k means that the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ is the same whether alternating or direct current is used?

The area under the curve of I against t is given by $\int_0^\pi \sin t\,dt = \left[-\cos t\right]_0^\pi = 2$. For a rectangle of width π to have the same area the height must be $\frac{2}{\pi}$.



You can look at integration as a quite sophisticated way of finding an average value of a function.

Mixed examination practice 17

Short questions

1. If $f'(x)=\sin x$ and $f\left(\frac{\pi}{3}\right)=0$, find $f(x)$. *[4 marks]*

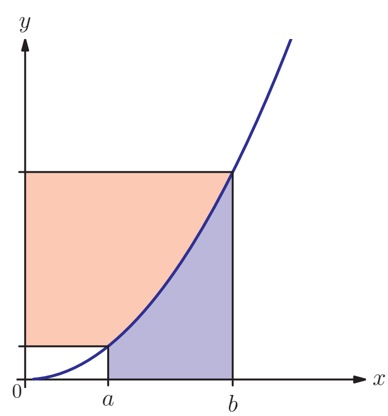
2. Calculate the area enclosed by the curves $y=\ln x$ and $y=e^x-e, x>0$.

[6 marks]

[© IB Organization 2003]

3. Find the area enclosed between the graph of $y=k^2-x^2$ and the x-axis, giving your answer in terms of k. *[6 marks]*

4. The diagram shows the graph of $y=x^n$ for $n>1$.



The red area is three times larger than the blue area. Find the value of n. *[6 marks]*

5. Find the indefinite integral:

$$\int \frac{1+x^2\sqrt{x}}{x}\,dx$$

[5 marks]

6. (a) Solve the equation:

$$\int_0^a x^3-x\,dx=0,\ a>0.$$

(b) For this value of a, find the total area enclosed between the x-axis and the curve $y=x^3-x$ for $0\le x\le a$. *[6 marks]*

7. Find the area enclosed between the graphs of $y = \sin x$ and $y = 1 - \sin x$ for $0 < x < \pi$. *[3 marks]*

8. (a) The function $f(x)$ has a stationary point at $(3,19)$ and $f''(x) = 6x + 6$.

What kind of stationary point is $(3,19)$? *[5 marks]*

(b) Find $f(x)$.

Long questions

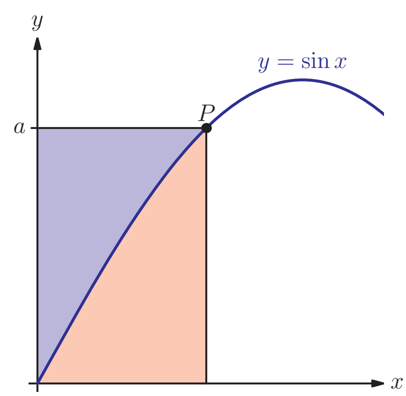
1. (a) Find the coordinates of the points of intersection of the graphs $y = 5a^2 + 4ax - x^2$ and $y = x^2 - a^2$.

(b) Find the area enclosed between these two graphs.

(c) Show that the fraction of this area above the axis is independent of a and state the value that this fraction takes. *[10 marks]*

2. (a) Use the identity $\cos^2 x + \sin^2 x = 1$ to show that $\cos(\arcsin x) = \sqrt{1 - x^2}$.

(b) The diagram below shows part of the curve $y = \sin x$.



Write down the x-coordinate of the point P in terms of a.

(c) Find the red shaded area in terms of a, writing your answer in a form without trigonometric functions.

(d) By considering the blue shaded area find $\int_0^a \arcsin x \, dx$ for $0 < a < 1$.

[12 marks]

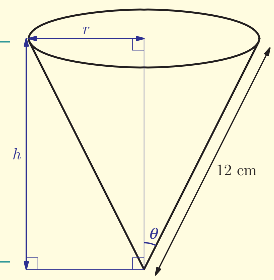
18 Further differentiation methods

In this chapter you will learn:

- how to differentiate composite functions
- how to differentiate reciprocal trigonometric functions: $\sec x, \csc x$ and $\cot x$
- how to differentiate products of functions
- how to differentiate quotients of functions
- how to differentiate functions that are not in the form $y = f(x)$
- how to differentiate exponential functions
- how to differentiate inverse trigonometric functions: $\arcsin x, \arccos x$ and $\arctan x$.

Introductory problem

Given a cone of fixed slant height 12 cm, find the maximum volume as apex angle θ varies.



In this chapter we will build on the techniques covered in chapter 16 so that we can differentiate a wider range of functions. Much of the work here will also be used in chapter 19 when we learn more integration techniques.

18A Differentiating composite functions using the chain rule

We can already differentiate functions such as $y = (3x^2 + 5x)^2$ by expanding the brackets and differentiating term by term:

$$y = (3x^2)^2 + 2(3x^2)(5x) + (5x)^2 = 9x^4 + 30x^3 + 25x^2$$

$$\therefore \frac{dy}{dx} = 36x^3 + 90x^2 + 50x = 2x(18x^2 + 45x + 25)$$

But what if the function is more complicated?

The same method would work, but it is clearly not practical to expand, for example, $y = (3x^2 + 5x + 2)^7$ and then differentiate each term. And what about functions such as $y = \sin 3x$ or $y = e^{x^2}$? While we can already differentiate $y = \sin x$ and $y = e^x$, we have no rules so far to tell us what to do when the argument is changed to $3x$ or x^2.

The functions $y=(3x^2+5x+2)^7$, $y=\sin 3x$ and $y=e^{x^2}$ may not seem related but do have something in common; they are all composite functions:

- $y=(3x^2+5x+2)^7$ is $y=u^7$ where $u(x)=3x^2+5x+2$
- $y=\sin 3x$ is $y=\sin u$ where $u(x)=3x$
- $y=e^{x^2}$ is $y=e^u$ where $u(x)=x^2$

There is a general rule for differentiating any composite function.

KEY POINT 18.1

The chain rule

If $y=g(u)$ where $u=f(x)$:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

We will accept the chain rule without proof, as it is very technical and requires differentiation from first principles. Let us apply the chain rule to the three functions above.

Worked example 18.1

Differentiate these functions:

(a) $y=(3x^2+5x+2)^7$ (b) $y=\sin(3x)$ (c) $y=e^{x^2}$

These are all composite functions so use chain rule

(a) $y=u^7$ where $u=3x^2+5x+2$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=7u^6\times(6x+5)$$

Write the answer in terms of x

$$=7(3x^2+5x+2)^6(6x+5)$$

(b) $y=\sin u$ where $u=3x$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\cos u\times(3)$$

Write the answer in terms of x

$$=3\cos(3x)$$

(c) $y=e^u$ where $u=x^2$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=e^u\times(2x)$$

Write the answer in terms of x in the conventional form

$$=2xe^{x^2}$$

Worked example 18.1(b) illustrates a special case of the chain rule when the 'inside' function is of the form $ax+b$.

KEY POINT 18.2

$$\frac{d}{dx} f(ax+b) = af'(ax+b)$$

For example,

$$\frac{d}{dx}(4x+1)^7 = 4\times 7(4x+1)^6 \text{ and } \frac{d}{dx}(e^{3-2x}) = -2e^{3-2x}$$

It is useful to remember this shortcut. In practice it is not necessary to keep specifying the function $u(x)$ each time and the chain rule calculation can be written down more directly as can be seen in the example below, i.e. imagine brackets around the inner function u and differentiate the outer function first, as if the bracketed expression were a single argument, and then multiply by the derivative of the bracketed expression.

Worked example 18.2

Differentiate these composite functions:

(a) $y = e^{x^2-3x}$ (b) $y = \frac{3}{\sqrt{x^3-5}}$

$e^{()}$ differentiates to $e^{()}$ and x^2-3x differentiates to $2x-3$

(a) $\frac{dy}{dx} = (2x-3)e^{(x^2-3x)}$

First rewrite the square root as a power

(b) $y = 3(x^3-5)^{-\frac{1}{2}}$

$3()^{-\frac{1}{2}}$ differentiates to $-\frac{3}{2}()^{-\frac{3}{2}}$ and x^3-5 differentiates to $3x^2$

$\frac{dy}{dx} = -\frac{3}{2}(x^3-5)^{-\frac{3}{2}}(3x^2) = -\frac{9x^2}{2}(x^3-5)^{-\frac{3}{2}}$

Sometimes it is necessary to apply the chain rule more than once.

Worked example 18.3

Differentiate $y = \cos^3(\ln 2x)$.

Remember that $\cos^3 A$ means $(\cos A)^3$

$y = (\cos(\ln 2x))^3$

This is a composite of three functions, so use chain rule

$(\ )^3$ differentiates to $3(\ )^2$

$\cos(\ )$ differentiates to $-\sin(\ )$

$\ln 2x$ differentiates to $2 \times \frac{1}{2x} = \frac{1}{x}$

$$\frac{dy}{dx} = 3(\cos(\ln 2x))^2 \times (-\sin(\ln 2x)) \times \frac{1}{x}$$

$$= -\frac{3}{x}\cos^2(\ln 2x)\sin(\ln 2x)$$

Now we can use the chain rule, we can add the derivatives of $y = \sec x$, $y = \csc x$ and $y = \cot x$ (see Key point 18.3 on the next page) to those of $y = \sin x$, $y = \cos x$ and $y = \tan x$ already established in chapter 16.

Worked example 18.4

Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$

Express $\sec x$ in terms of $\cos x$

$y = \sec x = (\cos x)^{-1}$

This is a composite function, so apply chain rule

$(\ )^{-1}$ differentiates to $-(\ )^{-2}$

$\cos(\ )$ differentiates to $-\sin(\ )$

$$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

We want the answer to contain $\tan x$, which is $\frac{\sin x}{\cos x}$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$

$= \sec x \tan x$ as required

The proofs for the other two reciprocal trigonometric functions follow the same pattern, giving the following results.

KEY POINT 18.3

$$y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$$
$$y = \csc x \Rightarrow \frac{dy}{dx} = -\csc x \cot x$$
$$y = \cot x \Rightarrow \frac{dy}{dx} = -\csc^2 x$$

Exercise 18A

1. Differentiate the following using the chain rule:

(a) (i) $(x^2 - 3x + 1)^7$ (ii) $(x^3 + 1)^5$

(b) (i) e^{x^2-2x} (ii) e^{4-x^3}

(c) (i) $(2e^x + 1)^{-3}$ (ii) $(2 - 5e^x)^{-4}$

(d) (i) $\sin(3x^2 + 1)$ (ii) $\cos(x^2 + 2x)$

(e) (i) $\cos^3 x$ (ii) $\sin^4 x$

(f) (i) $\ln(2x - 5x^3)$ (ii) $\ln(4x^2 - 1)$

(g) (i) $(4\ln x - 1)^4$ (ii) $(\ln x + 3)^{-5}$

2. Differentiate the following using the short cut from Key point 18.2:

(a) (i) $(2x + 3)^5$ (ii) $(4x - 1)^8$

(b) (i) $(5 - x)^{-4}$ (ii) $(1 - x)^{-7}$

(c) (i) $\cos(1 - 4x)$ (ii) $\cos(2 - x)$

(d) (i) $\ln(5x + 2)$ (ii) $\ln(x - 4)$

(e) (i) $\cot(3x)$ (ii) $\csc(5x)$

(f) (i) $\sec(2x + 1)$ (ii) $\tan(1 - x)$

3. Differentiate the following using the chain rule twice:

(a) (i) $\sec^2 3x$ (ii) $\tan^2 2x$

(b) (i) $e^{\sin^2 3x}$ (ii) $e^{(\ln 2x)^2}$

(c) (i) $(1 - 2\sin^2 2x)^2$ (ii) $(4\cos 3x + 1)^2$

(d) (i) $\ln(1 - 3\cos 2x)$ (ii) $\ln(2 - \cos 5x)$

4. Find the equation of the normal to the curve $y = \frac{1}{\sqrt{4x^2 + 1}}$ at the point where $x = \sqrt{2}$.

5. Find the exact coordinates of stationary points on the curve $y = e^{\sin x}$ for $x \in [0, 2\pi]$. *[5 marks]*

6. Given that $f(x) = \csc^2 x$:

(a) Find $f'(x)$.

(b) Solve the equation $f'(x) = 2f(x)$ for $-\pi < x < \pi$. *[7 marks]*

7. For what values of x does the function $f : x \mapsto \ln(x^2 - 35)$ have a gradient of 1? *[5 marks]*

8. (a) If a, b, p and q are positive with $a < b$ find the x-coordinate of the stationary point of the curve $y = (x-a)^p (x-b)^q$ in the domain $a < x < b$.

(b) Sketch the graph in the case when $p = 2$ and $q = 3$.

(c) By considering the graph or otherwise, determine a condition involving p and/or q to determine when this stationary point is a maximum. *[10 marks]*

9. A non-uniform chain hangs from two posts. Its height (h) satisfies the equation

$$h = e^x + \frac{1}{e^{2x}} \text{ for } -1 \le x \le 2.$$

The left post is positioned at $x = -1$. The right post is positioned at $x = 2$.

(a) State, with reasons, which post is taller.

(b) Show that the minimum height occurs when $x = \frac{1}{3}\ln 2$.

(c) Find the exact value of the minimum height of the chain. *[8 marks]*

Many people think that a chain hangs as a parabola but it can be proved that it actually hangs in the shape of the curve in question 9, called a *catenary*. To prove this requires a topic called *differential geometry*.

10. (a) Solve the equation $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

(b) Find the coordinates of the stationary points of the curve $y = \sin 2x - \sin x$ for $0 \le x \le 2\pi$.

(c) Hence sketch the curve $y = \sin 2x - \sin x$. *[8 marks]*

18B Differentiating products using the product rule

We now look at products of two functions. We can already differentiate some products, such as $y = x^4(3x^2 - 5)$, by expanding and differentiating term by term. However, like composite functions, this is tricky when the function becomes more complicated, for example $y = x^4(3x^2-5)^9$, and expanding is no help at all with functions such as $y = x^2 \cos x$ or $y = x \ln x$.

Just as there is a rule for differentiating composite functions, there is a rule for differentiating products.

KEY POINT 18.4

The product rule

If $y = u(x)v(x)$ then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

If you are interested in the proof, see Fill-in proof 21 on the CD-ROM.

Let us apply the product rule to the first function in the previous paragraph.

Worked example 18.5

Differentiate $y = x^4(3x^2 - 5)$.

This is a product so use the product rule. It doesn't make any difference which function is $u(x)$ and which is $v(x)$

Let $u = x^4$ and $v = 3x^2 - 5$

$\frac{du}{dx} = 4x^3$, $\frac{dv}{dx} = 6x$

Apply the product rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= (3x^2 - 5)4x^3 + x^4 \times 6x$$

$$= 12x^5 - 20x^3 + 6x^5$$

$$= 18x^5 - 20x^3$$

EXAM HINT

After applying the product rule you do not need to simplify the resulting expression unless the question clearly tells you to do so.

With a more complicated function, we may need the chain rule as well as the product rule.

Worked example 18.6

Differentiate $y = x^4(3x^2-5)^5$ and factorise your answer.

This is a product so use the product rule. It doesn't make any difference which function is $u(x)$ and which is $v(x)$

Let $u = x^4$ and $v = (3x^2-5)^5$

$\frac{du}{dx} = 4x^3$

$v(x)$ is a composite function, so use chain rule

$\frac{dv}{dx} = 5(3x^2-5)^4(6x)$

$= 30x(3x^2-5)^4$

Now apply the product rule

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (3x^2-5)^5 4x^3 + (x^4) \times 30x(3x^2-5)^4$

We are asked to factorise the answer, so look for common factors

$= 2x^3(3x^2-5)^4[2(3x^2-5)+15x^2]$

$= 2x^3(3x^2-5)^4(6x^2-10+15x^2)$

$= 2x^3(3x^2-5)^4(21x^2-10)$

Exercise 18B

1. Use the product rule to differentiate the following:

(a) (i) $y = x^2 \cos x$ (ii) $y = x^{-1}\sin x$

(b) (i) $y = x^{-2}\ln x$ (ii) $y = x\ln x$

(c) (i) $y = x^3\sqrt{2x+1}$ (ii) $y = x^{-1}\sqrt{4x}$

(d) (i) $y = e^{2x}\tan x$ (ii) $y = e^{x+1}\sec 3x$

2. Find $f'(x)$ and fully factorise your answer:

(a) (i) $f(x) = (x+1)^4(x-2)^5$ (ii) $f(x) = (x-3)^7(x+5)^4$

(b) (i) $f(x) = (2x-1)^4(1-3x)^3$ (ii) $f(x) = (1-x)^5(4x+1)^2$

3. Differentiate $y = (3x^2-x+2)e^{2x}$ giving your answer in the form $P(x)e^{2x}$ where $P(x)$ is a polynomial. *[4 marks]*

4. Given that $f(x) = x^2e^{3x}$, find $f''(x)$ in the form $(ax^2+bx+c)e^{3x}$. *[4 marks]*

5. Find the x-coordinates of the stationary points on the curve $y=(2x+1)^5 e^{-2x}$. *[5 marks]*

6. Find the exact values of the x-coordinates of the stationary points on the curve $y=(3x+1)^5(3-x)^3$. *[6 marks]*

7. Given that $y=x\sin 2x$ for $x\in[0,2\pi]$:

(a) show that the x-coordinates of the points of inflexion satisfy $\cos 2x = x\sin 2x$

(b) hence find the coordinates of the points of inflexion.

[6 marks]

8. Find the derivative of $\sin(xe^x)$ with respect to x. *[5 marks]*

9. (a) If $f(x)=x\ln x$, find $f'(x)$.

(b) Hence find $\int \ln x \, dx$. *[5 marks]*

10. Find the exact coordinates of the minimum point of the curve $y=e^{-x}\cos x,\ 0\le x\le \pi$. *[6 marks]*

11. Given that $f(x)=x^2\sqrt{1+x}$, show that $f'(x)=\dfrac{x(a+bx)}{2\sqrt{1+x}}$ where a and b are constants to be found. *[6 marks]*

12. (a) Write $y=x^x$ in the form $y=e^{f(x)}$.

(b) Hence or otherwise find $\dfrac{dy}{dx}$.

(c) Find the exact coordinates of the stationary points of the curve $y=x^x$. *[8 marks]*

18C Differentiating quotients using the quotient rule

A combination of the product rule and chain rule provides us with a method for differentiating quotients such as:

$$y=\frac{x^2-4x+12}{(x-3)^2}$$

We can express it as $y=(x^2-4x+12)(x-3)^{-2}$ then using the product rule and taking

$$u=(x^2-4x+12) \text{ and } v=(x-3)^{-2}$$

$$\Rightarrow \frac{du}{dx}=2x-4 \text{ and } \frac{dv}{dx}=(-2)(x-3)^{-3}$$

we have:

$$\frac{dy}{dx} = (x-3)^{-2}(2x-4) + (x^2-4x+12)(-2)(x-3)^{-3}$$

After tidying up the negative powers and fractions, this simplifies to $\frac{dy}{dx} = \frac{-2x-12}{(x-3)^3}$.

This process is laborious, but it can be applied to a general function of the form $\frac{u(x)}{v(x)}$ to produce a new rule for differentiating quotients.

The details are given in the Fill-in proof 22 on the CD-ROM, but you only need to know how to use the result.

KEY POINT 18.5

The quotient rule

If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Worked example 18.7

Use the quotient rule to differentiate $y = \frac{x^2-4x+12}{(x-3)^2}$. Simplify your answer as far as possible.

This is a quotient. Make sure to get u and v the right way round

$y = \frac{u}{v}$, $u = x^2 - 4x + 12$, $v = (x-3)^2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Use chain rule to differentiate v then substitute the appropriate values into the quotient rule

$$= \frac{(x-3)^2(2x-4) - (x^2-4x+12)2(x-3)}{\left[(x-3)^2\right]^2}$$

Cancel a factor of $(x - 3)$

$$= \frac{(2x-4)(x-3) - (x^2-4x+12)2}{(x-3)^3}$$

$$= \frac{2x^2-10x+12-2x^2+8x-24}{(x-3)^3} = \frac{-2x-12}{(x-3)^3}$$

In Section 16E we stated the result that the derivative of tan x is $\sec^2 x$. We can now use the quotient rule, together with the derivatives of $\sin x$ and $\cos x$, to prove this result.

Worked example 18.8

Prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

We know how to differentiate $\sin x$ and $\cos x$, so use them to express $\tan x$

$\tan x = \frac{\sin x}{\cos x}$, $u = \sin x$, $v = \cos x$

Use quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

The quotient rule, like the product rule, often leads to a long expression. You do not need to simplify this expression unless asked to do so. However, sometimes product and quotient rule questions are also used to test your skill with fractions and exponents, as in the following example.

Worked example 18.9

Differentiate $\frac{x}{\sqrt{x+1}}$, giving your answer in the form $\frac{x+c}{k\sqrt{(x+1)^p}}$ where $c, k, p \in \mathbb{N}$.

This is a quotient

$y = \frac{x}{\sqrt{x+1}}$, $u = x$, $v = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

continued . . .

Use quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x+1)^{-\frac{1}{2}}}{\left((x+1)^{\frac{1}{2}}\right)^2}$$

As we want a square root in the answer, turn the fractional powers back into roots

$$= \frac{\sqrt{x+1} - \frac{x}{2\sqrt{x+1}}}{x+1}$$

Remove 'fractions within fractions' by multiplying top and bottom by $2\sqrt{x+1}$

$$= \frac{2(x+1) - x}{2(x+1)\sqrt{x+1}}$$

Notice that $x\sqrt{x} = x^{\frac{3}{2}} = \sqrt{x^3}$

$$= \frac{x+2}{2\sqrt{(x+1)^3}}$$

Exercise 18C

1. Differentiate using the quotient rule:

(a) (i) $y = \dfrac{x-1}{x+1}$ (ii) $y = \dfrac{x+2}{x-3}$

(b) (i) $y = \dfrac{\sqrt{2x+1}}{x}$ (ii) $y = \dfrac{x^2}{\sqrt{x-1}}$

(c) (i) $y = \dfrac{1-2x}{x^2+2}$ (ii) $y = \dfrac{4-x^2}{1+x}$

(d) (i) $y = \dfrac{\ln 3x}{x}$ (ii) $y = \dfrac{\ln 2x}{x^2}$

2. Find the equation of the normal to the curve $y = \dfrac{\sin x}{x}$ at the point where $x = \dfrac{\pi}{2}$, giving your answer in the form $y = mx + c$ where m and c are exact. *[7 marks]*

3. Find the coordinates of the stationary points on the graph of $y = \frac{x^2}{2x-1}$. *[5 marks]*

4. The graph of $y = \frac{x-a}{x+2}$ has gradient 1 at the point $(a, 0)$ and $a \neq -2$. Find the value of a. *[5 marks]*

5. Find the exact coordinates of the stationary point on the curve $y = \frac{\ln x}{x}$ and determine its nature. *[6 marks]*

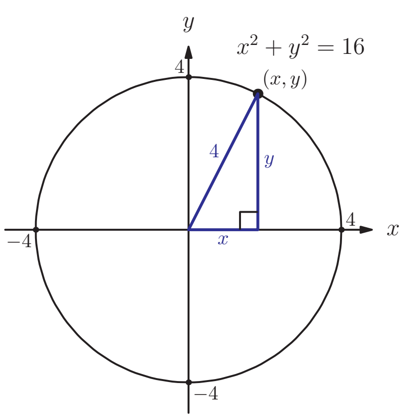
6. Find the range of values of x for which the function $f(x) = \frac{x^2}{1-x}$ is increasing. *[6 marks]*

7. Given that $y = \frac{x^2}{\sqrt{x+1}}$ show that $\frac{dy}{dx} = \frac{x(ax+b)}{2(x+1)^p}$, stating clearly the value of the constants a, b and p. *[6 marks]*

8. Show that if the curve $y = f(x)$ has a maximum stationary point at $x = a$ then the curve $y = \frac{1}{f(x)}$ has a minimum stationary point at $x = a$ as long as $f(a) \neq 0$. *[7 marks]*

18D Implicit differentiation

The functions we have differentiated so far have been of the form $y = f(x)$, but we will also meet functions that are not expressed in this form. For example, the coordinates of a point on the circle shown in the diagram satisfy the equation $x^2 + y^2 = 16$. Such functions are said to be **implicit** (and those in the form $y = f(x)$ are said to be explicit).



Rather than trying to rearrange the equation, we can just differentiate the equation term by term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

Note that care is needed when differentiating y^2 as it is a composite function. We will need the chain rule:

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y\frac{dy}{dx}$$

The chain rule will be needed when differentiating any terms involving y.

KEY POINT 18.6

When differentiating implicitly, we need to use:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

We can now find $\frac{dy}{dx}$ for the equation of the circle above.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Notice that the expression for $\frac{dy}{dx}$ will often be in terms of both x and y. Sometimes implicit differentiation may also need the product rule.

Worked example 18.10

Find an expression for $\frac{dy}{dx}$ if $e^x + x \sin y = \cos 2y$.

Differentiate term by term, using chain rule on all y terms

$$\frac{d}{dx}(e^x) + \frac{d}{dx}(x \sin y) = \frac{d}{dx}(\cos 2y)$$

$x \sin y$ is a product, so use the product rule and the chain rule on all y terms

$$\Rightarrow e^x + \left(x \times \cos y \frac{dy}{dx} + \sin y \times 1\right) = -2\sin 2y \frac{dy}{dx}$$

Group the terms involving $\frac{dy}{dx}$

$$\Rightarrow x\cos y \frac{dy}{dx} + 2\sin 2y \frac{dy}{dx} = -e^x - \sin y$$

$$\Rightarrow (x \cos y + 2\sin 2y)\frac{dy}{dx} = -e^x - \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^x - \sin y}{x\cos y + 2\sin 2y}$$

If we are only interested in the gradient at a particular point, or we are given the gradient and need to find the x- and y-coordinates, we can substitute given values into the differentiated equation without rearranging it.

Worked example 18.11

Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 27$.

Differentiate each term with respect to x but notice that the term $3xy^2$ will need the product rule

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(3xy^2) - \frac{d}{dx}(x^3) = \frac{d}{dx}(27)$$

Use the chain rule on all y terms

$$\Rightarrow 3y^2\frac{dy}{dx} + \left(3x \times 2y\frac{dy}{dx} + y^2 \times 3\right) - 3x^2 = 0$$

$$\Rightarrow 3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 - 3x^2 = 0$$

We know the value of $\frac{dy}{dx}$

For stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow 3y^2 - 3x^2 = 0$

$\Rightarrow (y - x)(y + x) = 0$

$\Rightarrow y = x$ or $y = -x$

We have found a relationship between x and y at the stationary points, to actually find the points substitute back into the original function

When $x = y$:

$x^3 + 3xx^2 - x^3 = 27$

$\Rightarrow 3x^3 = 27$

$\Rightarrow x^3 = 9$

$\Rightarrow x = \sqrt[3]{9}$

$\therefore (\sqrt[3]{9}, \sqrt[3]{9})$ is a stationary point

When $x = -y$:

$(-x)^3 + 3x(-x)^2 - x^3 = 27$

$\Rightarrow -x^3 + 3x^3 - x^3 = 27$

$\Rightarrow x^3 = 27$

$\Rightarrow x = 3$

$\therefore (3, -3)$ is a stationary point

One application of implicit differentiation is to differentiate exponential functions with a base other than e.

Worked example 18.12

Show that $\frac{d}{dx}(5^x) = 5^x \ln 5$.

Take ln of both sides to 'remove' the power

Let $y = 5^x$

Then $\ln y = x \ln 5$

We can differentiate implicitly

$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 5)$

Remember that ln*a* is a constant

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln 5$

$\Rightarrow \frac{dy}{dx} = y \ln 5 = 5^x \ln 5$

EXAM HINT

Although these results are given in the Formula booklet, you could be asked to prove them.

We can use this procedure, and a similar one for $y = \log_a x$ (using the change of base rule), to derive the following general results:

KEY POINT 18.7

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Exercise 18D

1. Find the gradient of each curve at the given point:

(a) (i) $x^2 + 3y^2 = 7$ at $(2, -1)$ (ii) $2x^3 - y^3 = -6$ at $(1, 2)$

(b) (i) $\cos x + \sin y = 0$ at $(0, \pi)$

(ii) $\tan x + \tan y = 2$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

(c) (i) $x^2 + 3xy + y^2 = 20$ at $(2, 2)$

(ii) $3x^2 - xy^2 + 3y = 21$ at $(-1, 3)$

(d) (i) $xe^y + ye^x = 2e$ at $(1, 1)$ (ii) $x \ln y - \frac{x}{y} = 2$ at $(-1, 1)$

2. Find $\frac{dy}{dx}$ in terms of x and y:

(a) (i) $3x^2 - y^3 = 15$ (ii) $x^4 + 3y^2 = 20$

(b) (i) $xy^2 - 4x^2y = 6$ (ii) $y^2 - xy = 7$

(c) (i) $\frac{x+y}{x-y} = 2y$ (ii) $\frac{y^2}{xy+1} = 1$

(d) (i) $xe^y - 4\ln y = x^2$ (ii) $3x\sin y + 2\cos y = \sin x$

3. Find the coordinates of stationary points on the curves given by these implicit equations:

(i) $-x^2 + 3xy + y^2 = 13$ (ii) $2x^2 - xy + y^2 = 28$

4. Find the exact value of the gradient at the given point:

(a) (i) $y = 3^x$ at $(1, 3)$ (ii) $y = 5^x$ at $(2, 25)$

(b) (i) $y = \left(\frac{1}{2}\right)^x$ when $x = -2$ (ii) $y = \left(\frac{1}{3}\right)^x$ when $x = -1$

(c) (i) $y = 2^{3x}$ when $x = -1$ (ii) $y = 4^{2x}$ when $x = \frac{1}{4}$

(d) (i) $y = 3^{3-x}$ when $x = 2$ (ii) $y = 5^{1-x}$ when $x = 2$

5. (a) On Fill-in proof 18 'Differentiating logarithmic functions graphically' on the CD-ROM we constructed an argument which suggested that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Use the fact that $\ln x$ is the inverse function of e^x and implicit differentiation to prove this result.

(b) Show that $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$.

(c) Differentiate $\ln kx$ and $\ln x^n$ using chain rule. What do you notice? Why is this the case? *[6 marks]*

6. Find the gradient of the curve with equation $x^2 - 3xy + y^2 + 1 = 0$ at the point (1, 2). *[6 marks]*

7. Find the equation of the tangent to the curve with equation $4x^2 - 3xy - y^2 = 25$ at the point (2, –3). *[6 marks]*

8. A curve has implicit equation $x2^y = \ln y$. Find an expression for $\frac{dy}{dx}$ in terms of x and y. *[6 marks]*

9. Find the coordinates of the stationary point on the curve given by $e^x + ye^{-x} = 2e^2$. *[6 marks]*

10. The line L is tangent to the curve C which has the equation $y^2 = x^3$ when $x = 4$ and $y > 0$.

(a) By rearranging the curve into the form $y = \pm f(x)$ or otherwise, sketch C.

(b) Find the equation of L.

(c) Show that L meets C again at the point P with an x-coordinate which satisfies the equation $x^3 - 9x^2 + 24x - 16 = 0$.

(d) Find the coordinates of the point P. *[10 marks]*

18E Differentiating inverse trigonometric functions

Implicit differentiation can also be used to find the derivatives of the inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$.

Worked example 18.13

If $y = \arcsin x$, find $\frac{dy}{dx}$ in terms of x.

We know how to differentiate sin, so express x in terms of y

$y = \arcsin x \quad \Rightarrow \sin y = x$

Differentiate each term with respect to x, remembering the chain rule

$\Rightarrow \cos y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

We want the answer in terms of x, so we need to change cos to sin

$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

We can establish the results for the inverse cos and tan functions similarly giving:

KEY POINT 18.8

$$y = \arcsin x \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arccos x \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan x \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

Worked example 18.14

Differentiate:

(a) $y = \arctan 4x$ (b) $y = \arccos\sqrt{x-3}$

Multiply the standard result by 4, the derivative of 4x (using chain rule)

(a) $\frac{dy}{dx} = \frac{1}{1+(4x)^2} \times 4$

$= \frac{4}{1+16x^2}$

Again using the chain rule multiply by $\frac{1}{2}(x-3)^{-\frac{1}{2}}$, the derivative of $\sqrt{x-3}$

(b) $\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\sqrt{x-3}\right)^2}} \times \frac{1}{2}(x-3)^{-\frac{1}{2}}$

$= \frac{-1}{\sqrt{1-(x-3)}} \times \frac{1}{2\sqrt{x-3}}$

$= \frac{-1}{2\sqrt{(4-x)(x-3)}}$

Exercise 18E

1. Find $\frac{dy}{dx}$ for each of the following:

(a) (i) $y = \arccos(3x)$ (ii) $y = \arccos(2x)$

(b) (i) $y = \arctan\left(\frac{x}{2}\right)$ (ii) $y = \arctan\left(\frac{2x}{5}\right)$

(c) (i) $y = x \arcsin x$ (ii) $y = x^2 \arccos x$

(d) (i) $y = \arctan(x^2+1)$ (ii) $y = \arcsin(1-x^2)$

2. Find the exact value of the gradient of the graph of $y = \arccos\left(\frac{x}{2}\right)$ at the point where $x = \frac{1}{3}$. *[5 marks]*

3. Given that $y = \arcsin\left(\frac{3x}{2}\right)$, show that $\frac{dy}{dx} = \frac{3}{\sqrt{4-9x^2}}$. *[5 marks]*

4. Given that $x \arctan y = 1$, find an expression for $\frac{dy}{dx}$. *[5 marks]*

5. (a) Find $\frac{d}{dx}(x \arcsin x)$. *[5 marks]*

(b) Hence find $\int \arcsin x \, dx$. *[6 marks]*

6. Show that the graph of $y = \arcsin(x^2)$ has no points of inflexion. *[6 marks]*

Summary

- The **chain rule** is used to differentiate composite functions.

  If $y = f(u)$ where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

- The **product rule** is used to differentiate two functions multiplied together.

  If $y = u(x)v(x)$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

- The **quotient rule** is used to differentiate one function divided by another.

  If $y = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

- The derivatives of the reciprocal trigonometric functions are:

  $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad \frac{d}{dx}(\csc x) = -\csc x \cot x$ $\quad \frac{d}{dx}(\cot x) = -\csc^2 x$

- The derivative of an exponential function is:

  $$\frac{d}{dx}(a^x) = a^x \ln a$$

- The derivative of a log function is:

  $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

- The derivatives of the inverse trigonometric functions are:

  $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

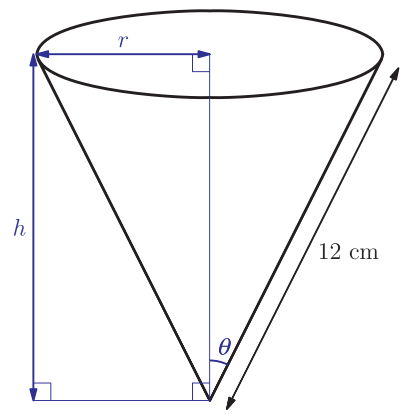
- In an **implicit equation**, differentiate each term separately noting that for functions of y the chain rule needs to be used:

  $\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$

Introductory problem revisited

Given a cone of fixed slant height 12 cm, find the maximum volume as apex angle θ varies.

First we need to write an expression for the volume of the cone. Then we can differentiate with respect to θ and solve $\frac{dV}{d\theta} = 0$ to find the value of θ at which the maximum occurs.



$V = \frac{1}{3}\pi r^2 h$

Using the right-angled triangle highlighted in the diagram:

$r = 12\sin\theta$

$h = 12\cos\theta$

Therefore, substituting into the formula for V we have:

$$V = \frac{1}{3}\pi(12\sin\theta)^2(12\cos\theta) = \frac{12^3}{3}\pi\sin^2\theta\cos\theta$$

For stationary points, $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = \frac{12^3}{3}\pi\left[(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta(-\sin\theta)\right]$$

$$= \frac{12^3}{3}\pi\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] = 0$$

$$\Rightarrow 2\sin\theta\cos^2\theta - \sin^3\theta = 0$$

$$\Rightarrow \sin\theta = 0 \quad \text{or} \quad 2\cos^2\theta - \sin^2\theta = 0$$

$\sin\theta = 0$ has no valid solutions, since for a cone, $0 < \theta < 90°$.

$2\cos^2\theta - \sin^2\theta = 0 \Rightarrow 2\tan^2 \theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$ ($\tan\theta = -\sqrt{2}$ has no solutions $0 < \theta < 90°$)

Therefore the maximum volume occurs when $\tan\theta = \sqrt{2}$, which means $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

Therefore, substituting into $V = \frac{12^3}{3}\pi\sin^2\theta\cos\theta$:

$$V_{max} = \frac{12^3}{3}\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) = \frac{12^3\, 2\sqrt{3}\pi}{3^3} = 4^3\, 2\sqrt{3}\pi = 128\sqrt{3}\pi$$

Mixed examination practice 18

Short questions

1. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = x^2 \arcsin x$

(b) $xe^y = 4y^2$ *[7 marks]*

2. Differentiate $f(x) = \arccos(1 - x^2)$. *[4 marks]*

3. Find the exact value of the gradient of the curve with equation $y = \frac{1}{4 - x^2}$ when $x = \frac{1}{2}$. *[5 marks]*

4. Find the equation of the normal to the curve with equation $4x^2 + xy^2 - 3y^3 = 56$ at the point (–5, 2). *[7 marks]*

5. Given that $y = \arctan(x^2)$ find $\frac{d^2y}{dx^2}$. *[5 marks]*

6. Find the gradient of the curve with equation $4\sin x \cos y + \sec^2 y = 5$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. *[6 marks]*

7. The graph of $y = xe^{-kx}$ has a stationary point when $x = \frac{2}{5}$.
Find the value of k. *[4 marks]*

8. A curve has equation $f(x) = \frac{a}{b + e^{-cx}}, a \neq 0, b, c > 0$.

(a) Show that $f''(x) = \frac{ac^2 e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3}$.

(b) Find the coordinates of the point on the curve where $f''(x) = 0$.

(c) Show that this is a point of inflexion. *[8 marks]*

9. Find the coordinates of stationary points on the curve with equation $(y - 2)^2 e^x = 4x$. *[7 marks]*

Long questions

1. A curve has equation $y = \frac{x^2}{1-2x}$.

(a) Write down the equation of the vertical asymptote of the curve.

(b) Use differentiation to find the coordinates of stationary points on the curve.

(c) Determine the nature of the stationary points.

(d) Sketch the graph of $y = \frac{x^2}{1-2x}$. *[15 marks]*

2. The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that $f'(x) = \frac{2x - x^2 \ln 2}{2^x}$.

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(b) (i) Find the exact value of x satisfying the equation $f'(x) = 0$.

(ii) Show that this value gives a maximum value for $f(x)$.

(c) Find the x-coordinates of the two points of inflexion on the graph of f.

[12 marks]

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3. Let $f(x) = \arccos(\sqrt{1-9x^2})$ for $0 < x < \frac{1}{3}$.

(a) Show that $f'(x) = \frac{3}{\sqrt{1-9x^2}}$.

(b) Show that $f''(x) > 0$ for all $x \in]0, \frac{1}{3}[$.

(c) Let $g(x) = \arccos(kx)$. If $g'(x) = -pf'(x)$ for $0 < x < \frac{1}{3}$, find the values of p and k. *[12 marks]*

4. A curve is given by the implicit equation $x^2 - xy + y^2 = 12$.

(a) Find the coordinates of the stationary points on the curve.

(b) Show that at the stationary points, $(x-2y)\frac{d^2y}{dx^2} = 2$.

(c) Hence determine the nature of the stationary points. *[16 marks]*

5. If $f(x) = \sec x$, $0 \le x \le \pi$ the inverse function is $f^{-1}(x) = \operatorname{arcsec} x$.

(a) Write down the domain of $\operatorname{arcsec} x$.

(b) Sketch the graph of $y = \operatorname{arcsec} x$.

(c) Show that the derivative of $\sec x$ is $\sec x \tan x$.

(d) Find the derivative of $\operatorname{arcsec} x$ with respect to x, justifying carefully the sign of your answer. *[12 marks]*

In this chapter you will learn:

- to integrate using known derivatives
- to use the chain rule in reverse
- to integrate using trigonometric identities
- to integrate using inverse trigonometric functions
- to use the product rule in reverse (integration by parts)
- to integrate using a change of variable (substitution)
- to integrate using the separation of a fraction into two fractions.

19 Further integration methods

Introductory problem

Use integration to prove that the area of a circle of radius r is equal to πr^2.

Having extended the range of functions we can differentiate, we now need to do the same for integration. Sometimes we will be able to use results from the previous chapter, but in other cases we will require new techniques. In this chapter we look at each of these in turn and then face the challenge of selecting the appropriate technique from the list of options we have.

19A Reversing standard derivatives

In chapter 17 we reversed a number of **standard derivatives** that had been established in chapter 16 to give us this list of functions we could integrate.

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \; n \neq -1$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

EXAM HINT

These are all given in the Formula booklet.

In chapter 18 (Key point 18.3) we differentiated sec x, csc x and cot x. We can now reverse these standard derivatives too and add them to our list:

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \csc x \cot x \, dx = -\csc x + c$$

$$\int \csc^2 x \, dx = -\cot x + c$$

The chain rule for differentiation (chapter 18A) allows us to go further and deal with integrals such as $\int 2\cos(2x)\,dx$. Here we think about integrating cos to sin and then consider what the chain rule would give us if we differentiated back. In this case the chain rule would give the 2 anyway as $\frac{d}{dx}(\sin 2x) = 2\cos 2x$ (2 is the derivative of $2x$) so we have the correct integration straight away:

$$\int 2\cos(2x)\,dx = \sin 2x + c$$

We may have a similar question in which we do not have the exact derivative and then we need to compensate by cancelling out any unwanted constant generated by the chain rule.

For example, in finding $\int (2x-3)^4\,dx$ we proceed as before integrating $(\;)^4$ to $\frac{1}{5}(\;)^5$ but now when we differentiate back the chain rule gives us an unwanted 2:

$$\frac{d}{dx}\left(\frac{1}{5}(2x-3)^5\right) = 2(2x-3)^4$$

so we divide by 2 to remove it:

$$\int (2x-3)^4\,dx = \frac{1}{2} \times \frac{1}{5}(2x-3)^5 + c = \frac{1}{10}(2x-3) + c.$$

You may notice a pattern here, we always divide by the coefficient of x. This is indeed a general rule, which follows simply by reversing the special case of the chain rule from Key point 18.2.

EXAM HINT

These are not given in the list of **standard integrals** in the Formula booklet, but can be deduced from the list of standard derivatives.

KEY POINT 19.1

The reverse chain rule

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$$

where $F(x)$ is the integral of $f(x)$.

EXAM HINT

This rule only applies when the 'inside' function is of the form $(ax + b)$!

With this shortcut we do not need to work through the chain rule every time.

Worked example 19.1

Find the following:

(a) $\int \frac{1}{2}e^{4x}dx$ (b) $\int \frac{2}{5-x}dx$

Integrate $e^{(\ )}$ to $e^{(\ )}$ and divide by the coefficient of x

(a) $\int \frac{1}{2}e^{4x}dx = \frac{1}{2}\times\frac{1}{4}e^{4x}+c$

$= \frac{1}{8}e^{4x}+c$

Integrate $\frac{1}{(\ )}$ to $\ln|\ \ |$ and divide by the coefficient of x

(b) $\int \frac{2}{5-x}dx = 2\left(\frac{1}{-1}\right)\ln|5-x|+c$

$= -2\ln|5-x|+c$

Exercise 19A

1. Find:

(a) (i) $\int 5(x+3)^4\,dx$ (ii) $\int (x-2)^5\,dx$

(b) (i) $\int (4x-5)^7\,dx$ (ii) $\int \left(\frac{1}{8}x+1\right)^3\,dx$

(c) (i) $\int 4\left(3-\frac{1}{2}x\right)^6\,dx$ (ii) $\int (4-x)^8\,dx$

(d) (i) $\int \sqrt{2x-1}\,dx$ (ii) $\int 7(2-5x)^{3/4}\,dx$

(e) (i) $\int \frac{1}{\sqrt[4]{2+\frac{x}{3}}}\,dx$ (ii) $\int \frac{6}{(4-3x)^2}\,dx$

2. Find these integrals:

(a) (i) $\int 3e^{3x}\,dx$ (ii) $\int e^{2x+5}\,dx$

(b) (i) $\int 4e^{\frac{2x-1}{3}}\,dx$ (ii) $\int e^{\frac{1}{2}x}\,dx$

(c) (i) $\int -6e^{-3x}\,dx$ (ii) $\int \frac{1}{e^{4x}}\,dx$

(d) (i) $\int \frac{-2}{e^{x/4}}\,dx$ (ii) $\int e^{-\frac{2}{3}x}\,dx$

3. Find:

(a) (i) $\int \frac{1}{x+4}\,dx$ (ii) $\int \frac{5}{5x-2}\,dx$

(b) (i) $\int \frac{2}{3x+4}\,dx$ (ii) $\int \frac{-8}{2x-5}\,dx$

(c) (i) $\int \frac{-3}{1-4x}\,dx$ (ii) $\int \frac{1}{7-2x}\,dx$

(d) (i) $\int 1-\frac{3}{5-x}\,dx$ (ii) $\int 3+\frac{1}{3-x}\,dx$

4. Integrate the following:

(a) $\int -\csc x \cot x\,dx$

(b) $\int 3\sec^2 3x\,dx$

(c) $\int \sin(2-3x)\,dx$

(d) $\int \csc^2\left(\frac{1}{4}x\right)dx$

(e) $\int 2\cos 4x\,dx$

(f) $\int \sec\frac{x}{2}\tan\frac{x}{2}\,dx$

5. Two students integrate $\int \frac{1}{3x}\,dx$ in two different ways.
Marina writes:

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx = \frac{1}{3}\ln|x| + c$$

Jack uses the special case of the reverse chain rule and divides by the coefficient of x:

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\ln|3x| + c$$

Who has the right answer?

6. Given that $0 < a < 1$ and the area between the x-axis, the lines $x = a^2$, $x = a$ and the graph of $y = \frac{1}{1-x}$ is 0.4, find the value of a correct to 3 significant figures. *[5 marks]*

19B Integration by substitution

The shortcut for reversing the chain rule works only when the derivative of the 'inside' function is a constant. This is because a constant factor can 'move through the integral sign', for example:

$$\int \cos 2x \, \mathrm{d}x = \int \frac{1}{2} \times 2\cos 2x \, \mathrm{d}x = \frac{1}{2}\int 2\cos 2x \, \mathrm{d}x = \frac{1}{2}\sin 2x + c$$

Another method for integrating products is integration by parts, the reverse of the product rule. We will meet this in Section 19F.

This cannot be done with a variable: $\int x \sin x \, \mathrm{d}x$ is not the same as $x\int \sin x \, \mathrm{d}x$. So we need a different rule for integrating a product of two functions. In some cases this can be done by extending the principle of reversing the chain rule, leading to the method of **integration by substitution**.

When using the chain rule to differentiate a composite function, we differentiate the outer function and multiply this by the derivative of the inner function; for example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(x^2 + 2\right)\right) = \cos(x^2 + 2) \times 2x$$

We can think of this as using a substitution $u = x^2 + 2$, and then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$.

Let us now look at $\int x\cos(x^2 + 2)\,\mathrm{d}x$.

As $\cos(x^2 + 2)$ is a composite function we can write it as $\cos u$, where $u = x^2 + 2$. So our integral becomes $\int x \cos u \, \mathrm{d}x$. We know how to integrate $\cos u$, so we want to change our variable to u. But then we need to be integrating with respect to u, so we should have $\mathrm{d}u$ instead of $\mathrm{d}x$. Those two are not the same thing, but they are related because $u = x^2 + 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$.

We can now 'rearrange' this to get $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$.

Substituting all this into our integral we now have

$$\begin{aligned}\int x\cos\left(x^2 + 2\right)\mathrm{d}x &= \int x\cos u\left(\frac{1}{2x}\right)\mathrm{d}u\\ &= \int \frac{1}{2}\cos u \, \mathrm{d}u\\ &= \frac{1}{2}\sin u + c\end{aligned}$$

This answer is in terms of u so we need to write it in terms of x.

$$\int x \cos(x^2 + 2)\,\mathrm{d}x = \frac{1}{2}\sin(x^2 + 2) + c$$

A word of warning here: $\frac{\mathrm{d}u}{\mathrm{d}x}$ is not really a fraction, so it is not clear that the above 'rearrangement' is valid. However, it can be shown that it follows from the chain rule that it is valid to replace $\mathrm{d}x$ by $\frac{1}{f'(x)}\mathrm{d}u$.

Worked example 19.2

Find the following:

(a) $\int \sin^5 x \cos x \, dx$ (b) $\int x^2 e^{x^3+4} \, dx$

It is helpful here to think of $\sin^5 x$ as $(\sin x)^5$. Therefore the inner function is $\sin x$

(a) Let $u = \sin x$

Then $\frac{du}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} du$

Make the substitution

$$\int (\sin x)^5 \cos x \, dx = \int u^5 \cos x \frac{1}{\cos x} du$$

$$= \int u^5 du$$

$$= \frac{1}{6} u^6 + c$$

Write the answer in terms of x

$$= \frac{1}{6} \sin^6 x + c$$

e^{x^3+4} is a composite function with inner function $x^3 + 4$

(b) Let $u = x^3 + 4$

Then $\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2} du$

Make the substitution

$$\int x^2 e^{x^3+4} dx = \int x^2 e^u \frac{1}{3x^2} du$$

$$= \int \frac{1}{3} e^u du$$

$$= \frac{1}{3} e^u + c$$

Write the answer in terms of x

$$= \frac{1}{3} e^{x^3+4} + c$$

You may have noticed in all of the above examples that, after making the substitution, the part of the integrand which was still in terms of x cancelled with a similar term coming from $\frac{du}{dx}$.

For example, in (b), $\int x^2 e^{x^3+4} dx = \int x^2 e^u \frac{1}{3x^2} du = \int \frac{1}{3} e^u du$.

This will always happen when one part of the integrand is an exact multiple of the derivative of the inner function, and can be explained by looking at the chain rule.

For example, consider the integral $\int (2x+3)(x^2+3x-5)^4\,dx$.

To find this integral, think about what we would need to differentiate to get $(2x+3)(x^2+3x-5)^4$. As $2x+3$ is the derivative of x^2+3x-5 we know that we would get $2x+3$ 'for free' when differentiating some power of x^2+3x-5 using the chain rule. In this case to end up with $(x^2+3x-5)^4$ we would want to be differentiating $\frac{1}{5}(x^2+3x-5)^5$, that is

$$\frac{d}{dx}\left(\frac{1}{5}(x^2+3x-5)^5\right)=(2x+3)(x^2+3x-5)^4$$

and therefore:

$$\int (2x+3)(x^2+3x-5)^4\,dx=\frac{1}{5}(x^2+3x-5)^5+c$$

This is the same answer we would get by using the substitution $u=x^2+3x-5$. If you notice that you can integrate an expression by reversing the chain rule, you can just write down the answer without any working. However, if you are not sure, it is safer to go through the whole process of substitution.

In some cases this cancelling of the remaining x-terms will not happen and you will have to express x in terms of u. The full method of substitution will then be as follows:

EXAM HINT

You will nearly always be told which substitution to use. If you are not, look for a composite function and take $u=$ 'inner' function.

KEY POINT 19.2

Integration by substitution

1. Select a substitution (if not already given).
2. Differentiate the substitution and write dx in terms of du.
3. Replace dx by the above expression, and replace any obvious occurrences of u.
4. Simplify as far as possible.
5. If any terms with x remain, write them in terms of u.
6. Work out the new integral in terms of u.
7. Write the answer in terms of x.

For the integral in the next example there are two possible substitutions. As there is a composite function $\sqrt{4x-1}$, we could use the 'inner' function: $u=4x-1$. However, we must always use the substitution we are given.

Worked example 19.3

Find $\int x\sqrt{4x-1}\,dx$ using the substitution $u=\sqrt{4x-1}$.

Differentiate the substitution	$u=\sqrt{4x-1}$ $\Rightarrow \frac{du}{dx}=\frac{4}{2\sqrt{4x-1}}$ $=\frac{2}{u}$
... and write dx in terms of du (Key point 19.2 Step 2)	$\therefore dx=\frac{1}{2}u\,du$
Replace those parts that we already have expressions for, and simplify if possible (Steps 3 and 4)	$\int x\sqrt{4x-1}\,dx=\int xu\frac{1}{2}u\,du$ $=\int\frac{1}{2}xu^2du$
There is still an x remaining, so replace it by using $u=\sqrt{4x-1}\Rightarrow x=\frac{u^2+1}{4}$ (Step 5)	$=\int\frac{1}{2}\frac{u^2+1}{4}u^2\,du$
Now everything is in terms of u so we can integrate (Step 6)	$=\frac{1}{8}\int u^4+u^2\,du=\frac{1}{8}\left(\frac{1}{5}u^5+\frac{1}{3}u^3\right)+c$
Write the answer in terms of x using $u=\sqrt{4x-1}$ (Step 7)	$=\frac{1}{8}\left(\frac{1}{5}\left(\sqrt{4x-1}\right)^5+\frac{1}{3}\left(\sqrt{4x-1}\right)^3\right)+c$

When limits are given, we must change them too. Then there is no need to change back to the original variable at the end.

KEY POINT 19.3

When evaluating a definite integral using substitution, add the following step to the process in Key point 19.2:

Step 3a. Write the limits in terms of u.

The next example shows one of the most common uses of substitution; integrating a quotient where the numerator is a multiple of the derivative of the denominator.

Worked example 19.4

Evaluate $\int_0^1 \frac{x-3}{x^2-6x+7}\,dx$ giving your answer in the form $a \ln p$.

This is of the form 'something' $\times \frac{1}{(\ )}$ so the 'inner' function is x^2-6x+7

Let $u = x^2-6x+7$.

Then $\frac{du}{dx} = 2x-6 \Rightarrow dx = \frac{1}{2x-6}du$

Write limits in terms of u

Limits: $x=0 \Rightarrow u=7,\ x=1 \Rightarrow u=2$

Make the substitution

$$\int_0^1 \frac{x-3}{x^2-6x+7}dx = \int_7^2 \frac{x-3}{u}\frac{1}{2x-6}du$$

Simplify $2x-6 = 2(x-3)$

$$= \int_7^2 \frac{1}{2u}du$$

$$= \left[\frac{1}{2}\ln|u|\right]_7^2$$

$$= \frac{1}{2}(\ln 2 - \ln 7)$$

$$= \frac{1}{2}\ln\left(\frac{2}{7}\right)$$

In the above example it is possible to write down the result of the integration without using the full substitution method, if we notice that $x-3$ is half of the derivative of x^2-6x+7, and so $(x-3)\times\frac{1}{x^2-6x+7}$ comes from differentiating $\frac{1}{2}\ln|x^2-6x+7|$.

This particular case of substitution, where the top of the fraction is the derivative of the bottom, is definitely worth remembering:

KEY POINT 19.4

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

The next example shows that the substitution can also be given as x in terms of u (or θ in this case). It also illustrates that substitutions can lead to integrals where the use of trigonometric identities is required.

We will see more examples of using trigonometric identities in the next section.

Worked example 19.5

Use the substitution $x = \sec\theta$ to find the exact value of $\int_{\sqrt{2}}^{2} (x^2 - 1)^{-(3/2)}\,dx$.

Differentiate the substitution and express dx in terms of dθ

$x = \sec\theta$

$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

Change the limits

When $x = \sqrt{2}$:

$\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}$

When $x = 2$:

$\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

continued . . .

Replace those parts that we already have expressions for. Remember that $\sec^2\theta - 1 = \tan\theta$

$$\int_{\sqrt{2}}^{2} (x^2-1)^{-\frac{3}{2}}\,dx = \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)^{-\frac{3}{2}} \sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\tan^2\theta)^{-\frac{3}{2}} \sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\tan\theta)^{-3} \sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\tan\theta)^{-2} \sec\theta\,d\theta$$

We seem to be stuck; writing everything in terms of sin and cos often helps

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Writing this as $\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$, we now have a standard derivative (Section 19A)

$$= \int_{\pi/4}^{\pi/3} \csc\theta\cot\theta\,d\theta$$

$$= [-\csc\theta]_{\pi/4}^{\pi/3} = \left(-\frac{2}{\sqrt{3}}\right) - (-\sqrt{2}) = \sqrt{2} - \frac{2}{\sqrt{3}}$$

Exercise 19B

1. Either by using a suitable substitution, or by considering the chain rule, find these integrals:

(a) (i) $\int x(x^2+3)^3\,dx$ (ii) $\int 3x(x^2-1)^5\,dx$

(b) (i) $\int (2x-5)(3x^2-15x+4)^4\,dx$

(ii) $\int (x^2+2x)(x^3+3x^2-5)^3\,dx$

(c) (i) $\int \frac{2x}{x^2+3}\,dx$ (ii) $\int \frac{6x^2-12}{x^3-6x+1}\,dx$

(d) (i) $\int 4\cos^5 3x\sin 3x\,dx$ (ii) $\int \cos 2x\sin^3 2x\,dx$

(e) (i) $\int 3xe^{3x^2-1}\,dx$ (ii) $\int 3xe^{x^2}\,dx$

(f) (i) $\int \frac{e^{2x+3}}{e^{2x+3}+4}\,dx$ (ii) $\int \frac{\cos x}{3+4\sin x}\,dx$

(g) (i) $\int 32\sec^2 2x \tan^3 2x \, dx$

(ii) $\int 6\sec^6\left(\frac{x}{4}\right)\tan\left(\frac{x}{4}\right) dx$

(h) (i) $\int \csc^4 x \cot x \, dx$

(ii) $\int \frac{\csc^2 2x}{(3+\cot 2x)} dx$

(i) (i) $\int \frac{x}{\sqrt{3-x^2}} dx$

(ii) $\int 2e^{-4x+1}\sqrt{e^{-4x+1}} \, dx$

2. Use a suitable substitution to show that $\int \tan x \, dx = \ln|\sec x| + c$.

EXAM HINT

The integral of tan is not given in the Formula booklet, and is worth remembering.

3. Find the following integrals using the given substitution:

(a) (i) $\int x\sqrt{x+1} \, dx$, $u = x+1$

(ii) $\int x^2\sqrt{x-2} \, dx$, $u = x-2$

(b) (i) $\int 2x(x-5)^7 \, dx$, $u = x-5$

(ii) $\int x(x+3)^5 \, dx$, $u = x+3$

4. Find the following using an appropriate substitution:

(a) (i) $\int x(2x-1)^4 \, dx$ (ii) $\int 9x(3x+2)^5 \, dx$

(b) (i) $\int x\sqrt{x-3} \, dx$ (ii) $\int (x+1)\sqrt{5x-6} \, dx$

(c) (i) $\int \frac{x^2}{\sqrt{x-5}} dx$ (ii) $\int \frac{4(x+5)}{(2x-3)^3} dx$

5. Use the given substitution to evaluate these definite integrals:

EXAM HINT

When an unusual substitution is required it will always be given in the question.

(a) (i) $\int_2^3 \left(\frac{x}{4-x}\right)^2 dx$, $u = 4-x$

(ii) $\int_1^3 \frac{x^3}{(x+2)^2} dx$, $u = x+2$

(b) (i) $\int_0^{\pi/6} \frac{\cos\theta}{1+\sin\theta} d\theta$, $u = \sin\theta$

(ii) $\int_0^{\pi/2} \frac{\sin\theta}{1+\cos\theta} d\theta$, $u = 1+\cos\theta$

(c) (i) $\int_0^{1/3} \frac{1}{\sqrt{4-9x^2}} dx$, $x = \frac{2}{3}\sin\theta$

(ii) $\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}} dx$, $x = \frac{1}{2}\cos\theta$

6. Find the exact value of $\int_0^2 (2x+1)e^{x^2+x-1}\,dx$. *[6 marks]*

7. Evaluate $\int_2^5 \frac{2x}{x^2-1}\,dx$, giving your answer in the form $\ln k$. *[4 marks]*

8. Use the substitution $u = x-2$ to find $\int \frac{x}{\sqrt{x-2}}\,dx$. *[6 marks]*

9. (a) Show that $(x-1)$ is a factor of x^3-1.

(b) Find $\int \frac{2x^2-x-1}{x^3-1}\,dx$. *[4 marks]*

10. Use the substitution $u = \ln x$ to find $\int \frac{\sec^2(\ln(x^2))}{2x}\,dx$. *[6 marks]*

11. Find $\int \frac{\cos x}{\sin^5 x}\,dx$. *[3 marks]*

12. Evaluate $\int_1^3 \frac{(2x-3)\sqrt{x^2-3x+3}}{x^2-3x+3}\,dx$. *[6 marks]*

19C Using trigonometric identities in integration

Sometimes it is necessary to rearrange the expression before reversing a standard derivative or using a substitution. In this section we will take a more systematic look at using trigonometric identities in order to integrate a wide range of functions.

As seen in the previous section, the presence of the $\cos x$ in $\int \sin^3 x \cos x\,dx$ makes it possible to apply the reverse chain rule (or a substitution) but how do we cope with just $\int \sin^3 x\,dx$? As a mixture of sin and cos helps us in the use of the reverse chain rule, we aim to introduce cos by using $\sin^2 x + \cos^2 x = 1$.

EXAM HINT

If you are asked to do an integral like this in the exam you will be given a hint, as in the example below.

Worked example 19.6

(a) Show that $\sin^3 x = \sin x - \cos^2 x \sin x$.

(b) Hence find $\int \sin^3 x \, dx$.

Introduce $\cos^2 x$ by using $\sin^2 x + \cos^2 x = 1$; to do this we need to 'split' $\sin^3 x$

(a) $\sin^3 x = \sin^2 x \sin x$
$= (1 - \cos^2 x)\sin x$
$= \sin x - \cos^2 x \sin x$

We can use the result from part (a)

(b) $\int \sin^3 x \, dx = \int \sin x - \cos^2 x \sin x \, dx$

Use a substitution $u = \cos x$

$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$\int \sin^3 x \, dx = -\cos x - \int u^2 \sin x \left(-\frac{1}{\sin x}\right) du$

$= -\cos x + \int u^2 du$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

The same trick does not work for $\int \sin^2 x \, dx$, as we can only rewrite it as $\int 1 - \cos^2 x \, dx$ which we also cannot integrate. Instead we notice that $\sin^2 x$ appears in one of the versions of the double-angle formulae for $\cos 2x$: $\cos 2x = 1 - 2\sin^2 x$, and we know how to integrate $\cos 2x$.

Double angle identities were covered in Section 12A.

Worked example 19.7

Find $\sin^2 x \, dx$.

Write an alternative expression for $\sin^2 x$ by using a double angle identity

$\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\therefore \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

Remember to divide by the coefficient of x when integrating $\cos 2x$

$= \frac{1}{2}x - \frac{1}{2}\frac{1}{2}\sin 2x + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

A similar method is used to integrate $\cos^2 x$ and should be learnt.

EXAM HINT
You will be expected to recall this method without hints.

KEY POINT 19.5

To integrate $\sin^2 x$, use $\cos 2x = 1 - 2\sin^2 x$.

To integrate $\cos^2 x$, use $\cos 2x = 2\cos^2 x - 1$.

The methods from Worked examples 19.6 and 19.7 can be extended to deal with any powers of $\sin x$ and $\cos x$. The method from Worked example 19.6 can be applied to any odd power, for example:

$$\sin^5 x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x = (1 - 2\cos^2 x + \cos^4 x)\sin x$$

which can be integrated using the reverse chain rule.

For even powers, we can use the identity from Key point 19.5, for example

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

and then the double angle identity has to be used again to relate $\cos^2 2x$ to $\cos 4x$.

See Worked example 15.25 in Section 15H to remind you of this method.

This becomes increasingly complicated for larger powers. Luckily, there is a ready-made alternative from the unlikely source of complex numbers. We saw in chapter 15 that we could use De Moivre's Theorem to generate expressions for powers of sin and cos in terms of multiple angles.

Worked example 19.8

(a) Show that $\cos^6 x = \frac{1}{16}\cos 6x + \frac{3}{8}\cos 4x + \frac{15}{16}\cos 2x + \frac{5}{8}$.

(b) Hence find $\int \cos^6 x \, dx$.

We derived similar identities in Section 15H, so we will not repeat it here

(a) See Section 15H for how to do this.

Use the result from part (a)

(b) $\int \cos^6 x \, dx = \int \frac{1}{16}\cos 6x + \frac{3}{8}\cos 4x + \frac{15}{16}\cos 2x + \frac{5}{8} \, dx$

Don't forget to divide by the coefficient of x

$= \frac{1}{96}\sin 6x + \frac{3}{32}\sin 4x + \frac{15}{32}\sin 2x + \frac{5}{8}x + c$

Don't worry if this seems complicated – a question like this will always be split into several parts, as in questions 10 and 11 in Worked exercise 19C at the end of this section.

We shall now integrate tan x and its powers. We have already integrated tan x as an application of the reverse chain rule in Exercise 19B, question 2: $\int \tan x \, dx = \ln|\sec x| + c$. However, this does not help when trying to integrate more complicated functions, for example $\tan^2 x$. We do, however, have an identity relating $\tan^2 x$ to something we know how to integrate: $1 + \tan^2 x = \sec^2 x$.

This identity was derived in chapter 12.

Worked example 19.9

Find $\int \tan^2 2x \, dx$.

We have an identity relating $\tan^2(\ )$ to $\sec^2(\ )$, which we know how to integrate

$$\int \tan^2 2x \, dx = \int \sec^2 2x - 1 \, dx$$

Use the standard result for integrating $\sec^2(\ )$, remembering to divide by the coefficient of x

$$= \frac{1}{2}\tan 2x - x + c$$

The same identity is used in integrating any power of tan x.

KEY POINT 19.6

To integrate $\tan^n x$ use the identity $1 + \tan^2 x = \sec^2 x$ and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Worked example 19.10

Find $\int \tan^3 x \, dx$.

Introduce $\sec^2 x$ by using $\tan^2 x = \sec^2 x - 1$

$$\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx$$
$$= \int (\sec^2 x - 1) \tan x \, dx$$

We integrate the two terms separately

$$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

We can apply the reverse chain rule (or a substitution $u = \tan x$) to $\sec^2 x \tan x$ because $\frac{d}{dx}(\tan x) = \sec^2 x$

$(\ )$ integrates to $\frac{1}{2}(\ )^2$

First integral:

$$\int \sec^2 x \tan x = \frac{1}{2}(\tan x)^2 + c$$

We found $\int \tan x$ in the previous section

Second integral:

$$\int \tan x \, dx = \ln|\sec x| + c$$

$$\therefore \int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + c$$

The above examples illustrate standard methods used to integrate powers of trigonometric functions. Many other trigonometric integrals can be rearranged into a form where we can simply reverse a standard derivative. Here we give one example of using trigonometric identities to do this.

Worked example 19.11

Find $\int \frac{\sin 4x}{\sin^3 2x} \, dx$.

As we have $4x$ and $2x$, apply the double angle identity for sine

$$\int \frac{\sin 4x}{\sin^3 2x} dx = \int \frac{2 \sin 2x \cos 2x}{\sin^3 2x} dx$$
$$= \int \frac{2 \cos 2x}{\sin^2 2x} dx$$

continued . . .

Check if this can be manipulated into the form of one of our standard derivatives. If we cannot see it immediately, it is a good idea to try and split the expression into a product of two trigonometric functions

$$= 2\int \frac{1}{\sin 2x}\frac{\cos 2x}{\sin 2x}\,dx$$

$$= 2\int \csc 2x \cot 2x \,dx$$

Remember to divide by the coefficient of x

$$= -\csc 2x + c$$

Exercise 19C

1. Simplify to get standard integrals, and then integrate:

(a) $\int \frac{\tan 3x}{\cos 3x}\,dx$

(b) $\int \frac{1}{\sin^2 x}\,dx$

(c) $\int \sin 5x \cos x - \cos 5x \sin x \,dx$

(d) $\int \frac{3 - \cos 2x}{\sin^2 2x}\,dx$

(e) $\int \frac{\cos 2x}{\cos x + \sin x}\,dx$

2. Use trigonometric identities before using a substitution (or reversing the chain rule) to integrate:

(a) $\int \cos^3 x \sin^2 x \,dx$

(b) $\int \frac{\cos^3 x}{\sin^2 x}\,dx$

(c) $\int \sin x \cos x e^{\cos 2x}\,dx$

(d) $\int \tan^4 3x + \tan^6 3x \,dx$

(e) $\int \frac{\sin 2x \cos 2x}{\sqrt{1 + \cos 4x}}\,dx$

3. Find the following integrals:

(a) (i) $\int 2\cos^2 x \,dx$ (ii) $\int \cos^2 3x \,dx$

(b) (i) $\int 2\tan^2\left(\frac{x}{2}\right)dx$ (ii) $\int \tan^2 3x \,dx$

4. Find the exact value of the following:

(a) (i) $\int_0^{\pi} \sin^2 2x \, dx$ (ii) $\int_0^{2\pi} \tan^2\left(\frac{x}{6}\right) dx$

(b) (i) $\int_0^{\pi/4} (\tan x - 1)^2 \, dx$ (ii) $\int_{\pi/4}^{\pi} (1+\cos 2x)^2 \, dx$

5. Three students integrate $\cos x \sin x$ in three different ways:

Amara uses reverse chain rule with $u = \sin x$:

$$\frac{du}{dx} = \cos x, so$$

$$\int \cos x \sin x \, dx = \int u \, du$$

$$= \frac{1}{2}\sin^2 x + c$$

Ben uses reverse chain rule with $u = \cos x$:

$$\frac{du}{dx} = -\sin x, \text{ so}$$

$$\int \cos x \sin x \, dx = \int -u \, du$$

$$= -\frac{1}{2}\cos^2 x + c$$

Carlos uses a double angle formula:

$$\int \cos x \sin x \, dx = \int \frac{1}{2}\sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + c$$

Who is right?

6. Find $\int \sin^2\left(\frac{x}{3}\right) dx$. *[5 marks]*

7. (a) Show that $\tan^3 x = \tan x \sec^2 x - \tan x$.

(b) Hence find $\int \tan^3 x \, dx$. *[6 marks]*

8. Given that $\int_0^{\pi/12} \tan^2(kx) \, dx = \frac{4-\pi}{12}$ find the value of k. *[6 marks]*

9. (a) Use the formula for $\cos(A+B)$ to show that $\cos 2x = 2\cos^2 x - 1$

(b) Hence find $\int \cos 2x \sin x \, dx$. *[7 marks]*

10. (a) Show that $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$.

(b) Hence find the exact value of $\int_0^{3\pi} \sin^3\left(\frac{x}{3}\right) dx$. *[7 marks]*

11. A complex number is defined by $z = \cos\theta + i\sin\theta$.

(a) (i) Show that $\frac{1}{z} = \cos\theta - i\sin\theta$.

(ii) Use De Moivre's Theorem to deduce that:

$z^n - \frac{1}{z^n} = 2i\sin n\theta$.

(b) (i) Expand $\left(z - \frac{1}{z}\right)^5$.

(ii) Hence find integers a, b and c such that:

$16\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$.

(c) Find $\int \sin^5 2x \, dx$. *[14 marks]*

19D Integration using inverse trigonometric functions

In the last section we saw examples of similar-looking integrals that required very different methods: $\int \sin^2 x \cos x \, dx$ could be done by reversing the chain rule. But $\int \sin^2 x \, dx$ required the more complex method of substituting with a trigonometric identity; without the derivative of $\sin x$ the integration was more difficult.

Similarly, consider $\int \frac{x}{\sqrt{1-x^2}} dx$ and $\int \frac{1}{\sqrt{1-x^2}} dx$.

The first integral features a function $(1-x^2)$ and a multiple of its derivative $(-2x)$, so we can apply the reverse chain rule: $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + c$. However, the absence of the derivative of $(1-x^2)$ in the second integral means that we need another method.

Fortunately, we have already met the expression $\frac{1}{\sqrt{1-x^2}}$ in chapter 18 as the derivative of $\arcsin x$. In the same chapter we saw that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. This means that:

These results are given in Key point 18.8.

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \text{ and } \int \frac{1}{1+x^2} dx = \arctan x.$$

We can extend these results slightly to include expressions of the form $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$. For example, to integrate $\frac{1}{16+x^2}$ we can take out a factor of $\frac{1}{16}$ to turn the denominator into the form $1+Y^2$ and then use a substitution:

$$\begin{aligned}\int \frac{1}{16+x^2}\,\mathrm{d}x &= \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,\mathrm{d}x\\ &= \frac{1}{16}\int \frac{1}{1+Y^2}\times 4\mathrm{d}Y \text{ where } Y=\frac{x}{4}\\ &= \frac{1}{4}\int \frac{1}{1+Y^2}\,\mathrm{d}Y\\ &= \frac{1}{4}\arctan Y + c\\ &= \frac{1}{4}\arctan\left(\frac{x}{4}\right)+c\end{aligned}$$

We can use this method to obtain the general result for the two integrals:

KEY POINT 19.7

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right)+c \quad (|x|<a)$$

It is worth noting several things about these; first of all, the arctan result has a factor of $\frac{1}{a}$ and the arcsin one does not. Secondly, the arcsin result only applies when $|x|<a$ because of the presence of the square root. Finally, you may be wondering why there is no corresponding result with arccos; if you look back at Key point 18.7 you will see that the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1-x^2}}$, which, according to the Key point above, would integrate to $-\arcsin x$. This is because the graphs of the arcin and the arccos functions are related through a reflection and a translation.

You may wonder whether there are rules for integrating $\frac{1}{\sqrt{x^2-a^2}}$ when $|x|>a$ or $\frac{1}{\sqrt{x^2+a^2}}$. These require the study of hyperbolic functions, which are in many ways similar to trigonometric functions, and can be used to describe some important curves, such as the shape of a hanging chain.

Worked example 19.12

Find the value of: $\int \frac{1}{1+9x^2}\,dx$

This is similar to the derivative of arctan x but x^2 has been replaced by $9x^2 = (3x)^2$
So reverse the standard derivative, remembering to divide by the coefficient of x

(a) $\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$

$= \frac{1}{3}\arctan(3x) + c$

When working through an integration, we may first need to put the expression into the correct form. As there is an x^2 term in the derivatives of both arcsin x and arctan x, this often involves **completing the square**. (For a reminder of this term, see glossary on CD-ROM.)

Worked example 19.13

Find $\int \frac{3}{\sqrt{-4x^2-4x+8}}\,dx$.

This is not a reverse chain rule integral and there is a square root in the denominator, so perhaps arcsin? The only way of producing $\sqrt{1-X^2}$ in the denominator is to start by completing the square to get $\sqrt{C-X^2}$

$\int \frac{3}{\sqrt{-4x^2-4x+8}}\,dx = \int \frac{3}{\sqrt{-(4x^2+4x-8)}}\,dx$

$= \int \frac{3}{\sqrt{-\left[(2x+1)^2-1-8\right]}}\,dx$

$= \int \frac{3}{\sqrt{9-(2x+1)^2}}\,dx$

Now reverse the standard derivative, remembering to divide by the coefficient of x which is 2

$= 3\int \frac{1}{\sqrt{3^2-(2x+1)^2}}\,dx$

$= \frac{3}{2}\arcsin\left(\frac{2x+1}{3}\right) + c$

Exercise 19D

1. Find the following:

(a) (i) $\int \frac{1}{1+2x^2}\,dx$ (ii) $\int \frac{1}{1+5x^2}\,dx$

(b) (i) $\int \frac{1}{\sqrt{1-3x^2}}\,dx$ (ii) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

(c) (i) $\int \frac{9}{x^2+9}\,dx$ (ii) $\int \frac{10}{x^2+10}\,dx$

(d) (i) $\int \frac{2}{\sqrt{25-x^2}}\,dx$ (ii) $\int \frac{5}{\sqrt{4-x^2}}\,dx$

2. By first completing the square, find the following:

(a) (i) $\int \frac{1}{x^2+4x+5}\,dx$ (ii) $\int \frac{1}{x^2-6x+10}\,dx$

(b) (i) $\int \frac{1}{\sqrt{8x-x^2-15}}\,dx$ (ii) $\int \frac{1}{\sqrt{2x-x^2}}\,dx$

(c) (i) $\int \frac{6}{x^2+10x+27}\,dx$ (ii) $\int \frac{5}{\sqrt{-4x^2-12x}}\,dx$

3. Find the exact value of $\int_0^{\sqrt{3}/2} \frac{3}{1+4x^2}\,dx$. *[4 marks]*

4. (a) Write $2x^2+4x+11$ in the form $2(x+p)^2+q$.

(b) Hence find $\int \frac{3}{2x^2+4x+11}\,dx$. *[5 marks]*

5. (a) Write $1+6x-3x^2$ in the form $a^2-3(x-b)^2$.

(b) Hence find the exact value of $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx$. *[5 marks]*

6. Show that $\int_3^{5.5} \frac{10}{4x^2-24x+61}\,dx = \frac{\pi}{4}$. *[6 marks]*

7. By using the substitution $u = e^x$, find the exact value of

$$\int_0^{\frac{1}{2}\ln 3} \frac{1}{e^x+e^{-x}}\,dx.$$ *[6 marks]*

19E Other strategies for integrating quotients

We now have a number of techniques to use when integrating a quotient of functions; we can reverse a standard derivative, apply the reverse chain rule or use an inverse trigonometric function. However, there are occasions when none of these methods seem to be useful. In such cases our only remaining option is to split the fraction into two separate fractions, each of which we *can* solve.

The easiest way to do this is to split the numerator.

Worked example 19.14

Find $\int \frac{x+1}{\sqrt{1-x^2}}\,dx$.

Split the fraction into two

$$\int \frac{x+1}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{1}{\sqrt{1-x^2}}dx$$

The first part can be integrated using the reverse chain rule or the substitution $u = 1 - x$

First integral:

Let $u = 1 - x^2$, so $\frac{du}{dx} = -2x \Rightarrow dx = -\frac{1}{2x}du$

$$\therefore \int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{x}{\sqrt{u}}\frac{1}{2x}du$$

$$= \int -\frac{1}{2}u^{-\frac{1}{2}}du$$

$$= -\sqrt{u} + c = -\sqrt{1-x^2} + c$$

We recognise the second part as the derivative of arcsin

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + c$$

$$\therefore \int \frac{x+1}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + \arcsin x + c$$

Sometimes it is not obvious how to split the numerator.

Worked example 19.15

Find $\int \frac{4x+19}{x^2+12x+41}\,dx$.

No obvious options, so split the fraction into two

However, simply writing this as:

$$\frac{4x}{x^2+12x+41} + \frac{19}{x^2+12x+41}$$

is no use as we still can't integrate

Instead, make the numerator of the first fraction a multiple of the derivative of the denominator in order to apply the reverse chain rule and then hope to be able to deal with the resulting second fraction

$$\int \frac{4x+19}{x^2+12x+41}dx$$

$$= \int \frac{4x+24}{x^2+12x+41}dx - \int \frac{5}{x^2+12x+41}dx$$

continued . . .

This works; second fraction can be integrated with the arctan function (after completing the square)

$$= \int \frac{4x+24}{x^2+12x+41}\,dx - \int \frac{5}{(x+6)^2+5}\,dx$$

$$= 2\int \frac{2x+12}{x^2+12x+41}\,dx - \int \frac{1}{\left(\frac{x+6}{\sqrt{5}}\right)^2+1}\,dx$$

Apply reverse chain rule to the first fraction (it is of the form $\frac{u'}{u}$) and arctan to the second

$$= 2\ln(x^2+12x+41) - \sqrt{5}\arctan\left(\frac{x+6}{\sqrt{5}}\right) + c$$

We should always check whether the fraction can be simplified before trying to split the numerator.

Worked example 19.16

Integrate $\int \frac{x+4}{12-5x-2x^2}\,dx$.

Check whether the polynomial factorises

$$\int \frac{x+4}{12-5x-2x^2}\,dx = \int \frac{x+4}{(3-2x)(x+4)}\,dx$$

$$= \int \frac{1}{3-2x}\,dx$$

We now have a standard integral, just remember to divide by the coefficient of x

$$= -\frac{1}{2}\ln|3-2x| + c$$

You will need the method of comparing coefficients from Section 3A.

The final type of functions to consider are improper fractions. These can be integrated by splitting them into a polynomial plus a proper fraction.

Worked example 19.17

(a) Find constants A, B and C such that:

$$\frac{x^2+5}{x+2} = Ax + B + \frac{c}{x+2}$$

(b) Hence find $\int \frac{x^2+5}{x+2}\,dx$.

We can multiply both sides by $x + 2$ to get rid of fractions

(a) $x^2 + 5 = (Ax + B)(x + 2) + C$

Setting $x = -2$ will eliminate the first term on the right, so we can find C

When $x = -2$:

$(-2^2) + 5 = (-2A + B)(0)$

$\therefore C = 9$

To find A and B we need to expand the brackets and compare coefficients

$x^2 + 5 = Ax^2 + 2Ax + Bx + 2B + 9$

x^2 terms: $1 = A$

x terms: $0 = 2A + B = 2 + B \therefore B = -2$

So

$$\frac{x^2+5}{x+2} = x - 2 + \frac{9}{x+2}$$

The result from part (a) allows us to use standard integrals

(b) $\int \frac{x^2+5}{x+2}\,dx = \int x - 2 + \frac{9}{x+2}\,dx$

$= \frac{1}{2}x^2 - 2x + 9\ln|x+2| + C$

Exercise 19E

1. By first simplifying, find:

(a) $\int \frac{(4x^2-9)^2}{(2x+3)^2}\,dx$ (b) $\int \frac{x+3}{6-13x-5x^2}\,dx$

2. Find the following by splitting the numerator:

(a) $\int \frac{5x+1}{x^2+6}\,dx$ (b) $\int \frac{x-3}{\sqrt{4-x^2}}\,dx$

(c) $\int \frac{8x+23}{x^2+8x+25}\,dx$ (d) $\int \frac{x-5}{\sqrt{-x^2+6x-7}}\,dx$

3. Find the following by splitting into a polynomial and a proper fraction:

(a) (i) $\int \frac{x+1}{x+2}\,dx$ (ii) $\int \frac{2x+3}{x-1}\,dx$

(b) (i) $\int \frac{x^2+2}{x-3}\,dx$ (ii) $\int \frac{x^2+2x-1}{x+5}\,dx$

(c) $\int \frac{x^2+5x+1}{x^2+3}\,dx$

4. (a) Show that $\frac{1}{x-2} = \frac{1}{x+3} = \frac{5}{x^2+x-6}$.

(b) Hence find $\int \frac{5}{x^2+x-6}\,dx$ giving your answer in the form $\ln(f(x))+c$. *[5 marks]*

5. Find the exact value of $\int_0^2 \frac{4}{x^2+4}\,dx$. *[4 marks]*

6. (a) Show that $\frac{5-x}{2+x-x^2} = \frac{1}{2-x} + \frac{2}{1+x}$.

(b) Given that $\int_0^1 \frac{5-x}{2+x-x^2}\,dx = \ln k$, find the value of k. *[6 marks]*

7. Find $\int \frac{4x+5}{\sqrt{1-x^2}}\,dx$. *[5 marks]*

8. (a) Write $2x^2-8x+17$ in the form $a(x-p)^2+q$.

(b) Hence find $\int \frac{2x+8}{2x^2-8x+17}\,dx$. *[7 marks]*

19F Integration by parts

In Section B we saw cases where we could integrate products of functions by using the reverse chain rule or a substitution, but we cannot yet solve integrals such as $\int x\sin x\,dx$ or $\int x^2e^x\,dx$.

In order to deal with these we return to the product rule for differentiation (Key point 18.4).

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating with respect to x we get:

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

KEY POINT 19.8

The **integration by parts** formula.

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

When using integration by parts, the challenge is deciding which of the functions is to be u and which $\frac{dv}{dx}$. The aim is to select them so that the product $v\frac{du}{dx}$ is easier to integrate than the original product. This often (but not always) means that you choose u to be a polynomial.

Worked example 19.18

Find $\int x\sin x\,dx$.

This is a product to which we can't apply the reverse chain rule, so try integration by parts
Choose u to be the polynomial part

$u = x$ and $\frac{dv}{dx} = \sin x$

$\Rightarrow \frac{du}{dx} = 1$ and $v = -\cos x$

Apply the formula

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int x\sin x\,dx = x(-\cos x) - \int(-\cos x)1\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

It may be necessary to use integration by parts more than once. As long as the integrals are becoming simpler each time, you are on the right track! The next example also shows you how to deal with the limits.

Worked example 19.19

Find the exact value of $\int_0^{\ln 2} x^2 e^x \, dx$.

This is a product to which we can't apply the reverse chain rule, so try integration by parts. Choose the polynomial as u

$u = x^2$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2x$ and $v = e^x$

Apply the formula. Put in the limits on the uv part straight away

$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int_0^{\ln 2} x^2 e^x \, dx = \left[x^2 e^x\right]_0^{\ln 2} - \int_0^{\ln 2} 2xe^x \, dx$

We have to integrate a product again, so use integration by parts again. Choose u to be the polynomial again (If we used $u = e^x$ and $\frac{dv}{dx} = 2x$ we would end up back where we started!)

$u = 2x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2$ and $v = e^x$

Apply the formula again and use the limits

So,

$\int_0^{\ln 2} 2xe^x \, dx = \left[2xe^x\right]_0^{\ln 2} - \int_0^{\ln 2} 2e^x \, dx = \left[2xe^x\right]_0^{\ln 2} - \left[2e^x\right]_0^{\ln 2}$

Put both integrals together, making sure to keep track of negative signs by using brackets appropriately

Therefore,

$$\int_0^{\ln 2} x^2 e^x \, dx = \left[x^2 e^x\right]_0^{\ln 2} - \left\{\left[2xe^x\right]_0^{\ln 2} - \left[2e^x\right]_0^{\ln 2}\right\}$$

$$= \left((\ln 2)^2 e^{\ln 2} - 0\right) - \left(2\ln 2 e^{\ln 2} - 0\right) + \left(2e^{\ln 2} - 2\right)$$

$$= 2(\ln 2)^2 - 4\ln 2 + 2$$

Sometimes it seems that we are getting nowhere, as the new integral resulting from integration by parts is no easier than the original. However, as long as things are not getting worse, they will eventually get better as the following example shows.

Worked example 19.20

Use integration by parts to find $\int e^x \cos x \, dx$.

This is one of the rare occasions when it makes no difference which way round we choose u and $\frac{dv}{dx}$

$u = \cos x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = -\sin x$ and $v = e^x$

Applying the formula, the new integral is no better but also no worse than the original

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

$$\int e^x \cos x \, dx = \cos x \, e^x - \int(-\sin x)e^x \, dx$$

$$= \cos x e^x + \int e^x \sin x \, dx$$

With no other option, we proceed with a second integration by parts

We need to be consistent and choose u and $\frac{dv}{dx}$ in the same way as before to avoid undoing what we have just done

$u = \sin x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = \cos x$ and $v = e^x$

It seems that no progress has been made as we have ended up with the integral we started with (except with a different sign)

So,

$$\int e^x \sin x \, dx = \sin x \, e^x - \int \cos x e^x \, dx$$

However, when we put everything together it becomes apparent that the different sign allows us to rearrange and find an expression for $\int e^x \cos x \, dx$

Therefore,

$$\int e^x \cos x \, dx = \cos x \, e^x + \left\{\sin x \, e^x - \int \cos x e^x \, dx\right\}$$

$$\Rightarrow 2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2}(\cos x + \sin x) + c$$

We are able to differentiate and integrate e^x but so far we have only been able to *differentiate* ln x.

In order to be able to integrate ln x, we can use integration by parts. This might not seem an obvious method at first because there is no product of functions here, but with a little creativity we can proceed.

Worked example 19.21

Find $\int \ln x \, dx$.

The seemingly trivial step of writing ln x as the product of 1 and ln x sets up integration by parts

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$

Cannot integrate $\ln x$ so let $u = \ln x$

$$u = \ln x \quad \text{and} \quad \frac{dv}{dx} = 1$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad v = x$$

Apply the parts formula

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\int 1 \times \ln x \, dx = (\ln x)x - \int \frac{1}{x} x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

Although this example only shows how to integrate ln x, in most other cases of integration by parts involving ln x we would still let ln $x = u$. The choices for u and v in the common cases are summarised below.

KEY POINT 19.9

When integrating $\int x^n f(x) \, dx$ by parts, choose $u = x^n$ in all cases except when $f(x) = \ln x$.

Exercise 19F

1. Use integration by parts to find the following:

(a) (i) $\int x\cos 2x \, dx$ (ii) $\int x\sin\left(\frac{x}{2}\right) dx$

(b) (i) $\int 4xe^{-2x} \, dx$ (ii) $\int xe^{4x} \, dx$

(c) (i) $\int 2x\ln 5x \, dx$ (ii) $\int x\ln x \, dx$

(d) (i) $\int x^2\cos 3x \, dx$ (ii) $\int x^2\sin x \, dx$

(e) $\int \frac{1}{4}x^2 e^{\frac{x}{4}} \, dx$

(f) $\int \frac{\ln x}{x^3} \, dx$

(g) $\int (\ln x)^2 \, dx$

2. Use integration by parts to find the following:

(a) $\int \arctan x \, dx$ (b) $\int \ln(2x+1) \, dx$

3. Evaluate the following exactly:

(a) $\int_0^{\pi/2} x\cos x \, dx$

(b) $\int_1^2 \frac{\ln x}{x^2} \, dx$

(c) $\int_0^{\pi/3} \sin x \ln(\sec x) \, dx$

4. When using the integration by parts formula, we start with $\frac{dv}{dx}$ and find v. Why do we not include a constant of integration when we do this? Try a few examples adding $+C$ to v and see what happens.

5. Find $\int 2xe^{-3x} \, dx$. *[5 marks]*

6. Evaluate $\int_1^e x^5 \ln x \, dx$. *[6 marks]*

7. (a) Show that $\int \tan x \, dx = \ln|\sec x| + c$.

(b) Hence find $\int \frac{x}{\cos^2 x} \, dx$. *[8 marks]*

8. Find the value of k such that $\int_0^k \arccos x \, dx = 0.5$ *[7 marks]*

9. Use the substitution $\sqrt{x+1} = u$ to find the exact value of $\int_{-1}^{3} \frac{1}{2} e^{\sqrt{x+1}}\, dx$. *[8 marks]*

Summary

- Look for **standard derivatives** before attempting any more complicated methods. They are given in the Formula booklet, but you may need to divide by the coefficient of x.
- If the expression contains both a function and its derivative and the 'inside' function is of the form $(ax + b)$ it is highly likely to be susceptible to the **reverse chain rule** or a substitution.
- **Integration by substitution** can also work in other situations, and you need to be able to use any given substitution. The steps of integration by substitution are given in Key point 19.2.
- When evaluating a definite integral using substitution, see Key point 19.3.
- A particular case of substitution, where the top of the fraction is the derivative of the bottom, is worth remembering:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

- Many integrals involving trigonometric functions can be simplified using identities.

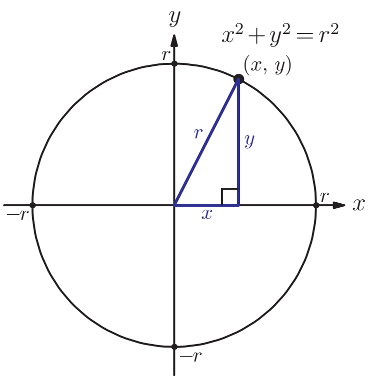
  Particularly useful identities are: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$; $\sin^2 x = \frac{1}{2}(1-\cos 2x)$; $\tan^2 x = \sec^2 x - 1$.

  For example, to integrate $\sin^2 x$, use $\cos 2x = 1 - 2\sin^2 x$; to integrate $\cos^2 x$, use $\cos 2x = 2\cos^2 x - 1$; to integrate $\tan^n x$, use $1 + \tan^2 x = \sec^2 x$ and that $\frac{d}{dx}(\tan x) = \sec^2 x$.
- The integral of tan is worth knowing: $\int \tan x\, dx = \ln|\sec x| + c$.
- Higher powers of sine and cosine can be integrated using De Moivre's Theorem.
- Integration can be done using inverse trigonometric functions: $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$;

  $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + c \; (|x| < a)$.
- It may be necessary to split a fraction that contains two (or more) terms in the numerator into two separate functions before integrating each with the above methods. If the degree of the numerator is at least as large as the degree of the denominator, then write it as a polynomial plus a proper fraction and compare coefficients.
- It may be possible to integrate a product of functions using the **integration by parts** formula:
- $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$
- The challenge with integration questions is often not in carrying out any of the above methods, but actually in selecting the correct method to use in the first place. In the exam you will often, but not always, be told which method to use. For extra practice see the Extension worksheet 17 'Basic integration'.

Introductory problem revisited

Use integration to prove that the area of a circle of radius r is equal to πr^2.

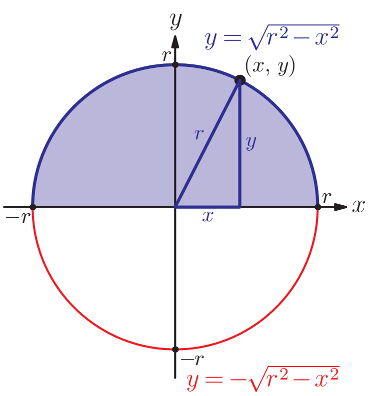


In order to use integration, we need to think of a circle as a graph of a function. We saw at the beginning of chapter 18 that the coordinates of a point on the circle with radius r satisfy the equation $x^2 + y^2 = r^2$. (You are not required to know this equation for the exam!) In chapter 18 we used implicit differentiation to find the gradient; in order to integrate we need an explicit expression for y in terms of x.

There is a small problem: the equation of the circle above is a relation, not a function – the graph of a circle does not pass the vertical line test.

You might find Key point 5.1 from chapter 5 useful here.

We can only integrate functions, but we can avoid the problem by considering only the top half of the circle and then doubling the answer we get for the area.



For the top half of the circle $y > 0$, so $y = \sqrt{r^2 - x^2}$. Now that you have done lots of integration practice, you may suspect that this one needs a substitution and it turns out that a useful substitution is $x = r\cos\theta$. This makes some sense, as we know that trigonometric functions are closely related to circles.

EXAM HINT

This is not one of the standard integrals, so if you have to do it in the exam you should be given a hint.

Now that we have decided on the strategy we can carry out the integration.

Write down the integral to be evaluated

The area of the top half of the circle is given by:

$$\frac{A}{2} = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

State the method to be used

Substitution: $x = r\cos\theta$

continued . . .

Differentiate

$$\frac{dx}{d\theta} = -r\sin\theta$$

$$\Rightarrow dx = -r\sin\theta\, d\theta$$

Express the integrand in terms of θ

$$r^2 - (r\cos\theta)^2 = r^2(1-\cos^2\theta) = r^2\sin^2\theta$$

Notice that $\sin\theta$ is positive on the top half of the circle

$$\Rightarrow \sqrt{r^2 - x^2} = r\sin\theta$$

Change the limits

Limits:

when $x = -r$, $\cos\theta = -1$ so $\theta = \pi$

when $x = r$, $\cos\theta = 1$ so $\theta = 0$

Put everything together

$$\frac{A}{2} = \int_{\pi}^{0} (r\sin\theta)(-r\sin\theta)\, d\theta$$

Remove the minus sign by swapping limits and take the constant outside the integral

$$= r^2\int_0^{\pi} \sin^2\theta\, d\theta$$

Use the double angle formula to integrate $\sin^2\theta$

$$= r^2\int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= r^2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= r^2\left\{\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right] - [0]\right\}$$

$$= \frac{\pi r^2}{2}$$

Hence the area of the whole circle is $2 \times \frac{\pi r^2}{2} = \pi r^2$, as required.

Mixed examination practice 19

Short questions

1. Find the exact value of $\int_0^{\pi} \cos^2(3x)\,dx$. *[6 marks]*

2. Use integration by parts to find $\int x\cos 2x\,dx$. *[6 marks]*

3. Given that $\int_0^m \frac{dx}{3x+1}\,dx = 1$ calculate, to 3 significant figures, the value of m. *[6 marks]*

4. Find the exact value of $\int_0^{\pi/12} \frac{1}{\cos^2 4x}\,dx$. *[5 marks]*

5. Find the following integrals:

(a) $\int \frac{1}{1-3x}\,dx$ (b) $\int \frac{1}{(2x+3)^2}\,dx$ *[6 marks]*

6. Find $\int \ln x\,dx$. *[5 marks]*

7. (a) Simplify $\frac{e^{-4x}+3e^{-2x}}{e^{-4x}-9}$ (b) Hence find $\int \frac{e^{-4x}+3e^{-2x}}{e^{-4x}-9}\,dx$. *[6 marks]*

8. Find $\int \frac{6x+4}{x^2+4}\,dx$. *[5 marks]*

9. (a) Show that $\frac{x+5}{(x+1)(x+2)}$ can be written as $\frac{2}{x-1}-\frac{1}{x+2}$.

(b) Hence find, in the form $\ln k$, the exact value of $\int_5^7 \frac{x+5}{(x-1)(x+2)}\,dx$. *[8 marks]*

10. Find $\int \frac{1}{x\ln x}\,dx$. *[6 marks]*

11. Using the substitution $u = \frac{1}{2}x - 1$, or otherwise, find $\int \frac{x}{\sqrt{\frac{1}{2}x-1}}\,dx$. *[5 marks]*

12. Find the exact value of $\int_2^5 \frac{x-1}{x+2}\,dx$. *[6 marks]*

13. Use integration by parts to find $\int \arctan x\,dx$. *[7 marks]*

14. Given that $\int_{-a}^{a} \frac{2}{1-x^2}\,dx = 1$, find the exact value of a. *[7 marks]*

Long questions

1. (a) Show that $\dfrac{4-3x}{(x+2)(x^2+1)}$ can be written in the form $\dfrac{A}{x+2}+\dfrac{1-Bx}{x^2+1}$ finding the constants A and B.

(b) Hence find $\displaystyle\int \frac{4-3x}{(x+2)(x^2+1)}\,\mathrm{d}x$.

(c) Find the exact value of $\displaystyle\int_0^{\sqrt{3}/2} \frac{4-3x}{\sqrt{1-x^2}}\,\mathrm{d}x$. *[15 marks]*

2. Let $I=\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$ and $J=\displaystyle\int \frac{\cos x}{\sin x+\cos x}\,\mathrm{d}x$.

(a) Find $I+J$.

(b) By using the substitution $u=\sin x+\cos x$, find $J-I$.

(c) Hence find $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$. *[9 marks]*

3. Let $t=\tan\left(\dfrac{x}{2}\right)\mathrm{d}x$.

(a) Find $\dfrac{\mathrm{d}t}{\mathrm{d}x}$ in terms of t.

(b) (i) Show that $\sin 2\theta=\dfrac{2\tan\theta}{\sec^2\theta}$.

(ii) Hence show that $\sin x=\dfrac{2t}{1+t^2}$.

(c) Use the substitution $t=\tan\left(\dfrac{x}{2}\right)$ to evaluate $\displaystyle\int_0^{\pi/2}\frac{1}{1+\sin x}\,\mathrm{d}x$. *[14 marks]*

4. Consider the complex number $z=\cos\theta+\mathrm{i}\sin\theta$.

(a) Using De Moivre's Theorem show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$.

(b) By expanding $\left(z+\dfrac{1}{z}\right)^4$, show that $\cos^4\theta=\dfrac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$.

(c) Let $g(a)=\displaystyle\int_0^a \cos^4\theta\,\mathrm{d}\theta$.

(i) Find $g(a)$.

(ii) Solve $g(a)=1$. *[11 marks]*

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20 Further applications of calculus

In this chapter you will learn:

- to write real world problems as equations involving variables and their derivatives
- how to relate different rates of change
- to apply calculus to problems involving motion (kinematics)
- to find volumes of shapes rotated around an axis
- to maximise or minimise functions with constraints.

Introductory problem

A forest fire spreads in a circle at the speed of 12 km/h. How fast is the area affected by the fire increasing when its radius is 68 km?

Did you know that if you are in a sealed box you cannot measure your velocity but you can measure your acceleration? Or that Newton's second law says that force is the rate of change of momentum? These are two examples where a rate of change is easier to find than the underlying variable. To get from this rate of change to the underlying variable requires the use of integration. This chapter will look at various applications of the calculus you have met in the previous four chapters, with a particular emphasis on real-world applications of rates of change.

20A Related rates of change

When blowing up a balloon we can control the amount of gas in the balloon (V), but we may want to know how fast the radius (r) is increasing. These are two different rates of change, but they are linked – the faster the gas fills the balloon the faster the radius will increase. We need to link two derivatives: $\frac{dV}{dt}$ and $\frac{dr}{dt}$. This is done by using the chain rule and the geometric context.

Worked example 20.1

A spherical balloon is being inflated with air at a rate of 200 cm^3 per minute. At what rate is the radius increasing when the radius is 8 cm?

Define variables

V = volume of air in balloon in cm^3
r = radius of balloon in cm
t = time in minutes

Write the given rate of change and the required rate of change

$$\frac{dV}{dt} = 200$$

$$\frac{dr}{dt} = ?$$

Relate these rates of change using the chain rule

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

So we need to find $\frac{dV}{dr}$

Use geometric context

Since the balloon is spherical, $V = \frac{4}{3}\pi r^3$,

$$\Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

Put into the chain rule

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 200, \ r = 8$$

$$\therefore 200 = 256\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = 0.249 \text{ (3SF)}$$

So radius is increasing at about 0.249 cm/minute

EXAM HINT

Don't use units in the working, as long as the units in the information are consistent. Always give units with your final answer.

The rate required may be linked to several other variables.

Worked example 20.2

As a conical icicle melts the rate of decrease of height h is 1 cm^{-1} and the rate of decrease of the radius of the base, r, is 0.1 cm h^{-1}. At what rate is the volume (V) of the icicle decreasing when the height is 30 cm and the base radius is 4 cm?

Write the given rates of change and the required rates of change
Remember that decrease means negative derivative

$$\frac{dh}{dt} = -1$$

$$\frac{dr}{dt} = -0.1$$

$$\frac{dV}{dt} = ?$$

Use geometry to link the variables

$$V = \frac{1}{3}\pi r^2 h$$

Differentiate both sides with respect to t, requiring the product rule and the chain rule

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2\right)h + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r\frac{dr}{dt}h + \frac{1}{3}\pi r^2\frac{dh}{dt}$$

$$r = 30,\ h = 4,\ \frac{dr}{dt} = -0.1,\ \frac{dh}{dt} = -1$$

Put in given values

$$\therefore \frac{dV}{dt} = \frac{2}{3}\pi \times 4 \times (-0.1) \times 30 + \frac{1}{3}\pi \times 4^2 \times (-1)$$

$$= -41.9 \text{ cm}^3\text{h}^{-1}$$

The volume is decreasing at 41.9 cm^3 per hour

Exercise 20A

1. In each case, find an expression for $\frac{dz}{dx}$ in terms of x.

(a) (i) $z = 4y^2,\ y = 3x^2$ (ii) $z = y^2, y = x^3 + 1$

(b) (i) $z = \cos y,\ y = 3x^2$ (ii) $z = \tan y,\ y = x^2 + 1$

2. (a) (i) Given that $z = y^2 + 1$ and $\frac{dy}{dx} = 5$, find $\frac{dz}{dx}$ when $y = 5$.

(ii) Given that $z = 2y^3$ and $\frac{dy}{dx} = -2$, find $\frac{dz}{dx}$ when $y = 1$.

(b) (i) If $w = \sin x$ and $\frac{dw}{dt} = -3$, find $\frac{dx}{dt}$ when $x = \frac{\pi}{3}$.

(ii) If $P = \tan h$ and $\frac{dP}{dx} = 2$, find $\frac{dh}{dx}$ when $h = \frac{\pi}{4}$.

(c) (i) Given that $V = 12r^3$, $\frac{dr}{dt} = 1$ and $\frac{dV}{dt} = 4$, find the possible values of r.

(ii) Given that $H = 3S^{-2}$, find the value of S for which

$\frac{dH}{dx} = 3$ and $\frac{dS}{dx} = 4$.

3. (a) (i) Given that $V = 3r^2h$, find $\frac{dV}{dt}$ when $r = 3$, $h = 2$, $\frac{dr}{dt} = 2$

and $\frac{dh}{dt} = -1$.

(ii) Given that $N = kx^4$, find $\frac{dN}{dt}$ when

$x = 2$, $k = 5$, $\frac{dk}{dt} = 1$ and $\frac{dx}{dt} = 1$.

(b) (i) Given that $m = \frac{S}{n}$ and that

$S = 100$, $\frac{dS}{dt} = 20$, $n = 50$ and $\frac{dn}{dt} = 4$, find $\frac{dm}{dt}$.

(ii) Given that $\rho = \frac{m}{V}$ and that

$m = 24$, $\frac{dm}{dt} = 2$, $V = 120$ and $\frac{dV}{dt} = 6$, find $\frac{d\rho}{dt}$.

4. A circular stain is spreading so that the radius is increasing at the constant rate of 1.5 cm s^{-1}. Find the rate of increase of the area when the radius is 12 cm. *[5 marks]*

5. The area of a square is increasing at the constant rate of 50 $cm^2\,s^{-1}$. Find the rate of increase of the side of the square when the length of the side is 12.5 cm. *[5 marks]*

6. The surface area of a closed cylinder is given by $A = 2\pi r^2 + 2\pi rh$, where h is the height and r is the radius of the base. At the time when the surface area is increasing at the rate of $20\pi\ \text{cm}^2\ \text{s}^{-1}$ the radius is 4 cm, the height is 1 cm and is decreasing at the rate of $2\ \text{cm s}^{-1}$. Find the rate of change of radius at this time. *[6 marks]*

7. A spherical balloon is being inflated at a constant rate of $500\ \text{cm}^3\ \text{s}^{-1}$. The radius at time t seconds is r cm.

Find the radius of the balloon at the time when it is increasing at the rate of $0.5\ \text{cm s}^{-1}$. *[6 marks]*

8. A ship is 5 km east and 7 km North of a lighthouse. It is moving North at a rate of $12\ \text{kmh}^{-1}$ and East at a rate of $16\ \text{kmh}^{-1}$. At what rate is its distance from the lighthouse changing? *[7 marks]*

20B Kinematics

Kinematics is the study of movement – especially position, speed and acceleration. We first need to define some terms carefully:

Time is normally given the symbol t. We can normally define $t = 0$ at any convenient time.

In a 400 m race athletes run a single lap so, despite running 400 m they have returned to where they started. This distance is how much ground someone has covered, whilst the **displacement** is how far away they are from a particular position. The symbol s is normally used to represent displacement.

The rate of change of displacement with respect to time is called **velocity**, and it is normally given the symbol v.

In the IB you will only have to deal with motion in one dimension. However, motion is often in two or three dimensions. To deal with this requires a combination of vectors and calculus called (unsurprisingly) vector calculus.

KEY POINT 20.1

Velocity is given by: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$.

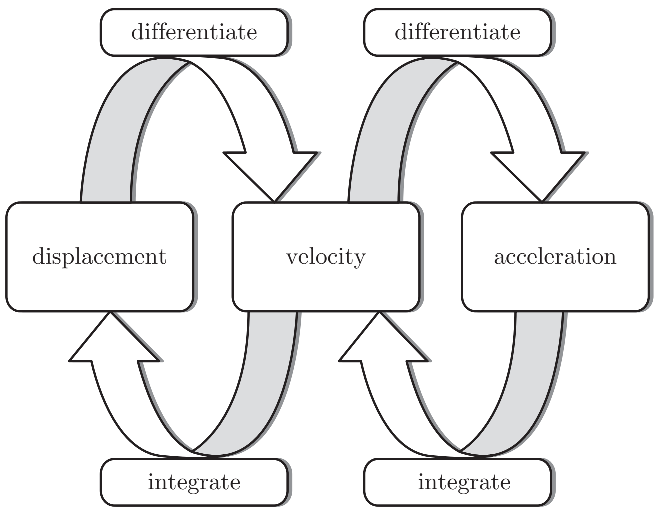
Speed is the magnitude of the velocity: $|v|$.

The rate of change of velocity with respect to time is called **acceleration**, and it is given the symbol a.

KEY POINT 20.2

Acceleration is given by: $a = \frac{\mathrm{d}v}{\mathrm{d}t}$.

To reverse the process – going from acceleration to velocity to displacement – is done by integration.



There is an important difference between finding distance and displacement between times a and b.

See Section 17H for more on the differences between areas and integrals.

KEY POINT 20.3

Displacement is the integral: $\int_a^b v\,\mathrm{d}t$

Distance travelled is the area: $\int_a^b |v|\,\mathrm{d}t$

Worked example 20.3

The velocity (ms^{-1}) of a car at time t seconds after passing a flag is modelled by $v = 17 - 4t$, for $0 \le t < 5$.

(a) What is the initial speed of the car?

(b) Find the acceleration of the car.

(c) What is the maximum displacement of the car from the flag?

continued . . .

(d) Find the distance the car travels.

(a) When $t = 0$, $v = 17\ \text{ms}^{-1}$

(b) $a = \dfrac{dv}{dt} = -4\ \text{ms}^{-2}$

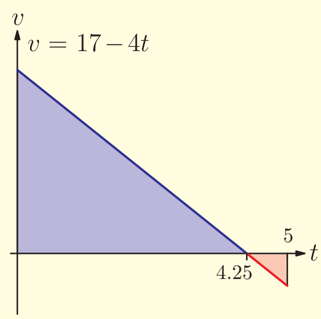
(c) When $v = 0$, $t = 4.25$

Maximum displacement occurs when $\dfrac{ds}{dt} = 0$, which is the same as $v = 0$

$s = \int_0^{4.25} v\,dt = \int_0^{4.25} 17 - 4t\,dt$

$= 36.125m$ (from GDC)

(d)



Distance is actual area between graph and x-axis of v-t graph

$\int_0^5 |v|\,dt = 37.25$ (from GDC)

So total distance is 37.25 m

In all the examples so far, velocity and acceleration were given as functions of time. But there are many practical situations where it is easier to see how velocity depends on the displacement. For example, a speed camera records a car's speed as it passes over certain marks along the road. From this data we may produce an equation for the speed of the car, such as $v = 60 - \sqrt{20000s}$ (v is km and s in km/h). Is it possible to deduce the acceleration of the car from this equation? We know that $a = \dfrac{dv}{dt}$, but as v depends on s we cannot differentiate it with respect to t. Using related rates of change allows us to get around this problem:

KEY POINT 20.4

$$a = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}$$

Worked example 20.4

A car is braking from the initial speed of 60 km/h. The speed of the car depends on the distance travelled since the brakes were applied, and is given by the equation $v = 60 - \sqrt{20000s}$. Find the acceleration of the car after it has been braking for 50 m.

We have v in terms of s and want to find a, so use $a = v\frac{dv}{ds}$

$$a = v\frac{dv}{ds}$$

$$= \left(60 - \sqrt{20\,000s}\right) \times \frac{-1}{2}(20\,000s)^{-\frac{1}{2}} \times 20\,000$$

$$= \frac{-10\,000\left(60 - \sqrt{20\,000s}\right)}{\sqrt{20\,000s}}$$

Remember that s should be in kilometres!

When $s = 0.05$:

$a = -8970 \text{ km/h}^2$

The car is decelerating at 8970 km/h^2.

This is about 0.7 m/s^2

Exercise 20B

1. Find the expressions for the velocity and acceleration in terms of time if the displacement is given by the equation:

   (a) (i) $s = 4e^{-2t}$ (ii) $s = 5 - 2e^{3t}$

   (b) (i) $s = 5\sin\left(\frac{t}{2}\right)$ (ii) $s = 2 - 3\cos(2t)$

2. A particle moves with the given velocity. The particle is at the origin at $t = 0$. Find the displacement in terms of t:

   (a) (i) $v = 3t^2 - 1$ (ii) $v = \frac{1}{2}(1 - t^3)$

   (b) (i) $v = 2e^{-t}$ (ii) $v = 1 + e^{2t}$

   (c) (i) $v = \frac{3}{t+2}$ (ii) $v = 3 - \frac{1}{t+1}$

3. For the given velocity function, find the distance travelled between the given times:

(a) (i) $v = 2e^{-t}$ between $t = 0$ and $t = 2$
(ii) $v = 4(\ln t)^3$ between $t = 2$ and $t = 3$

(b) (i) $v = 1 - 5\cos t$ between $t = 0.2$ and $t = 0.9$
(ii) $v = 2\cos(3t)$ between $t = 1$ and $t = 1.5$

(c) (i) $v = t^2 - 2$ between $t = 0$ and $t = 2.3$
(ii) $v = 5\sin(2t)$ between $t = 0.5$ and $t = 2.5$

4. An object moves in a straight line so that the velocity is a function of the displacement. Find the acceleration of the object for the given value of the displacement or the velocity.

(a) (i) $v = e^{-2s}$, $s = \ln 3$ (ii) $v = 3\sin 2s$, $s = \frac{\pi}{24}$

(b) (i) $v = \frac{s-1}{s+2}$, $v = \frac{2}{5}$ (ii) $v = 3\ln(2s)$, $v = 10$

5. Use integration to derive these constant acceleration formulae for an object moving with constant acceleration a, and initial velocity u, where s is the displacement from the initial position.

(a) $v = u + at$
(b) $s = ut + \frac{1}{2}at^2$
(c) $v^2 = u^2 + 2as$

6. An object moves in a straight line so that its velocity at time t is given by $v = \frac{t}{t^2 + 1}$.

(a) Find an expression for the acceleration of the object at time t.

(b) Given that the object is initially at the origin, find its displacement from the origin when $t = 5$. *[6 marks]*

7. A ball is projected vertically upwards so that its velocity v ms^{-1} at time t s is given by $v = 12 - 9.8t$. Find the distance travelled by the ball in the first 2 seconds of motion. *[5 marks]*

8. The velocity of an object, in ms^{-1}, is given by $v = 5\cos\left(\frac{t}{3}\right)$.

(a) Find the displacement of the object from the starting point when $t = 6$.

(b) Find the total distance travelled by the object in the first 6 seconds. *[6 marks]*

9. The displacement of an object varies with time as

$s = -\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4t$, for $0 \le t \le 5$. Find the maximum velocity of the object. *[5 marks]*

10. An object moves in a straight line so that its velocity, v, is a function of the displacement, s, given by $v = \ln(s+2)$. Find the acceleration of the object when $v = 4$. *[5 marks]*

11. The velocity of an object, in ms^{-1}, is given by $v = \frac{10(s-2)}{s^2+4}$, where s is the displacement in metres.

(a) Find the maximum velocity of the object.

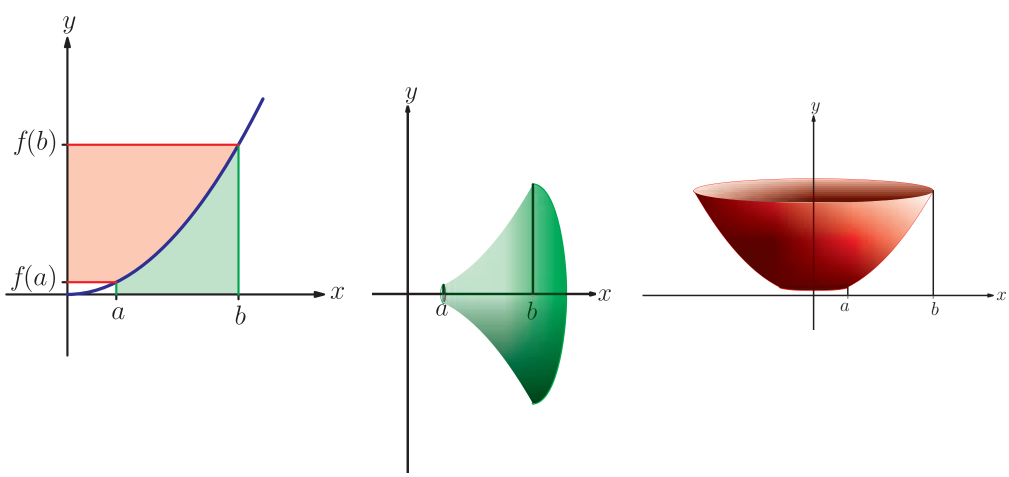
(b) Find the acceleration of the object when $s = 3$. *[6 marks]*

20C Volumes of revolution

You might find Key point 17.8 and Key point 17.9 useful here.

In chapter 17 you saw that the area between a curve and the x-axis from $x = a$ to $x = b$ is given by $\int_a^b y \, dx$ as long as $y > 0$, and also that the area between a curve and the y-axis from $y = c$ to $y = d$ is given by $\int_c^d x \, dy$. In this section we will use a similar formula to find the volume of a shape formed by rotating the curve around either the x-axis or the y-axis.

If a curve is rotated fully around the x-axis or the y-axis the resulting shape is called a **volume of revolution**.



KEY POINT 20.5

The volume of revolution around the x-axis is given by:

$$\int_{x=a}^{x=b} \pi y^2 \, dx$$

The volume of revolution around the y-axis is given by:

$$\int_{y=c}^{y=d} \pi x^2 \, dy$$

EXAM HINT

Notice that the limits use the variable you are integrating with respect to.

The formulae are derived on the Fill-in proof sheet 23 'Volumes of revolution' on the CD-ROM.

Worked example 20.5

The graph of $y = \sin 2x$, $0 \le x \le \frac{\pi}{2}$, is rotated 360° around the x-axis. Find the volume of the solid generated, in terms of π.

Use the formula for the volume of revolution

$$V = \int_0^{\pi/2} \pi (\sin 2x)^2 \, dx$$

$$= \pi \int_0^{\pi/2} \sin^2 2x \, dx$$

Integrate $\sin^2 2x$ using the double-angle formula

$$= \pi \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) \, dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{8}\sin 4x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left\{ \left(\frac{\pi}{4} - \frac{1}{8}\sin 2\pi \right) - \left(0 - \frac{1}{8}\sin 0 \right) \right\}$$

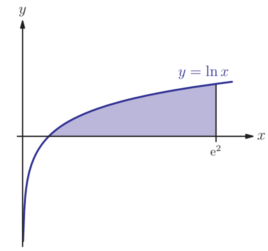
$$= \pi \left(\frac{\pi}{4} - 0 - 0 \right)$$

$$= \frac{\pi^2}{4}$$

The formulae in Key point 20.5 apply when the curve is rotated through a full turn (2π radians) around an axis. You can also form a solid by rotating the curve through a part of the full turn, most commonly π radians (half a turn).

Worked example 20.6

Find the volume of revolution generated when the shaded region is rotated π radians around the y-axis.



First establish limits in terms of y

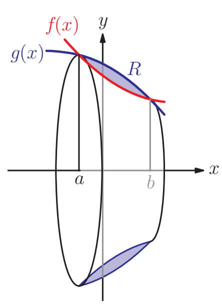
When $x = 1,\ y = 0$
When $x = e^2,\ y = 2$

The volume when rotated by π is half the volume when rotated by 2π

$$V = \frac{1}{2}\int_0^2 \pi x^2 \, dy$$

Rearrange equation of line to get x^2 in terms of y

$$x = e^y$$
$$x^2 = \left(e^y\right)^2 = e^{2y}$$
$$V = \frac{\pi}{2}\int_0^2 e^{2y} \, dy$$
$$= \frac{\pi}{2} \times 53.6 \quad \text{(from GDC)}$$
$$= 84.2 \quad \text{(3SF)}$$



You might also be asked to find a volume of revolution of an area between two curves. We can apply a similar argument to the one we used for areas in Section 17J.

From the diagram we can see that the volume formed when the region R is rotated around the x-axis is given by the volume of revolution of $g(x)$ minus the volume of revolution of $f(x)$.

KEY POINT 20.6

The volume of revolution of the region between curves $g(x)$ and $f(x)$ is

$$\int_a^b \pi\left(g(x)^2 - f(x)^2\right)dx$$

where $g(x)$ is above $f(x)$ and the curves intersect at $x = a$ and $x = b$.

EXAM HINT

Do not fall into the trap of saying that the volume is:

$$\int_a^b \pi\left[g(x) - f(x)\right]^2 dx$$

Worked example 20.7

The region bounded by the curves $y = x^2 + 6$ and $y = 8x - x^2$ is rotated $360°$ about the x-axis.

(a) Show that the volume of revolution is given by $4\pi\int_1^3 13x^2 - 4x^3 - 9\,dx$.

(b) Evaluate this volume, correct to 3 significant figures.

The limits of integration are the intersection points

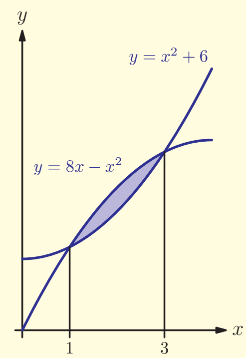
(a) Intersections:

$x^2 + 6 = 8x - x^2$

$2x^2 - 8x + 6 = 0$

$2(x-1)(x-2) = 0$

$x = 1$ or 3



Use $V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$

Draw a sketch to see which curve is above

$$V = \pi\int_1^3 (8x - x^2)^2 - (x^2 + 6)^2\,dx$$

$$= \pi\int_1^3 (64x^2 - 16x^3 + x^4) - (x^4 + 12x^3 + 36)\,dx$$

$$= \pi\int_1^3 52x^2 - 16x^3 - 36\,dx$$

$$= 4\pi\int_1^3 13x^2 - 4x^3 - 9\,dx$$

We can evaluate the integral using GDC

(b) Using GDC,

$V = 184$ (3SF)

There are also formulae for finding the surface area of a solid formed by rotating a region around an axis. Some particularly interesting examples arise if we allow one end of the region to tend to infinity. For example, rotating the region formed by the lines $y = \frac{1}{x}$, $x = 1$ and the x-axis results in a solid called the Gabriel's Horn, or Torricelli's trumpet. Areas and volumes can still be calculated using something called improper integrals, and it turns out that it is possible to have a solid of finite volume but infinite surface area!

EXAM HINT

Notice that the calculation for $\int [f(x)]^2 - [g(x)]^2\,dx$ is easier than doing $\int [f(x)]^2\,dx - \int [g(x)]^2\,dx$

Exercise 20C

1. Find the volume of revolution formed when the curve $y = f(x)$, with $a \le x \le b$, is rotated through 2π radians about the x-axis.

   (a) (i) $f(x) = x^2 + 6, \quad a = -1, \quad b = 3$
   (ii) $f(x) = 2x^3 + 1, \quad a = 0, \quad b = 1$

   (b) (i) $f(x) = e^{2x} + 1, \quad a = 0, \quad b = 1$
   (ii) $f(x) = e^{-x} + 2, \quad a = 0, \quad b = 2$

   (c) (i) $f(x) = \sqrt{\sin x}, \quad a = 0, \quad b = \pi$
   (ii) $f(x) = \sec x, \quad a = -\frac{\pi}{4}, \quad b = \frac{\pi}{4}$

2. The part of the curve $y = g(x)$ with $a \le x \le b$, is rotated 360° around the y-axis. Find the volume of revolution generated, correct to 3 significant figures:

   (a) (i) $g(x) = 4x^2 + 1, \quad a = 0, \quad b = 2$
   (ii) $g(x) = \frac{x^2 - 1}{3}, \quad a = 1, \quad b = 4$

   (b) (i) $g(x) = \ln x + 1, \quad a = 1, \quad b = 3$
   (ii) $g(x) = \ln(2x - 1), \quad a = 1, \quad b = 5$

   (c) (i) $g(x) = \cos x, \quad a = 0, \quad b = \frac{\pi}{2}$
   (ii) $g(x) = \tan x, \quad a = 0, \quad b = \frac{\pi}{4}$

3. The part of the graph of $y = \ln x$ between $x = 1$ and $x = 2e$ is rotated 360° around the x-axis. Find the volume generated. *[4 marks]*

4. The part of the curve $y^2 = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated 2π radians around the x-axis. Find the exact volume of the solid generated. *[4 marks]*

5. The part of the curve $y = \ln(x^2)$ between $x = 1$ and $x = e^2$ is rotated 360° around the y-axis. Find the exact value of the resulting volume or revolution. *[6 marks]*

6. (a) (i) Find an equation of the straight line passing through points $(0, h)$ and $(r, 0)$.

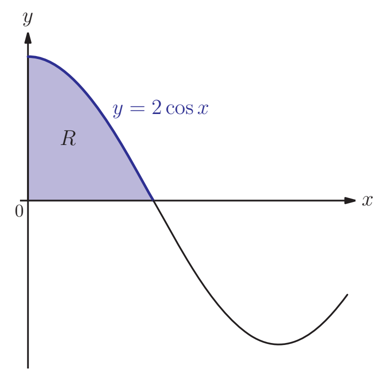
(ii) By finding the volume of revolution formed when the line is rotated around the y-axis, show that the volume of a cone is $\frac{1}{3}\pi r^2 h$.

(b) A circle of radius r and the centre at the origin has equation $x^2 + y^2 = r^2$, where $-r \le x, y \le r$. By rotating the circle around the x-axis prove that the volume of a sphere is $\frac{4}{3}\pi r^3$. *[9 marks]*

7. Region R is bounded by the curve $y = 2\cos x$ and the coordinate axes, as shown in the diagram.

Find the volume generated when R is rotated 2π radians about the y-axis.

[5 marks]



8. Find the exact volume generated when the region between the graph of $y = \sqrt{x}$, the y-axis and the line $y = 3$ is rotated π radians about the the y-axis. *[7 marks]*

9. The part of the curve $y = \frac{3}{\sqrt{x}}$ between $x = 1$ and $x = a$ rotated 2π radians around the x-axis. The volume of the resulting solid is $\pi \ln\left(\frac{64}{27}\right)$. Find the exact value of a. *[7 marks]*

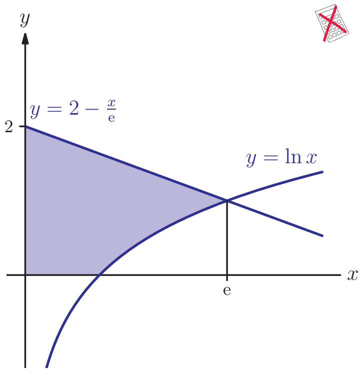
10. The region bounded by the curve $y = e^{2x} - 1$, the y-axis and the line $y = 3$ is rotated π radians around the y-axis. Find the volume of the solid generated. *[5 marks]*

11. (a) Find the coordinates of the points of intersection of the curves $y = 4\sqrt{x}$ and $y = x + 3$.

(b) The region between the curves $y = 4\sqrt{x}$ and $y = x + 3$ is rotated 2π radians around the y-axis. Find the volume of the solid generated. *[7 marks]*

12. (a) Find the coordinates of the points of intersection of curves $y = x^2 + 3$ and $y = 4x + 3$.

(b) Find the volume of revolution generated when the region between the curves $y = x^2 + 3$ and $y = 4x + 3$ is rotated $360°$ around the x-axis. *[7 marks]*



13. The diagram shows the curve $y = \ln x$ and the line $y = -\frac{1}{e}x + 2$. The two graphs intersect at (e, 1). The shaded region is rotated 360° around the y-axis.

Find the exact value of the volume of revolution. *[8 marks]*

In many real world situations we have information about the rate of change of a quantity. These are called differential equations, and they are introduced in Fill-in proof 20 'Fundamental theorem of calculus'.

20D Optimisation with constraints

In this section we shall look at how to maximise or minimise a function that appears to depend upon two different variables. However, these two variables will always be related by a constraint which will allow one of them to be eliminated. We can then follow the normal procedure for finding maxima or minima.

See Section 16J for the procedure for finding maxima and minima.

Worked example 20.8

Find the maximum value of $F = xy - y$ given that $x + 3y = 7$.

Define variables

We wish to maximise $F = xy - y$

Write F in terms of only one variable

$x = 7 - 3y$

$\therefore F = (7 - 3y)y - y$

$= 6y - 3y^2$

Find stationary points

$\Rightarrow \frac{dF}{dy} = 6 - 6y$

But $\frac{dF}{dy} = 0$ at a maximum point

$\therefore 6 - 6y = 0, \ y = 1$

$\Rightarrow x = 7 - 3y = 4$

$\therefore F = 4 \times 1 - 1 = 3$

continued . . .

Classify stationary points

$\frac{d^2F}{dx^2} = -6 < 0$

so $F = 3$ is a local maximum

Check endpoints and asymptotes

There are no asymptotes and when $|y|$ is large F becomes negative so 3 is the global maximum

Sometimes the constraint is not explicitly given, and needs to be deduced from the context. The two common types of constraints are:

- A shape has a fixed perimeter, area or volume – this gives an equation relating different variables (height, length, radius...)
- A point lies on a given curve – this gives a relationship between x and y.

Worked example 20.9

A rectangle has perimeter 100 cm. What is the largest its area can be?

The area of the rectangle is *length* × *width*. Introduce those as variables so we can write equations

Let $x = length$ and $y = width$.
Then $Area = xy$

It is impossible to see from this equation alone what the maximum possible value of the area is. But x and y are related: We can write an equation to express the fact that we know the perimeter

$Perimeter = 2x + 2y = 100$

continued . . .

This means that we can express the area in terms of only one of the variables

$2y = 100 - 2x$

$\Rightarrow y = 50 - x$

$\therefore Area = x(50 - x)$

$= 50x - x^2$

We can now use differentiation to find the maximum point

$A = 50x - x^2$

$\Rightarrow \frac{dA}{dx} = 50 - 2x$

For stationary points:

$\frac{dA}{dx} = 0$

$\Rightarrow 50 - 2x = 0$

$\Rightarrow x = 25$

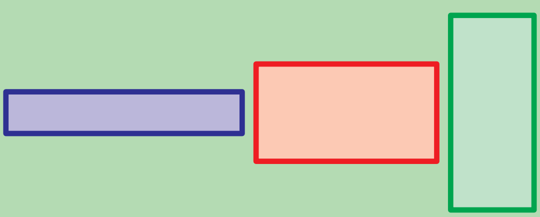
We need to check whether this is a minimum or a maximum by using the second derivative

$\frac{d^2A}{dx^2} = -2 < 0$, so this is a maximum point.

The maximum area is

$A = 50 \times 25 - 25^2 = 625\text{cm}^2$.

It is intuitively clear that a long and thin or a short and wide rectangle will have a very small area, so the largest area should be somewhere in between.



A related problem is finding the minimum possible surface area for an object of a fixed volume. Examples of this can be seen in nature: Snakes have evolved to be long and thin in order to maximise their surface area for heat absorption, while polar bears avoid losing too much heat by adopting a rounder shape which minimises the surface area for their volume.

You may have noticed in the above example that the rectangle with the largest area is actually a square ($x = y = 25$). It turns out that out of all plane shapes with a fixed perimeter, the circle has the largest possible area. This is called 'the isoperimetric problem', and has several intriguing proofs and many applications.

Worked example 20.10

Find the point on the curve $y = x^3$ closest to the point (2, 0).

Define variables

L is the length from the point (2, 0) to the point $P(x, y)$

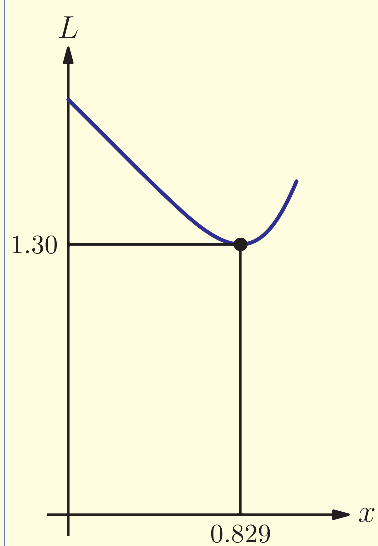
So $L = \sqrt{(x-2)^2 + y^2}$

Write L in terms of only one variable

If P lies on the curve then $y = x^3$

$\therefore L = \sqrt{(x-2)^2 + x^6}$

Find stationary points. This looks complicated and there is no requirement for exact answers so use GDC



From GDC, the minimum is when $x = 0.829$ (3SF) and $y = 0.569$ (3SF).

Exercise 20D

1. (a) (i) Find the maximum value of xy given that $x + 2y = 4$.

(ii) Find the maximum possible value of xy given that $3x + y = 7$.

(b) (i) Find the minimum possible value of $a + b$ given that $ab = 3$ and $a, b > 0$.

(ii) Find the minimum possible value of $2x + y$ given that $ab = 4$ and $a, b > 0$.

(c) (i) Find the maximum possible value of $4r^2h$ if $2r^2 + rh = 3$ and $r, h > 0$.

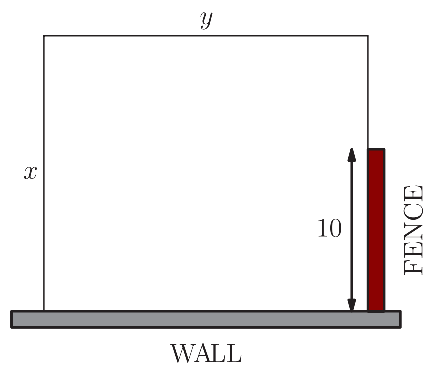
(ii) Find the maximum possible value of rh^2 if $4r^2 + 3h^2 = 12$ and $r, h > 0$.

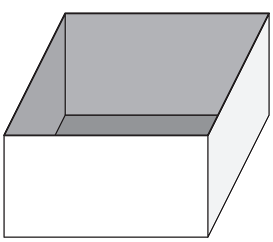
2. A farmer wishes to fence off a rectangular area adjacent to a wall. There is an existing piece of fence, 10 m in length, and perpendicular to the wall, as shown in the diagram.

Let x and y be the dimensions of the enclosure. Given that the length of the new fencing is to be 200 m:

(a) Write down an expression for the area of the enclosure in terms of x only.

(b) Hence find the values of x and y to create the maximum possible area.





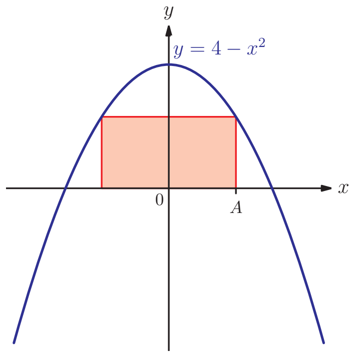
3. A square sheet of card of side 12 cm has four squares of side x cm cut from the corners. The sides are then folded to make a small open box.

(a) Find an expression for the volume of the box in terms of x.

(b) Find the value of x for which the volume is maximum possible, and prove that it is a maximum. *[6 marks]*

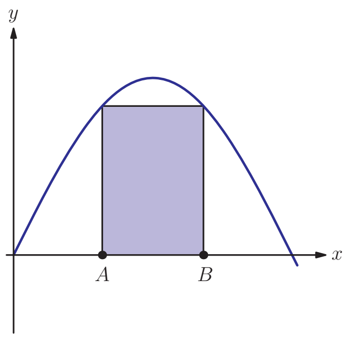
4. An open box in the shape of a square-based prism has volume 32 cm². Find the minimum possible surface area of the box. *[6 marks]*

5. A rectangle is drawn inside the region bounded by the curve $y = 4 - x^2$ and the x-axis, so that two of the vertices lie on the axis and the other two on the curve.



Find the coordinates of vertex A so that the area of the rectangle is a maximum. *[6 marks]*

6. A rectangle is drawn inside the region bounded by the curve $y = \sin x$ and the x-axis, as shown in the diagram. The vertex A has coordinates $(x, 0)$.



(a) (i) Write down the coordinates of point B.

(ii) Find an expression for the area of the rectangle in terms of x.

(b) Show that the rectangle has maximum area when $2\tan x = \pi - 2x$.

(c) Find the maximum possible area of the rectangle.

[8 marks]

7. What is the largest possible capacity of a closed cylindrical cuboid with surface area 450 cm? *[6 marks]*

8. What is the largest possible capacity of a closed square based cuboid with surface area 450 cm? *[6 marks]*

9. The sum of two numbers, x and y, is 6, and $x, y \geq 0$. Find the two numbers if the sum of their squares is:

(a) the minimum possible

(b) the maximum possible. *[7 marks]*

10. A cone of radius r and height h has volume 81π.

(a) Show that the curved surface area of the cone is given by $S = \frac{\pi}{r}\sqrt{r^6 + 243^2}$.

(b) It is required to make the cone so that the curved surface area is the minimum possible. Find the radius and the height of the cone. *[7 marks]*

11. A 20 cm piece of wire is bent to form an isosceles triangle with base b.

(a) Show that the area of the triangle is given by:

$A = \frac{b}{2}\sqrt{100 - 10b}$.

(b) Show that the area of the triangle is the largest possible when the triangle is equilateral. *[6 marks]*

12. The sum of the square of the two positive numbers is a. Prove that their product is the maximum possible when the two numbers are equal. *[6 marks]*

13. Find the coordinates of the point on the curve $y = x^2$, $x \geq 0$, closest to the point (0, 4). *[7 marks]*

Summary

- If there are more than two variables involved in a question, you may need to relate their rates of change using the chain rule, e.g. $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$
- Do not confuse distance (how much ground has been covered) and **displacement** (how far away from a particular position), or **velocity** (rate of change of displacement with respect to time: $v = \frac{ds}{dt}$) and **speed** (magnitude of velocity: $|v|$).
- **Acceleration** is the rate of change of velocity with respect to time: $a = \frac{dv}{dt}$.
- In **kinematics**, differentiate to go from displacement to velocity to acceleration. Integrate to go from acceleration to velocity to displacement.
- The displacement between times a and b is $\int_a^b v \, dt$.
- The distance between times a and b is $\int_a^b |v| \, dt$.
- If the velocity depends on displacement we need to use $a = v\frac{dv}{ds}$.

- If a curve is rotated fully around the x- or y-axis, the resulting shape is called a **volume of revolution**.
- The volume of revolution is given by

$$V = \int_{x=a}^{x=b} \pi y^2 \mathrm{d}x \text{ for rotation around the } x\text{-axis}$$

$$V = \int_{y=c}^{y=d} \pi x^2 \mathrm{d}y \text{ for rotation around the } y\text{-axis}$$

- The volume formed by rotating the region between two curves $g(x)$ and $f(x)$, where $g(x)$ is above $f(x)$ and the curves intersect at $x = a$ and $x = b$, is:

$$\int_a^b \pi\left[g(x)^2 - f(x)^2\right]\mathrm{d}x$$

- When solving optimisation problems that involve a function which depends on two variables, the variables will be related by a constraint that will allow one variable to be eliminated before differentiating to find stationary points. Two common types of constraint are:
  - a shape has a fixed perimeter, area or volume (this gives an equation relating different variables)
  - a point lies on a given curve (this gives a relationship between x and y).

Introductory problem revisited

A forest fire spreads in a circle at the speed of 12 km/h. How fast is the area affected by the fire increasing when its radius is 68 km?

Let r be the radius of the region affected by the fire and let A be its area. We are told that $\frac{\mathrm{d}r}{\mathrm{d}t} = 12$, where t is the time since the start of the fire, measured in hours. We need to find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 68$. To do this, we need to relate the rate of change of A to the rate of change of r.
Using the chain rule:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

Since the region is a circle, we know that $A = \pi r^2$, so $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$. Hence,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times 12 = 24\pi r.$$

When $r = 68$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 5127$, so the area affected by the fire is increasing at the rate of about 5130 km^2/h.

Mixed examination practice 20

Short questions

1. The region bounded by the curve $y = ax - x^2$ is rotated 360° around the x-axis. Find, in terms of a, the resulting volume of revolution. *[7 marks]*

2. An object moves in a straight line so that its velocity, in ms^{-1} is given by $v = t^3 - 6t^2 + 8t$, where t is measured in seconds.

(a) Find the displacement from the initial position when $t = 5$.

(b) Find the total distance travelled in the first 5 seconds. *[6 marks]*

3. The sum of the squares of two positive numbers is 32. Find the two numbers so that their sum is the maximum possible. *[6 marks]*

4. A circular stain is spreading so that the rate of increase of radius is inversely proportional to the square root of the radius. Initially, the radius of the stain is 4 cm and it is increasing at the rate of 2 cms^{-1}. Find the radius of the stain at the time when its area is increasing at the rate of 115 cm^2s^{-1}. *[6 marks]*

5. An object moves in a straight line so that its displacement, s, is given by the equation $s = 3e^{-t}\sin t$, where t is time.

(a) Calculate the velocity of the object when $t = 3$.

(b) Sketch the graph of $v(t)$ for $0 \leq t \leq 3$. *[6 marks]*

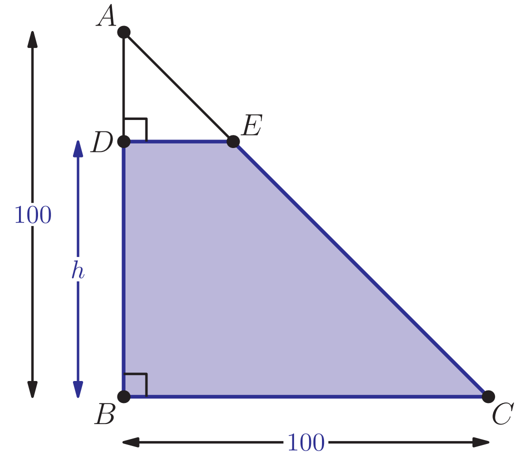
6. The diagram shows an isosceles right-angled triangle of side 100 cm. Point D is moving along the side AB towards point B so that the area of the trapezium $DBCE$ is decreasing at the constant rate of $18\,\text{cm}^3\text{s}^{-1}$. Let $BD = h$.

(a) Write down an expression for the area of the trapezium $DBCE$ in terms of h.

(b) Show that $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{18}{h-100}$.

Initially point D is at vertex A.

(c) Given that $h = 100 - k\sqrt{t}$, find the value k. *[8 marks]*



7. An aeroplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level.

At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the aeroplane is moving towards the observer. *[6 marks]*

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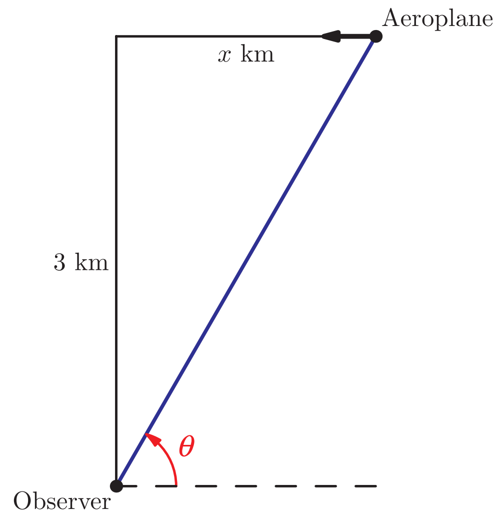
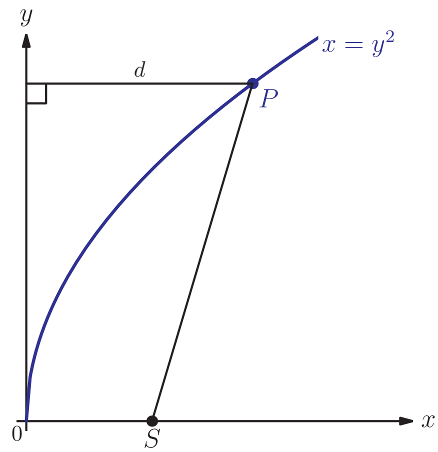


8. The diagram shows a part of the curve with equation $x = y^2$ and a fixed point $S(1, 0)$. Point P lies on the curve and has y-coordinate k ($k \geq 0$).

Let d denote the distance of P from the y-axis, and let r denote the ratio $\dfrac{d}{SP}$

(a) Show that $r = \dfrac{k^2}{\sqrt{k^4 - k^2 + 1}}$. *[7 marks]*

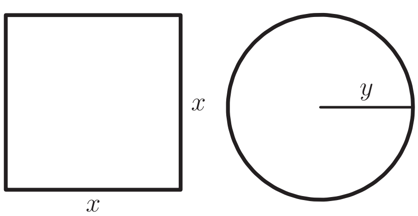
(b) Find the maximum possible value of r.



9. The acceleration of an object depends on its velocity as $a = \dfrac{v^2 + 4}{2v}$. The initial velocity is 3. Show that $v^2 = 13e^t - 4$. *[6 marks]*

Long questions

1. The diagram shows a square with side x cm and a circle with radius y cm.



(a) Write down an expression for the perimeter:

(i) of the square (ii) of the circle

The two shapes are made out of a piece of wire of total length 8 cm.

(b) Find an expression for x in terms of y.

(c) Show that the total area of the two shapes is given by:

$$A = \frac{\pi}{4}(\pi + 4)y^2 - 2\pi y + 4$$

(d) If the total area of the two shapes is the minimum possible, what percentage of the wire is used for the circle? *[10 marks]*

2. Consider two curves with equations $y = x^2 - 8x + 12$ and $y = 12 + x - x^2$ for $0 \le x \le 5$.

(a) Write down the coordinates of the points of intersection of the two curves.

(b) Find the greatest vertical distance between the two curves.

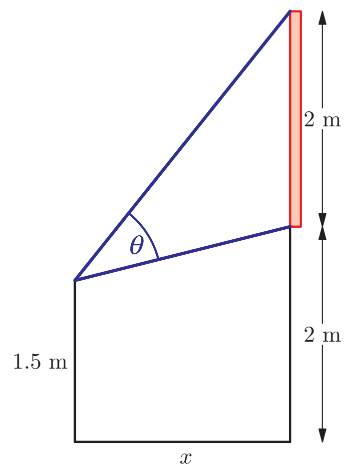
(c) The region between the curves is rotated 360° around the x-axis.

(i) Write down an expression for the volume of the solid generated.

(ii) Evaluate the volume, giving your answer to the nearest integer.

[10 marks]

3. A painting of height 2 m is hanging on the wall of an art gallery so that the bottom of the painting is 2 m above the floor. A visitor is sitting on a stool so that his eyes are at the height of 1.5 m. The stool is at the distance x m from the wall.



(a) Show that the angle at which the visitor sees the painting is:

$$\theta = \arctan\frac{2.5}{x} - \arctan\frac{0.5}{x}$$

(b) Find how far from the wall the stool should be placed so that the painting appears as large as possible. Give your answer in the form $\frac{\sqrt{p}}{q}$, where p and q are integers.

[9 marks]

4. (a) Show that $\int \ln x \, dx = x\ln x - x + c$.

(b) An object is initially at the origin, and moves with velocity $v = 3\ln(t+1)$

(i) Find the acceleration of the object after 5 seconds.

(ii) Find an expression for the displacement in terms of t.

(iii) Find the distance travelled by the object in the first 5 seconds.

(c) A second object has velocity given by $v = 8 - t$. It is also initially at the origin.

(i) The second object has greater velocity for $0 \le t \le a$. Find the value of a.

(ii) Find the greatest speed of the second object during the first 20 seconds.

(iii) After how long have the two objects travelled the same distance?

[16 marks]

5. Triangle ABC is made out of a piece of elastic string. Vertices A and B are being pulled apart so that the length of the base AB is increasing at the rate of 3 cms^{-1} and the height, h, is decreasing at the rate of 2 cms^{-1}. Initially, $AB = 20$ cm and $h = 30$ cm.

(a) Show that $AB = 20 + 3t$.

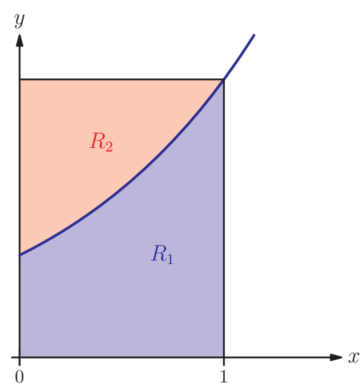
(b) Find an expression for h in terms of t.

(c) Find an expression for the rate of change of the area of the triangle in terms of t. *[12 marks]*

(d) Find the rate at which the area of the triangle is changing when $AB = 26$ and $h = 26$.

6. (a) Use integration by parts to show that $\int (\ln x)^2 \, dx = x\left((\ln x)^2 - \ln x + 1\right)$.

(b) Consider the graph of $y = e^x$ between $x = 0$ and $x = 1$. Regions R_1 and R_2 are defined as shown on the diagram. Region R_1 is rotated around the x-axis and region R_2 is rotated around the y-axis to form volumes V_1 and V_2 respectively. Find the exact value of the ratio $\frac{V_1}{V_2}$. *[14 marks]*



7. Particle A moves in a straight line, starting from O_A, such that its velocity in metres per second for $0 \le t \le 9$ is given by:

$$v_A = -\frac{1}{2}t^2 + 3t + \frac{3}{2}$$

Particle B moves in a straight line, starting from O_B, such that its velocity in metres per second for $0 \leq t \leq 9$ is given by:

$$v_B = e^{0.2t}$$

(a) Find the maximum value of v_A, justifying that it is a maximum.

(b) Find the acceleration of B when $t = 4$.

The displacements of A and B from O_A and O_B respectively, at time t are s_A metres and s_B metres. When $t = 0$, $s_A = 0$ and $s_B = 5$.

(c) Find an expression for s_A and for s_B, giving your answers in terms of t.

(d) (i) Sketch the curves of s_A and s_B, on the same diagram.

(ii) Find the values of t at which $s_A = s_B$.

[23 marks]

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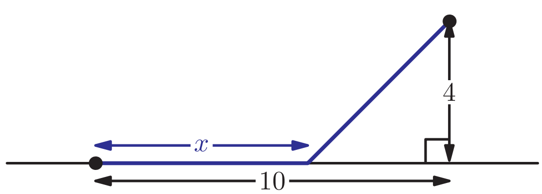
8. John needs to get from his house, which is on the main road, to his friend's house, which is in the field 10 km along the road and 4 km away from the road, as shown in the diagram. John can either cycle along the road, at the speed of 10 kmh^{-1} or walk through the field, at the speed of 5 kmh^{-1}.



John decides to cycle for the first x km and then walk the rest of the way in a straight line.

(a) Show that the time it takes John to get to his friend's house is given by:

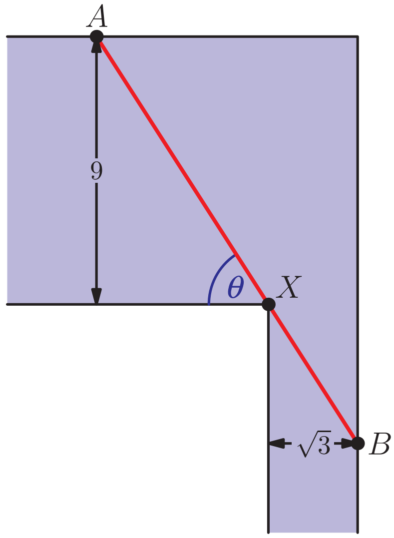
$$T = \frac{x}{10} + \frac{1}{5}\sqrt{16 + (10 - x)^2}$$

(b) John wishes to get to his friend's house in the shortest possible time.

(i) Show that the distance, x, he should cycle satisfies $3(10 - x)^2 = 16$.

(ii) Hence find how far John should cycle. *[10 marks]*

9. A ladder is carried around a corner from a corridor of width 9 m into a corridor of width $\sqrt{3}$ m as shown in the diagram.



(a) AXB is a straight line making angle θ with the first corridor, as shown.

(i) Write AX and XB in terms of θ.

(ii) Find the minimum length of AB.

(b) Find the maximum length of a ladder that can be around the corner.

[8 marks]

21 Summarising data

In this chapter you will learn:

- about the difference between a sample and the entire set of data
- about different types of data
- different ways to measure how spread out the data is
- how to work with data that has been summarised

Introductory problem

The magnetic dipole of an electron is measured in a very sensitive experiment 3 times. The values are 2.000 001 5, 2.000 000 12 and 2.000 000 9. Does this support the theory that the magnetic dipole is 2?

Huge amounts of data are collected in scientific experiments and social surveys. Data can be very difficult to interpret, so we often summarise the important features using statistics – numbers or diagrams which represent something about the dataset, such as where its centre lies or how spread out it is.

21A Some important concepts in statistics

Perhaps the most important concept in statistics is the difference between a sample and the population.

The **population** is the entire group which we are interested in. This might be everyone in the world, or just all 18-year-old girls in a school.

The **sample** is a group of data collected from the population. It is often assumed that any statistics of the sample reflect the statistics of the population. This may not be the case for three reasons:

1. The sample may be unrepresentative of the population. If you want to know the average height of boys in a school and you only ask the basketball team, you may get an overestimate.
2. Even with a good sample, it is unlikely you will get exactly the same statistic as for the whole population.

Although statistics are often helpful, there are many ways in which they can be misleading.

3. Perhaps the most subtle reason is the fact that for some statistics we can show mathematically that the statistic for the sample will systematically tend to be different from the statistic of the population.

To avoid the first of these issues it is generally good practice to collect a **random sample**. This means that each member of the population is equally likely to be selected for the sample and the probability of selecting any member of the population is independent of the selection of any other member. This can be very difficult to achieve.

The other two issues are more difficult to avoid. In particular, even with a random sample, it is possible to get a correct but unusual data value. Such a value is called an **outlier**, and is different from an **anomaly**, which refers to a measurement error.

Quantum theory, an area of physics, suggests that at the atomic level all physical variables are actually discrete! Surprisingly, evidence to support this can be found by looking very carefully at the colour of objects radiating heat at different temperatures. If you want to know more about this, try researching the ultra-violet catastrophe.

Another important distinction is between different types of numerical data which may be collected:

Discrete data is data that takes only particular, predefined values. Shoe size is an example of discrete data. Data does not have to take only integer values to be discrete.

Continuous data is data which can take any value in a given range. Foot length is an example of continuous data. To work with continuous data we need to group it together, so we would not count how many people have a foot length of 20.034 621... cm; we might instead count how many people have a foot length between 20.0 and 20.1 cm.

Exercise 21A

1. Decide which of these variables are continuous and which are discrete:
   (a) days in a month
   (b) shoe size
   (c) length of foot
   (d) weight of a gerbil
   (e) length of arm to the nearest centimetre
   (f) age in completed years.
2. For each of the following suggest why the sample taken is not random:
   (a) a questionnaire to find out attitudes towards a new shopping development taken on the high street at 4 p.m. on a Saturday

(b) interviewing students in a lesson about truancy rates

(c) an internet survey of voting intentions

(d) a cholesterol test from one person in each household in a street.

21B Measures of spread

Many statistical calculations, such as the mean, median and mode, focus on finding a measure of the centre of the data. Others, such as the range and the interquartile range, give you a measure of how spread out the data is. The problem with these two measures is that they do not take into account all of the data.

Section Y of Prior Learning reminds you how to calculate each of these statistics.

What are the pros and cons of representing a complicated set of data by a single number?

In everyday language the word average is often used without specifying whether it refers to the mean, the mode or the median. Potentially this can lead to some confusing newspaper headlines. In mathematics it is important to be precise.

A commonly used measure that does use all the data is called the **standard deviation**, and for a sample it is given the symbol s_n. This measures the average distance of data items from the mean.

KEY POINT 21.1

Standard deviation formula

$$s_n^2 = \sum_i \frac{(x_i - \bar{x})^2}{n}$$

Where x_i is the ith data value, $\bar{x}$ is the mean of all the data values and n is the number of data items.

Although this is the defining formula for the standard deviation, it is not normally the formula used to calculate it. Fill-in proof 24 'An alternative formula for variance' shows that an alternative and much more useful formula is:

KEY POINT 21.2

$$s_n^2 = \overline{x^2} - \bar{x}^2$$

The word *range* has several different meanings in mathematics, both technical and informal. In particular, the range of a function (chapter 5) means the set of values that the function can take; but the range of data is a *single number* representing the length of the interval in which the data lies.

EXAM HINT

$\overline{x^2}$ is the average of data items after they have been squared. $\bar{x}^2$ is the square of the average of the original data items. These two things are not the same! In fact, the first one is always larger, so s_n^2 cannot be negative.

It may help you to remember this formula as 'the mean of the squares minus the square of the mean'.

The square of the standard deviation is called the **variance**. It does not give any direct measure of spread, but if you study the Statistics option (Topic option 7) you will see that it is, algebraically, very convenient to work with.

EXAM HINT

Make sure you know whether a question is asking for standard deviation or variance.

Worked example 21.1

Find the range, interquartile range and standard deviation of the following numbers:

1, 12, 9, 9, 15, 7, 5

range = 15 − 1 = 14

Order the data then split into two halves

Data in order: 1, **5**, 7, 9, 9, **12**, 15

LQ = 5, UQ = 12 so IQR = 7

Find $\bar{x}$ and $\overline{x^2}$

	x	x^2
	1	1
	5	25
	7	49
	9	81
	9	81
	12	144
	15	225
Mean:	8.29	86.57

$s_n = \sqrt{86.57 - 8.29^2}$

$= \sqrt{17.92}$

≈ 4.23 (3SF)

EXAM HINT

Most graphical calculators have methods for finding all these statistics. See Calculator skills sheets 11 and 12. However, you can still be asked to do it without a calculator.

As a rough guide, about two thirds of the data will be less than one standard deviation away from the mean. In the above example two data items (1 and 15) out of seven are more than 4.23 away from the mean (8.29). This can be a useful quick check. In a large data set nearly all the data will be within two standard deviations from the mean, and anything more than three standard deviations from the mean is very unusual.

Sometimes you may have to work with summary statistics concerning the data, but do not have the original data. In this situation, it is often best to work with the sum of all the data values and the sum of the all the squared data values.

Worked example 21.2

Six trophies have a mean height of 28 cm and a standard deviation of 6 cm. A seventh trophy is added with height 38 cm. Find the mean and standard deviation of the seven trophies.

Find the total height of the first six trophies. This is because a total is very easy to combine with a new piece of data

$\Sigma x = 28 \times 6 = 168$

Use the formula for variance to find the total of the squares of the heights of the original six trophies

$6^2 = \frac{(\Sigma x^2)}{6} - 28^2$

$\Sigma x^2 = 4920$

Find the new totals

$\Sigma x = 168 + 38 = 206$

$\Sigma x^2 = 4920 + 38^2 = 6364$

Find the new mean

$x = \frac{206}{7} = 29.4 (3SF)$

Find the new variance

$s_n^2 = \frac{6364}{7} - 29.4^2 = 43.1 (3SF)$

$s_n = 6.57 (3SF)$

Exercise 21B

1. For each set of data below calculate the standard deviation:

   (a) (i) 19.0, 23.4, 36.2, 18.7, 15.7

   (ii) 0.4, −1.3, 7.9, 8.4, −9.4

   (b) (i) 28, 31, 54, 28, 17, 30

   (ii) 60, 18, 42, 113, 95, 23

2. For each set of data below calculate the standard deviation:

   (a) 1, 1, 2, 3, 5 (b) 3, −2, 4, −2, 5, 2

3. The ordered set of data 5, 5, 7, 8, 9, x, 13 has interquartile range equal to 7.

   (a) Find the value of x.

   (b) Find the standard deviation of the data set. *[5 marks]*

4. Consider the five numbers, 2, 5, 9, x and y. The mean of the numbers is 5 and the variance is 6. Find the value of xy. *[7 marks]*

5. In five tests Suewan has an average of 23 marks and a standard deviation of 4 marks. In her sixth test she scores 32. What is the overall standard deviation of her marks? *[7 marks]*

6. The mean of a set of 15 data items is 600 and the standard deviation is 12. Another piece of data is discovered and the new mean is 600.25. What is the new standard deviation? *[7 marks]*

7. If the sum of 20 pieces of data is 1542 find the smallest possible value of $\sum x_i^2$. *[4 marks]*

8. (a) Explain why for any set of data $x_i - \bar{x}$ is no greater than the range.

   (b) By considering the formula $s_n = \sqrt{\sum_i \frac{(x_i - \bar{x})^2}{n}}$,

   prove that the standard deviation is always less than or equal to the range. *[4 marks]*

21C Frequency tables and grouped data

It is common to summarise a large quantity of data in a **frequency distribution** table. This is a list of all the values the data takes, along with how often they occur. We could convert this into a list of all the data values and calculate the statistics as we had before, but it is enough to just imagine writing out a list, 16 ones, two twos, etc.

Worked example 21.3

Find the mean number of passengers observed in cars as they passed a school.

Passengers	Frequency
0	32
1	16
2	2
3 or more	0

(none in the first group, 16 in the second group 2, and 4 in the third group)

total number of passengers =

$(32 \times 0) + (16 \times 1) + (2 \times 2) + 0 = 20$

$\text{mean} = \frac{20}{50} = 0.4$

EXAM HINT

The mean does not have to be an achievable value; do not round to the nearest whole number.

This method suggests an important formula.

KEY POINT 21.3

Finding the mean from a frequency table:

$$\bar{x} = \frac{\sum x_i f_i}{n}$$

where f_i is the frequency of the ith data value and

$n = \sum_i f_i$ is the total number of data items.

We can work out $\overline{x^2}$ in a similar way, which gives the following formula for standard deviation.

KEY POINT 21.4

Standard deviation from a frequency table:

$$s_n^2 = \frac{\sum_i x_i^2 f_i}{n} - \bar{x}^2$$

EXAM HINT

The notation $[a,c[$ means the same as $a < x < c$. You might also see this written as $[a, c)$.

In Worked example 21.3, we knew the exact data values, but when we are dealing with grouped data, we do not have this level of precision. In order to work out the mean and standard deviation, our best and simplest assumption is that all the original values in a particular group are located at the centre of the group, called the mid-interval value. To find the centre of the group we take the mean of the largest and the smallest possible values in the group, called the upper and lower interval boundaries.

Worked example 21.4

Find the mean and standard deviation of the weight of eggs produced by a chicken farm. Explain why these answers are only estimates.

Weight of eggs, in g	Frequency
[100, 120[	26
[120, 140[	52
[140, 160[	84
[160, 180[	60
[180, 200[	12

Make a table using the mid-interval value for each group

x_i	f_i	$x_i f_i$	$x_i^2 f_i$
110	26	2860	314 600
130	52	6760	878 800
150	84	12 600	1 890 000
170	60	10 200	1 734 000
190	12	2280	433 200
Sum:	234	34 700	5 250 600

Apply the formulae

$$\bar{x} = \frac{\sum x_i f_i}{n} = \frac{34700}{234} = 148.3\text{g (3SF)}$$

$$s_n^2 = \frac{\sum x_i^2 f_i}{n} - \bar{x}^2 = \frac{5250600}{234} - 148.3^2 = 448.4$$

Therefore $s_n = 21.2\text{g (3SF)}$

These answers are only estimates because we have assumed that all the values in each group are at the centre, rather than using the actual data.

EXAM HINT

Whenever you find a mean or a standard deviation it is always worth checking that the numbers make sense in context. Given the data, an average of about 150 g here seems reasonable.

Sometimes the endpoints of the intervals shown in the table are not the actual smallest and largest possible values in that group. For example, when measuring length in centimetres it is common to round the values to the nearest integer, so 10–15 actually means [9.5, 15.5]. To find the mid-interval values we must first identify the actual interval boundaries.

Worked example 21.5

Estimate the mean of this data:

Age	Frequency
10 to 12	27
13 to 15	44
16 to 19	29

Carefully decide on the upper and lower interval boundaries. There should be no 'gaps' between the groups, because age is continuous data. You are 12 years old until your 13th birthday

Group	x_i	f_i	$x_i f_i$
[10, 13[	11.5	27	310.5
[13, 16[	14.5	44	638
[16, 20[	18	29	522
Sum:		100	1470.5

$$\bar{x} = \frac{\sum x_i f_i}{n} = \frac{1470.5}{100} = 14.7 \text{ (3SF)}$$

Exercise 21C

1. Calculate the mean and standard deviation of each data set:

(a)

x	Frequency
0	16
1	22
2	8
3	4
4	0

(b)

x	Frequency
–1	10
0	8
1	5
2	1
3	1

2. Calculate the mean and standard deviation for each data set:

(a)

x	Frequency
10	7
12	19
14	2
16	0
18	2

(b)

x	Frequency
0.1	16
0.2	15
0.3	12
0.4	9
0.5	8

3. A group is described as '17 – 20'. State the upper and lower boundaries of this group if it is measuring:

(a) age in completed years

(b) number of pencils

(c) length of a worm to the nearest centimetre

(d) hourly earnings, rounded up to whole dollars.

4. Find the mean and standard deviation of each of the following sets of data:

(a) (i) x is the time taken to complete a puzzle in seconds

x	Frequency
$[0,15[$	19
$[15,30[$	15
$[30,45[$	7
$[45,60[$	5
$[60,90[$	4

(ii) x is the weight of plants in grams

x	Frequency
$[50,100[$	17
$[100,200[$	23
$[200,300[$	42
$[300,500[$	21
$[500,1000[$	5

(b) (i) x is the length of fossils found in a geological dig, to the nearest centimetre

x	Frequency
0 to 4	71
5 to 10	43
11 to 15	22
16 to 30	6

(ii) x is the power consumption of light bulbs, to the nearest watt

x	Frequency
90 to 95	17
96 to 100	23
101 to 105	42
106 to 110	21
111 to 120	5

(c) (i) x is the age of children in a hospital ward

x	Frequency
0 to 2	12
3 to 5	15
6 to 10	7
11 to 16	6
17 to 18	3

(ii) x is the amount of tips paid in a restaurant, rounded down to the nearest dollar

x	Frequency
0 to 5	17
6 to 10	29
11 to 20	44
21 to 30	16
31 to 50	8

5. In a sample of 50 boxes of eggs, the number of broken eggs per box is shown below:

Number of broken eggs	0	1	2	3	4	5	6
Number of boxes	17	8	7	7	6	5	0

(a) Calculate the median number of broken eggs per box.

(b) Calculate the mean number of broken eggs per box. *[4 marks]*

6. The mean of the data in the table is 32 and the variance is 136. Find the possible values of p and q.

x	Frequency
20	12
40	q
p	8

[8 marks]

Summary

- Most of statistics is based on trying to infer properties of a **population** based upon a **sample** from that population.
- To get a representative sample of the population, it is good practice to collect a **random sample**, where each member of the population is equally likely to be selected for the sample and the probability of selecting a member of the population is independent.
- An **outlier** is a correct but unusual data value.
- An **anomaly** is an unusual data value caused by a measurement error.
- **Discrete data** takes only a predefined value (it does not have to be an integer!).
- **Continuous data** can take any value in a given range. This type of data is generally grouped before we can work with it.
- **Standard deviation** (s_n) is a measure of how spread out the data is relative to the data's mean, and it takes into account all of the data.
- The square of the standard deviation is called the **variance** and it has the formula:

$s_n^2 = \sum_i \frac{(x_i - \bar{x})^2}{n}$ or more commonly: $s_n^2 = \overline{x^2} - \bar{x}^2$

- Large datasets are summarised in **frequency distribution** tables and the mean and standard deviations can be calculated from these tables using the formulae:

$$\bar{x} = \frac{\sum x_i f_i}{n}$$

where f_i is the frequency of the ith data value and $n = \sum_i f_i$ is the total number of data items, and:

$$s_n^2 = \frac{\sum_i x_i^2 f_i}{n} - \bar{x}^2$$

- When investigating grouped data we must assume that every element has the mid-interval value of the group (the mean of the upper and lower boundaries).
- The methods for calculating statistics for grouped data vary slightly depending upon whether the data is discrete or continuous.

Introductory problem revisited

The magnetic dipole of an electron is measured in a very sensitive experiment 3 times. The values are 2.000 000 15, 2.000 000 12 and 2.000 000 9. Does this support the theory that the magnetic dipole is 2?

The average magnetic dipole is 2.000 000 12 which is pretty close to 2, but the standard deviation in the measurements is 0.000 000 245, and so the mean is approximately 5 sample standard deviations away from 2. Within the natural variation observed, the magnetic dipole cannot be said to be 2.

The difference between 2.000 001 2 and 2 might seem trivial, but it was this difference which inspired Richard Feynman to create a new theory of physics called Quantum electrodynamics, which did indeed predict this tiny difference from 2! This is an example of theory driving experiment which in turn creates new theory – the interplay between theoretical mathematics and reality.

Mixed examination practice 21

Short questions

1. A student takes the bus to school every morning. She records the length of the time, in minutes, she waits for the bus on 12 randomly chosen days. The data is summarised by:

$$\sum_{i=1}^{12} x_i = 49 \text{ and } \sum_{i=1}^{12} x_i^2 = 305.7$$

Calculate:

(a) the mean time she spends waiting for the bus

(b) the standard deviation of the times. *[5 marks]*

2. The average wavelength of light in nanometres emitted by a glowing wire is measured on 50 different occasions and the results are given below:

Wavelength in nm (λ)	Frequency
600–640	22
640–680	18
680–720	x
720–760	y

The mean of λ is calculated from this table as 653.6.

(a) Find the values of x and y.

(b) Calculate an estimate of the variance.

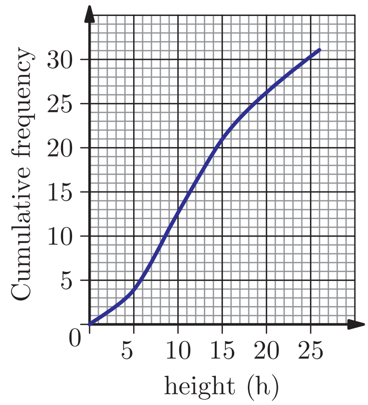
(c) Explain why this is only an estimate. *[9 marks]*

3. An experiment was conducted on the reaction times of 15 students (t) in seconds. The results were that the average reaction time was 0.2 s and the variance was 0.0025 s^2. A 16th student is observed later. She has a reaction time of 0.16 s. Find the new mean and standard deviation. *[8 marks]*

4. The variance of two data items is k. Find an expression in terms of k for the range. *[5 marks]*

Long questions

1. The following is the cumulative frequency diagram for the heights of 30 plants, given in centimetres.



(a) Use the diagram to estimate the median height.

(b) Complete the following frequency table:

Height (h)	Frequency
$0 < h \leq 5$	4
$5 < h \leq 10$	9
$10 < h \leq 15$	
$15 < h \leq 20$	
$20 < h \leq 25$	

(c) Hence estimate the mean height. *[8 marks]*

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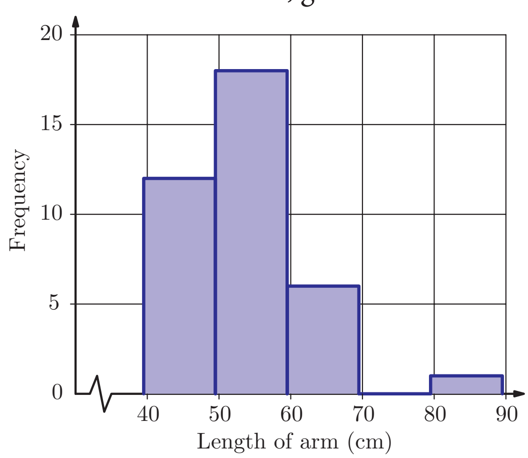
2. The following histogram shows the length of the arms of 37 children in a classroom in Lithuania, given to the nearest cm.



(a) Explain why the bar representing the first group goes below 40.

(b) Complete the following frequency distribution:

Length	Frequency
40–49	12
50–59	
60–69	
70–79	
80–89	

(c) Use this data to estimate the mean and the standard deviation of the data.

(d) Give one reason to explain why the average arm length of all children in Lithuania might be different from the value found above.

3. The frequency distribution of the number of cars in households on a street is given below:

Number of cars	Frequency
1	a
2	b

(a) Find an expression for the mean of the number of cars and show that the variance is given by $\frac{ab}{(a+b)^2}$.

(b) Prove that it is impossible for the mean to equal the variance.

(c) If the number of households with one car is three times larger than the number of households with two cars find the mean and the standard deviation in the number of cars.

22 Probability

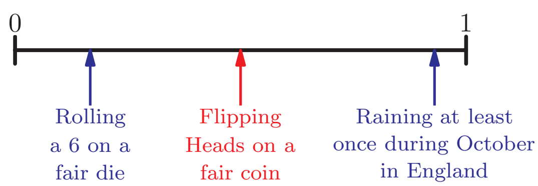
In this chapter you will learn:

- how probability can be estimated from data
- how probability can be predicted theoretically
- how to work out probabilities when you are interested in more than one outcome
- how to work out the probability of a sequence of events occurring
- how to work out probabilities of simple functions of independent random events
- how counting principles can be used to calculate probabilities
- how being given additional information changes our estimate of a probability
- how to infer information from changes in probabilities.

Introductory problem

A woman gives birth to non-identical twins. One of them is a girl. What is the probability that the other one is a girl?

In real life we often deal with uncertain events, but not all events are equally uncertain. It is not certain that the next Steven Spielberg film will be a big hit, nor is it certain that it will snow in India next summer. However, intuitively, these two events do not seem equally likely. We can put events on a scale with impossible at one end and certain at the other and assign a number between 0 and 1 to indicate where the event is on this line. This number is called the probability of the event.



For probability to be useful, it must be more than just a reflection of past experience. We also have to be able to predict probabilities of events in the future, and use these predictions to make decisions. This is the focus of this chapter.

Probability is one of the most recent additions to the field of mathematics. It was formalised by the mathematician Pierre de Fermat (1601–1665) in response to a request from his patron, the Chevalier de Méré, a notorious gambler who wanted help at the gambling table.

22A Introduction to probability

We can estimate the probability of an event practically by doing an experiment repeatedly; this is known as empirical or experimental probability:

KEY POINT 22.1

The probability of an event 'A' occurring is denoted by $P(A)$. From observation, we can estimate $P(A)$ as:

$$P(A) = \frac{\text{Number of times } A \text{ occurs}}{\text{Number of times } A \text{ could have occurred}}.$$

Probability is revised in Prior learning section H on the CD-ROM.

However, in some situations it is possible to predict the probability *before* the experiment (theoretical probability). To do this we need to be able to list all the possible outcomes. This list is called the **sample space**.

For example, when you toss a coin there are two possible outcomes: Heads or Tails. Since both are equally likely (for a 'fair' coin), the probability of each must be one half. This leads to a theoretical definition of probability:

You might think it seems obvious that these two definitions are equivalent. However, it is quite tricky to prove. If you would like to see how it is done you might like to research the law of large numbers.

KEY POINT 22.2

$$P(A) = \frac{\text{number of times } A \text{ occurs in the sample space}}{\text{number of items in the sample space}}$$

Again we have $0 \le P(A) \le 1$. We can now add an interpretation to $P(A)$. If $P(A) = 0$ then the event A is impossible. If $P(A) = 1$ then event A is certain. As $P(A)$ rises the likelihood of A occurring increases.

There are two possible outcomes when you enter a lottery. Either you win or you do not win, but this does not mean that the probability of winning is one half, since there is no reason to believe that both outcomes are equally likely. Many mistakes in probability come from this type of error.

Worked example 22.1

(a) For a family with two children, list the sample space for the sexes of the children, assuming no twins.

(b) Hence find the theoretical probability that the two children are a boy and a girl.

A table provides a systematic way to list the possibilities

(a)

First child	Second child
Boy	Boy
Boy	Girl
Girl	Boy
Girl	Girl

There are four outcomes, and two of them are a boy and a girl

(b) $P(\text{a boy and a girl}) = \frac{2}{4} = \frac{1}{2}$

EXAM HINT

Although the 'boy first, girl second' and 'girl first, boy second' cases can both be described as 'a boy and a girl', we must count them separately in the sample space.

EXAM HINT

'Die' is the singular of 'dice'.

Worked example 22.2

What is the probability of getting a prime number when you roll a six-sided die?

We can list all the possible outcomes and they are all equally likely	Possible outcomes are 1, 2, 3, 4, 5, 6
We can identify how many of them are prime	2, 3 and 5 are prime, which is 3 outcomes out of a sample space of 6.
We can write this as a probability	So probability is $\frac{3}{6} = \frac{1}{2}$

An event either happens or it does not. Everything other than the event happening is called the **complement** of the event. For example, the complement of rolling a 6 on a die is rolling 'not a 6' so 1, 2, 3, 4 or 5. The complement of A is given the symbol A'.

An event and its complement are mutually exclusive (they can't both happen), but we know that one of the two must happen so they have a total probability of 1. We can therefore deduce a formula linking the probability of an event and its complement.

KEY POINT 22.3

$$P(A) + P(A') = 1$$

Suppose that in a board game, players face a penalty whenever the sum of two rolled dice is 7. We do not care how the 7 is achieved – it may be from a 1 and a 6, or a 3 and a 4. In such a situation it is useful to use a probability grid diagram which lists the sample space with each possibility of the first event on one axis, each possibility of the second event on the other axis, giving all possible results in the cells of the table.

Saying that an event either happens or it does not is called the law of the excluded middle, and it is a basic axiom of standard logic. However, there is an alternative logical system called fuzzy logic where an event could be in a state of 'maybe happening'. As well as lots of real world applications there is a philosophical physics problem called Schrödinger's Cat which applies this idea.

Worked example 22.3

What is the probability of getting a sum of 7 when two dice are rolled?

Draw a probability grid diagram showing all possible totals when two dice are rolled

Die B \ Die A	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Count how many items are in the sample space

36 items in the sample space, each of equal probability.

Count how many sums of 7 there are

6 ways to get 7 therefore the probability of a sum of 7 is $\frac{6}{36} = \frac{1}{6}$

EXAM HINT

Notice in the grid in Worked example 22.3 that a '3' on die A and a '1' on die B has to be counted as a separate event from a '1' on die A and '3' on die B. However, a '3' on die A and a '3' on die B is only one event. This often causes confusion.

We are often interested in more complicated events, such as 'getting a grade *A* in Extended Essay or TOK'.

In everyday language the word 'or' can be ambiguous. If you say 'Everyone at the party is a doctor or a lawyer' you generally do not mean that each person could be both. However, in a game, if you say 'I win if I get a black number or an even number' you would also expect to win if you got a black even number. In probability we use the word 'or' in the second sense: '*A* or *B*' means *A* or *B or both* could happen.

Worked example 22.4

When a die is rolled find the probability that the outcome is odd or a prime number.

List the sample space

Possible outcomes: 1, 2, 3, 4, 5, 6

List the events which satisfy the condition

Odd or prime: 1, 2, 3, 5

Probability $= \frac{4}{6} = \frac{2}{3}$

Exercise 22A

1. List the sample space for each of the following:

(a) a fair 6 sided die

(b) arrangements of the letters 'RED'

(c) the sexes of a 3-child family

(d) a six sided die with 3 sides labelled '1' and the remaining sides labelled 2, 3, 4.

2. In a standard pack of 52 playing cards there are 4 different suits (red hearts, red diamonds, black clubs and black spades). In each suit there are number cards from 2 to 10, then four 'picture cards'; jack, queen, king and ace. Find the probability that a randomly chosen card is:

(a) (i) red (ii) a spade

(b) (i) a jack (ii) a picture card

(c) (i) a black number card (ii) a club picture card

(d) (i) not a heart (ii) not a picture card

(e) (i) a club or a picture card

(ii) a red card or a number card

(f) (i) a red number card strictly between three and nine

(ii) a picture card that is not a jack

EXAM HINT

You can give probability as a fraction, a decimal or as a percentage.

3. A bag contains three different kinds of marble: six are red, four are blue and five are yellow. One marble is taken from the bag. Calculate the probability that it is:

(a) (i) red (ii) yellow

(b) (i) not blue (ii) not red

(c) (i) blue or yellow (ii) red or blue

(d) (i) green (ii) not green

(e) (i) neither red nor yellow (ii) neither yellow nor blue

(f) (i) red or green (ii) neither blue nor green

(g) (i) red and green (ii) red and blue

4. By means of an example show that $P(A)+P(B)=1$ does not mean that B is the complement of A.

5. Two fair six-sided dice numbered one to six are rolled. By drawing probability grid diagrams find the probability that:

(a) the sum is 8

(b) the product is greater than or equal to 8

(c) the product is 24 or 12

(d) the largest value is 4

(e) the largest value is more than twice the smaller value

(f) the value on the first die divided by the value on the second die is a whole number.

6. A fair four-sided die (numbered 1 to 4) and a fair eight-sided die (numbered 1 to 8) are rolled. Find the probability that:

(a) the sum is 8

(b) the product is greater than or equal to 8

(c) the product is 24 or 12

(d) the largest value is 4

(e) the largest value is more than twice the smaller value

(f) the value on the eight-sided die divided by the value on the four-sided die is a whole number.

7. Two fair six-sided dice are thrown and the score is the highest common factor of the two outcomes. If this were done 180 times how many times would you expect the score to be 1? *[5 marks]*

8. Three fair six-sided dice are thrown, and the score is the sum of the three results. What is the probability that the score is less than 6? *[5 marks]*

22B Combined events and Venn diagrams

In this section we shall generalise the sample space method so we can efficiently calculate probabilities when we are interested in more than one outcome.

Which is more likely when you roll a die once:

- getting a prime number *and* an odd number?
- getting a prime number *or* an odd number?

The first possibility is restrictive; we have to satisfy both conditions. The second has more possibilities – we can satisfy either condition. So the second must be more likely.

These are examples of two of the most common ways of combining events: Intersection (in normal language 'and') and Union (in normal language 'or'). These are given the following symbols, which might be used in the examination, so get used to them!

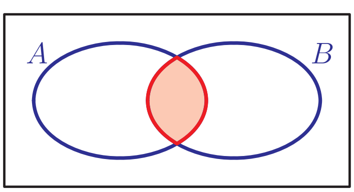
$A \cap B$ is the **intersection** of A and B, meaning when both A and B happens.

$A \cup B$ is the **union** of A and B, meaning when either A happens, or B happens, or both happen.

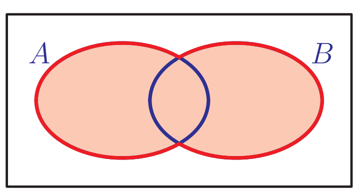
If we have neither apples nor pears then we have no apples and no pears. In set notation this can be written as $(A \cup B)' = A' \cap B'$. This is one of De Morgan's laws - a description of some of the algebraic rules obeyed by sets and hence probability. You will meet these again if you take the Discrete mathematics option (Topic option 10) or the Sets, relations and groups option (Topic option 8).

Why do mathematicians not just use simple words? One of the reasons for this is that everyday language can be ambiguous. If I say that I play rugby or hockey some people may think this means I do not play both. Mathematicians hate ambiguity.

We can use **Venn diagrams** to illustrate the concepts of union and intersection:



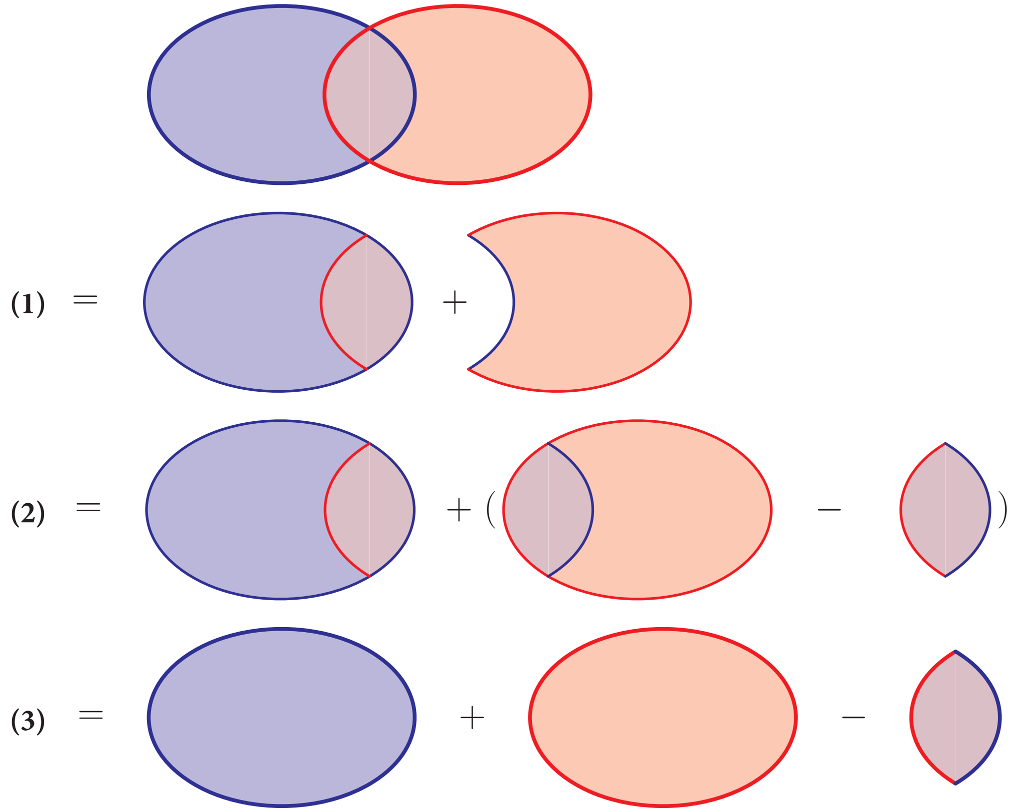
$A \cap B$



$A \cup B$

See Prior learning J on the CD-ROM which explains the concept of Venn diagrams.

Venn diagrams can help us calculate the probability of the union of two events. In the following diagram, the region of the overlap can be seen to be composed of three sections, which can be broken down into their different regions.



This illustrates a very important formula.

KEY POINT 22.4

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We can interpret this formula as 'If you want to count the number of ways of getting A or B, count the number of ways of getting A and add to that the number of ways of getting B. However, we have then counted the number of ways of getting A and B twice, so we need to compensate by subtracting it'.

If there is no possibility of A and B occurring at the same time, then $P(A \cap B) = 0$. We call these events **mutually exclusive**, and the formula reduces to $P(A \cup B) = P(A) + P(B)$.

EXAM HINT

Be careful that you do not use the formula for mutually exclusive events unless you are sure that the events cannot both occur simultaneously.

Worked example 22.5

A chocolate is selected randomly from a box. The probability of it containing nuts is $\frac{1}{4}$. The probability of it containing caramel is $\frac{1}{3}$. The probability of it containing both nuts and caramel is $\frac{1}{6}$. What is the probability of a randomly chosen chocolate containing either nuts or caramel or both?

Use the formula

$$P(\text{nuts} \cup \text{caramel}) = P(\text{nuts}) + P(\text{caramel}) - P(\text{nuts} \cap \text{caramel})$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} = \frac{5}{12}$$

Exercise 22B

1. (a) (i) $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$.

Find $P(A \cup B)$.

(ii) $P(A) = \frac{3}{10}$, $P(B) = \frac{4}{5}$ and $P(A \cap B) = \frac{1}{10}$.

Find $P(A \cup B)$.

(b) (i) $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{8}$ and $P(A \cup B) = \frac{5}{8}$.

Find $P(A \cap B)$.

(ii) $P(A) = 0.2$, $P(B) = 0.1$ and $P(A \cup B) = 0.25$.

Find $P(A \cap B)$.

(c) (i) $P(A \cap B) = 20\%$, $P(A \cup B) = 0.4$ and $P(A) = \frac{1}{3}$.

Find $P(B)$.

(ii) $P(A \cup B) = 1$, $P(A \cap B) = 0$ and $P(B) = 0.8$.

Find $P(A)$.

(d) (i) Find $P(A \cup B)$ if $P(A) = 0.4$, $P(B) = 0.3$ and A and B are mutually exclusive.

(ii) Find $P(A \cup B)$ if $P(A) = 0.1$, $P(B) = 0.01$ and A and B are mutually exclusive.

2. (a) (i) When a fruit pie is selected at random,

$P(\text{it contains pears}) = \frac{1}{5}$ and $P(\text{it contains apples}) = \frac{1}{4}$.

10% contain both apples and pears. What is the probability that it contains either apples or pears?

(ii) In a library 80% of books are classed as fiction and 70% were written in the 20th century. Half of the books are 20th century fiction. What proportion of the books are either fiction or from the 20th century?

(b) (i) 95% of pupils in a school play either football or tennis. The probability of a randomly chosen pupil playing football is $\frac{6}{10}$ and the probability of playing tennis is $\frac{5}{8}$. What percentage of pupils play both football and tennis?

(ii) 2 in 5 people in a school study Spanish and 1 in 3 study French. Half of the school study either French or Spanish. What fraction study both French and Spanish?

(c) (i) 90% of pupils in a school have a Facebook account and 3 out of 5 have a Twitter account. One twentieth of pupils have neither a Facebook account nor a Twitter account. What percentage are on both Facebook and Twitter?

(ii) 25% of teams in a football league have French players and a third have Italian players. 60% have neither French nor Italian players. What percentage have both French and Italian players?

3. Simplify the following expressions where possible:

(a) $P(x > 2 \cap x > 4)$

(b) $P(y \leq 3 \cup y < 2)$

(c) $P(a < 3 \cap a > 4)$

(d) $P(a < 5 \cup a \geq 0)$

(e) $P(\text{apple} \cup \text{fruit})$

(f) $P(\text{apple} \cap \text{fruit})$

(g) $P(\text{multiple of } 4 \cap \text{multiple of } 2)$

(h) $P(\text{square} \cup \text{rectangle})$

(i) $P(\text{blue} \cap (\text{blue} \cup \text{red}))$

(j) $P(\text{blue} \cap (\text{blue} \cap \text{red}))$

4. In a survey, 60% of people are in favour of building a new primary school and 85% are in favour of building a new library. Half of all those surveyed would like both a new primary school and a new library. What percentage supported neither a new library nor a new primary school? *[5 marks]*

5. If $P(A) = 0.2$, $P(A \cap B) = 0.1$ and $P(A \cup B) = 0.7$, find $P(B')$. *[5 marks]*

6. Events A and B satisfy $P((A \cap B)') = 0.2$, $P(A) = P(B) = 0.5$. Find $P(A \cap B')$. *[5 marks]*

7. An integer is chosen at random from the first one thousand positive integers. Find the probability that the integer chosen is:

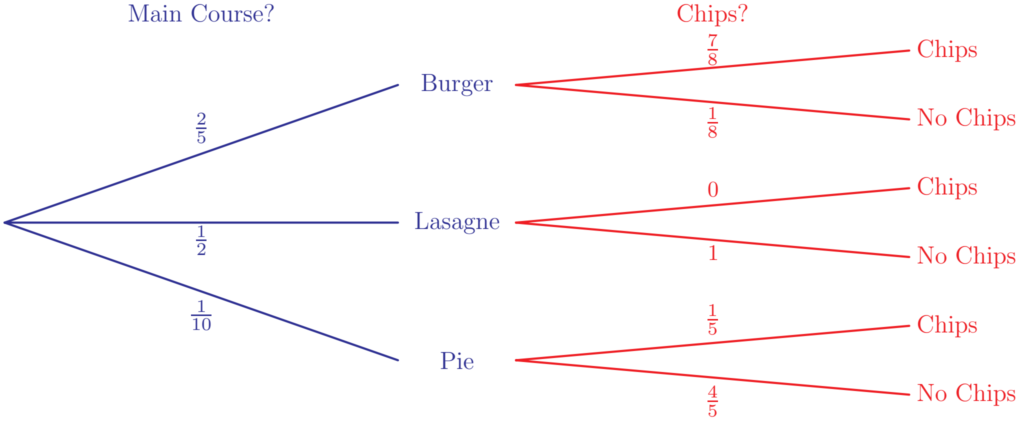
(a) a multiple of 6

(b) a multiple of both 6 and 8. *[5 marks]*

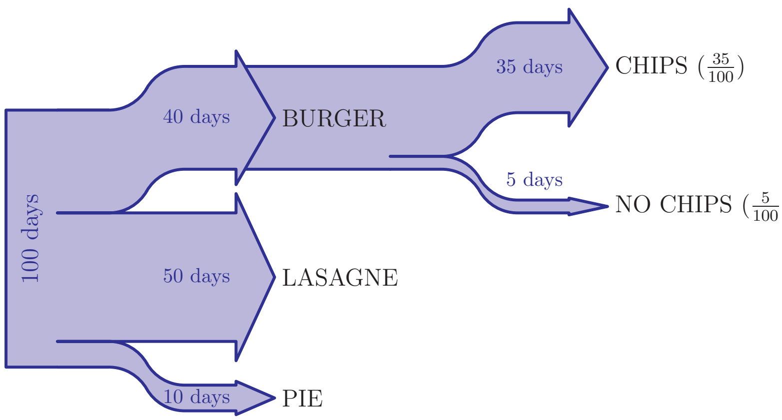
22C Tree diagrams and finding the intersection

Venn diagrams provide a formula linking intersection and union. However they do not help us find a formula for either intersection or union by themselves. To do this it is useful to consider a second method: tree diagrams. When several events happen, either in succession or simultaneously, a **tree diagram** is a way of listing all the possible outcomes. It starts with branches for all the possible outcomes for one of the events, then from each branch we list all possible outcomes for the next event. Along each branch we write the probability of taking that branch.

Tree diagrams have an advantage over the sample space method as they can cope with more than two events, and with outcomes which are not equally likely.



We can consider there to be a filtering process at each branch. Suppose we started with 100 days of school food. On $\frac{2}{5}$ of these days (i.e. 40 days) there will be burgers. On $\frac{7}{8}$ of these burger days there will also be chips. So overall there will be 35 out of 100 days with burgers and chips – or a probability of $\frac{7}{20}$.



To find the probability of travelling along each branch we have found a fraction of a fraction. To do this we need to multiply the two probabilities; but what do these two probabilities represent? In our burger and chips case the $\frac{2}{5}$ represents the probability of burgers, and it might be tempting to say that the $\frac{7}{8}$ is the probability of chips. However, this is not true. Looking at the completed tree diagram we can see that 37 out of 100 days have chips, and this is certainly not $\frac{7}{8}$. The $\frac{7}{8}$ represents the probability of having chips if you already know that there are burgers. This is called **conditional probability**. We use the notation P(chips | burgers), read as 'the probability of chips given burgers'. This leads to a very important rule.

KEY POINT 22.5

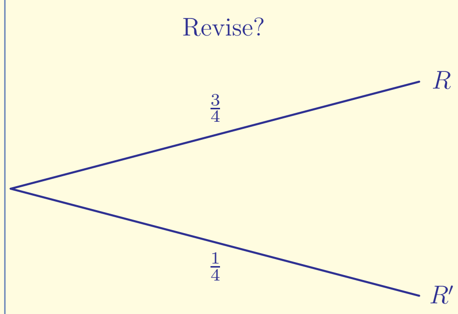
Conditional probability

$$P(A \cap B) = P(B)P(A|B)$$

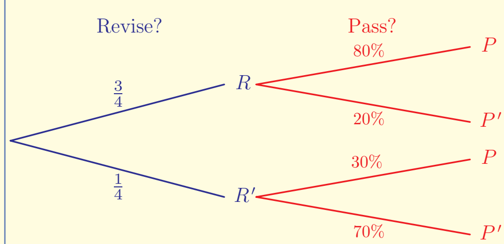
Worked example 22.6

If I revise there is an 80% chance I will pass the maths test, but if I do not revise there is only a 30% chance of passing. I revise for $\frac{3}{4}$ of tests. What proportion of tests do I pass?

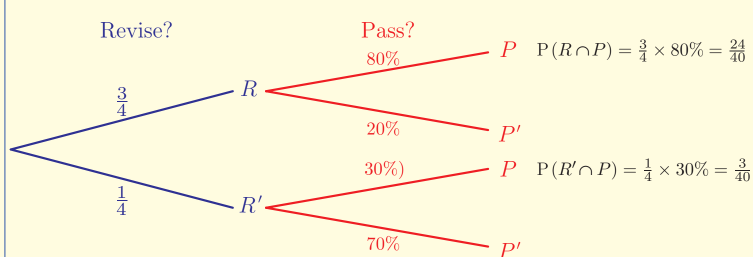
Decide which probability (revise or pass) is not conditional. Start the tree diagram with this event. The probability of passing the test is conditional on revision, so the revision branches have to come first



Add the conditional event



Identify which branches result in passing the test Multiply to find the probability at the end of each branch



$$\mathrm{P}(\text{passing}) = \mathrm{P}(\text{revising} \cap \text{passing}) + \mathrm{P}(\text{not revising} \cap \text{passing})$$

$$= \frac{24}{40} + \frac{3}{40}$$

$$= \frac{27}{40}$$

EXAM HINT

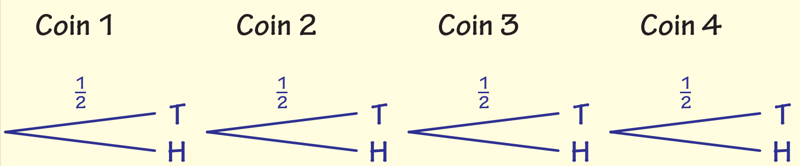
The question in Worked example 22.6 uses the words chance and proportion. These are just other words for probability. Make sure you do not get put off by unusually worded questions.

Sometimes a question may ask for the probability of an event which is quite difficult to find directly, but it is much easier to find the probability of its complement.

Worked example 22.7

Find the probability of getting at least 1 head when you toss four fair coins.

We could draw a tree diagram to find the probability of 1, 2, 3 or 4 heads, but it is easier to find the complement (0 heads)



$$P(0\text{heads}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$P(\geq 1\text{heads}) = 1 - P(0\text{heads})$$

$$= \frac{15}{16}$$

EXAM HINT

When you are asked to find the probability of 'at least...' or 'at most...' finding the complement is frequently a good idea.

Exercise 22C

1. (a) (i) $P(A) = 0.4$ and $P(B \mid A) = 0.3$. Find $P(A \cap B)$.

   (ii) $P(X) = \frac{3}{5}$ and $P(Y \mid X) = 0$. Find $P(X \cap Y)$.

   (b) (i) $P(A) = 0.3$, $P(B) = 0.2$ and $P(B \mid A) = 0.8$.

   Find $P(A \cap B)$.

(ii) $P(A) = 0.4,\ P(B) = 0.8$ and $P(A \mid B) = 0.3$.

Find $P(A \cap B)$.

(c) (i) $P(A) = \frac{2}{5},\ P(B) = \frac{1}{3}$ and $P(A \mid B) = \frac{1}{4}$.

Find $P(A \cup B)$.

(ii) $P(A) = \frac{3}{4},\ P(B) = \frac{1}{4}$ and $P(B \mid A) = \frac{1}{3}$.

Find $P(A \cup B)$.

2. A class contains 6 boys and 8 girls. Two are picked at random. What is the probability that they are both boys? *[4 marks]*

3. A bag contains 4 red balls, 3 blue balls and 2 green balls. A ball is chosen at random from the bag and is not replaced. A second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order. *[5 marks]*

4. Given that $P(X) = \frac{1}{3},\ P(Y \mid X) = \frac{2}{9}$ and $P(Y \mid X') = \frac{1}{3}$, find:

(a) $P(Y')$

(b) $P(X' \cup Y')$ *[6 marks]*

5. A factory has two machines making widgets. The older machine has larger capacity, so it makes 60% of the widgets, but 6% are rejected by quality control. The newer machine has only a 3% rejection rate. Find the probability that a randomly selected widget is rejected. *[5 marks]*

6. The school tennis league consists of 12 players. Daniel has a 30% chance of winning any game against a higher-ranked player, and a 70% chance of winning any game against a lower-ranked player. If Daniel is currently in third place, find the probability that he wins his next game against a random opponent. *[5 marks]*

7. There are 36 disks in a bag. Some of them are black and the rest are white. Two are simultaneously selected at random. Given that the probability of selecting two disks of the same colour is equal to the probability of selecting two disks of different colour, how many black disks are there in the bag? *[6 marks]*

22D Independent events

We can now evaluate the intersection and union of two events A and B if we know $P(A|B)$ or $P(B|A)$, but finding this can be quite difficult. There is one important exception, when the two events do not affect each other. These are called **independent events**. In this case knowing that B has occurred has no impact on the probability of A occurring, so for independent events $P(A|B) = P(A)$. However, we know that $P(A \cap B) = P(A|B)\, P(B)$.

KEY POINT 22.6

For independent events: $P(A \cap B) = P(A)P(B)$.

As well as being true for all independent events, Key point 22.6 is actually a defining feature of independent events. So to show that two events are independent we need to show that they satisfy this equation.

All strawberries are red, but not all red things are strawberries. There can be problems distinguishing between a property and a defining feature. A circle has a constant width, but does having a constant width make a plane shape a circle?

Worked example 22.8

$P(A) = 0.5$, $P(B) = 0.2$ and $P(A \cup B) = 0.6$. Are the events A and B independent?

Use the information to find $P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\Rightarrow 0.6 = 0.5 + 0.2 - P(A \cap B)$
$\Rightarrow P(A \cap B) = 0.1$

Evaluate $P(A)P(B)$

$P(A)P(B) = 0.1$

Compare the two

$\therefore P(A \cap B) = P(A)P(B)$
So A and B are independent

If we know that two events are independent we can use this to help calculate other probabilities.

Worked example 22.9

A and B are independent events with $P(A \cup B) = 0.8$ and $P(A) = 0.2$. Find $P(B)$.

Write $P(A \cap B)$ in terms of other probabilities

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\Rightarrow 0.8 = 0.2 + P(B) - P(A \cap B)$$

Use independence

$$= 0.2 + P(B) - 0.2 \times P(B)$$

$$\Rightarrow 0.6 = 0.8 \times P(B)$$

$$\Rightarrow P(B) = \frac{3}{4} \text{ (or 0.75)}$$

Exercise 22D

1. Events A and B are independent.

(a) (i) $P(A) = 0.3$ and $P(B) = 0.7$. Find $P(A \cap B)$.

(ii) $P(A) = \frac{1}{5}$ and $P(B) = \frac{1}{3}$. Find $P(A \cap B)$.

(b) (i) $P(A) = \frac{4}{5}$ and $P(A \cap B) = \frac{3}{7}$. Find $P(B)$.

(ii) $P(A \cap B) = 0.5$ and $P(B) = 0.9$. Find $P(A)$.

(c) (i) $P(A) = 40\%$ and $P(B) = 16\%$. Find $P(A \cup B)$.

(ii) $P(A) = 0.2$ and $P(B) = \frac{1}{4}$. Find $P(A \cup B)$.

(d) (i) $P(A \cup B) = 0.6$ and $P(A) = 0.4$. Find $P(B)$.

(ii) $P(A \cup B) = 0.5$ and $P(A) = 0.1$. Find $P(B)$.

2. Determine which of the following pairs of events are independent:

(a) (i) $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$

(ii) $P(A) = 0.8$, $P(B) = 0.1$ and $P(A \cap B) = 0.05$

(b) (i) $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{2}{5}$

(ii) $P(A) = 56\%$, $P(B) = 32\%$ and $P(A \cup B) = 72\%$

3. The independent events A and B are such that $P(A) = 0.6$ and $P(A \cup B) = 0.72$. Find:

(a) $P(B)$

(b) the probability that either A occurs or B occurs, but not both. *[6 marks]*

4. A school has two photocopiers, one for teachers and one for pupils. The probability of the teachers' photocopier working is 92%. The probability of the students' photocopier working is 68%. The two outcomes do not affect each other. What is the probability that:

(a) both photocopiers are working

(b) neither photocopier is working

(c) at least one photocopier is working. *[7 marks]*

5. As part of a promotion a toy is put in each packet of crisps sold. There are eight different toys available. Each toy is equally likely to be found in any packet of crisps.

David buys four packets of crisps.

(a) Find the probability that the four toys in these packets are all different.

(b) Of the eight toys in the packets, his favourites are the yo-yo and the gyroscope. Find the probability that he finds at least one of his favourite toys in these four packets. *[7 marks]*

6. Given that events A and B are independent with $P(A \cap B) = 0.3$ and $P(A \cap B') = 0.6$, find $P(A \cup B)$. *[5 marks]*

22E Counting principles in probability

In chapter 1 you learnt how to count the number of ways in which events can occur.

Counting principles can be used to calculate probabilities in some problems that would be extremely difficult in any other way. A random arrangement of a collection of objects forms a sample space so we can find the probability of particular condition A occurring.

KEY POINT 22.7

$$P(A) = \frac{\text{number of ways in which } A \text{ occurs}}{\text{total number of ways}}$$

Worked example 22.10

A committee of 4 is randomly chosen from 6 girls and 5 boys. What is the probability that the committee contains exactly 3 girls?

First decide how many different committees can be made up. This is a selection where order does not matter

Total number of committees $= \binom{11}{4} = 330$

Then see how may committees have exactly three girls

Choosing 3 girls from 6 and 1 boy from 5 can be done in $\binom{6}{3} \times \binom{5}{1} = 100$ ways

Find the ratio

$P(\text{exactly } 3 \text{ girls}) = \frac{100}{330} = \frac{10}{33}$

Exercise 22E

1. A set of four alphabet cubes bearing the letters A, R, S and T are dropped in a line at random. What is the probability that they spell out one of the words STAR, RATS or ARTS? *[4 marks]*

2. Consider the word PARTING. What is the probability that a sequence of four letters chosen from this word contains the letter P? *[4 marks]*

3. (a) A team of 11 is chosen randomly from a squad of 18. What is the probability that both the captain and the vice captain are selected?

(b) Two of the squad are goalkeepers and one of them must be chosen. If neither of the goalkeepers is captain or vice captain, what now is the probability that both the captain and the vice captain are selected? *[6 marks]*

4. A team of five students is to be chosen at random to take part in a debate. The team is to be chosen from a group of six history students and three philosophy students. Find the probability that:

(a) only history students are chosen

(b) all three philosophy students are chosen. *[6 marks]*

5. Six boys sit at random in six seats arranged in a row. Two of the boys are brothers. Find the probability that the brothers:

(a) sit at the ends of the row

(b) sit next to each other. *[6 marks]*

See Worked example 1.13 for a similar problem.

6. A rugby team consists of 8 forwards, 7 backs and 5 substitutes. They all line up at random in one row for a picture. What is the probability that:

(a) the forwards are all next to each other?

(b) no two forwards are next to each other? *[8 marks]*

22F Conditional probability

Estimate the probability that a randomly chosen person is a dollar millionaire. Would your estimate change if you were told that they live in a mansion?

When we get additional information, probabilities change. In the above example, P(millionaire) is very different to P(millionaire|lives in a mansion). The second is a conditional probability, and we used it in Section 22 C when looking at tree diagrams.

One important method for finding conditional probabilities is called restricting the sample space. We write out a list of all the equally likely possibilities before we are given any information, and then cross out any possibilities the information rules out.

Worked example 22.11

Given that the number rolled on a die is prime, show that the probability that it is odd is $\frac{2}{3}$.

Write out sample space for a single roll of a die	On one roll we could get 1, 2, 3, 4, 5 or 6
But we are told that the number is prime	If it is prime it can only be 2, 3 or 5
Decide how many of these are odd	Two of these are odd, so the probability is $\frac{2}{3}$

EXAM HINT

Note that P $(A \cap B)$ = P $(B \cap A)$, therefore:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

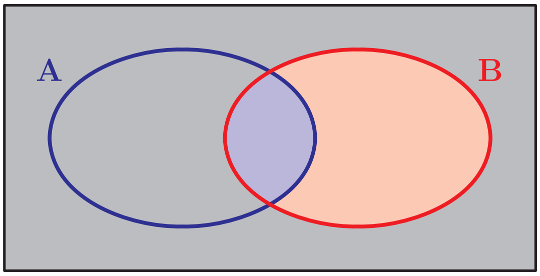
In Key point 22.5 we saw that $P(A \cap B) = P(B)P(A \mid B)$. This formula can be rearranged to get a very important formula for conditional probability.

KEY POINT 22.8

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

This can be visualised using Venn diagrams.



If B has been given we can ignore all of the Venn diagram except B. The portion of this region in which A occurs is $A \cap B$. We will look at this in more detail in Section 22G.

When calculating a probability it is not always obvious which sample space needs to be used. Supplementary sheet 11 'Measuring risk', on the CD-ROM explores this difficulty in the context of estimating the risk of a transport accident.

Worked example 22.12

The probability that a randomly chosen resident of a city in Japan is a millionaire is $\frac{1}{10000}$. The probability that a randomly chosen resident lives in a mansion is $\frac{1}{30000}$. Only 1 in 40 000 are millionaires who live in mansions. What is the probability of a randomly chosen individual being a millionaire given that they live in a mansion?

Write required probability in 'given' notation and apply the formula

$$P(\text{millionaire} \mid \text{mansion}) = \frac{P(\text{millionaire} \cap \text{mansion})}{P(\text{mansion})}$$

$$= \frac{\left(\frac{1}{40000}\right)}{\left(\frac{1}{30000}\right)}$$

$$= \frac{3}{4}$$

EXAM HINT

It can sometimes be difficult to interpret questions to decide whether they want conditional probability or combined probability. For example, if the question tells you that a boy has green eyes and asks what is the probability that he also has brown hair, it is tempting to find P(green eyes $\cap$ brown hair). However the fact that he has green eyes has been given, so we should actually find P(brown hair | green eyes).

Exercise 22F

1. For each of the questions below write the probability required in mathematical notation. An expression rather than a number is required.
   (a) Find the probability that the outcome on a dice is prime and odd.
   (b) Find the probability that a person is from either Senegal or Taiwan.
   (c) A student is studying the International Baccalaureate®. Find the probability that he is also studying French.
   (d) If a playing card is a red card, find the probability that it is a heart.
   (e) What proportion of German people live in Munich?
   (f) What is the probability that someone is wearing neither black nor white socks?
   (g) What is the probability that a vegetable is a potato if it is not a cabbage?
   (h) What is the probability that a ball drawn is red given that the ball is either red or blue.

2. (a) (i) If $\mathrm{P}(X)=0.3$ and $\mathrm{P}(X \cap Y)=\frac{1}{5}$, find $\mathrm{P}(Y \mid X)$.
   (ii) If $\mathrm{P}(Y)=0.8$ and $\mathrm{P}(X \cap Y)=\frac{3}{7}$, find $\mathrm{P}(X \mid Y)$.
   (b) (i) If $\mathrm{P}(X)=0.4$, $\mathrm{P}(Y)=0.7$ and $\mathrm{P}(X \cap Y)=\frac{1}{4}$, find $\mathrm{P}(X \mid Y)$.
   (ii) If $\mathrm{P}(X)=0.6$, $\mathrm{P}(Y)=0.9$ and $\mathrm{P}(X \cap Y)=\frac{1}{2}$, find $\mathrm{P}(Y \mid X)$.

3. The events A and B are such that
   $\mathrm{P}(A)=0.6,\ \mathrm{P}(B)=0.2,\ \mathrm{P}(A \cup B)=0.7$.
   (a) (i) Find the value of $\mathrm{P}(A \cap B)$.
   (ii) Hence show that A and B are not independent.
   (b) Find the value of $\mathrm{P}(B \mid A)$. *[7 marks]*

4. Let A and B be events such that $\mathrm{P}(A) = \frac{2}{3}$, $\mathrm{P}(B \mid A) = \frac{1}{2}$ and $\mathrm{P}(A \cup B) = \frac{4}{5}$.

(a) Find $\mathrm{P}(A \cap B)$.

(b) Find $\mathrm{P}(B)$.

(c) Show that A and B are not independent. *[7 marks]*

5. $\mathrm{P}(A) = \frac{2}{3}$, $\mathrm{P}(A \mid B) = \frac{1}{5}$ and $\mathrm{P}(A \cup B) = \frac{4}{5}$. Find $\mathrm{P}(B)$. *[6 marks]*

22G Further Venn diagrams

If we have information about the number of people in many overlapping groups, a Venn diagram is a very useful way of representing the information.

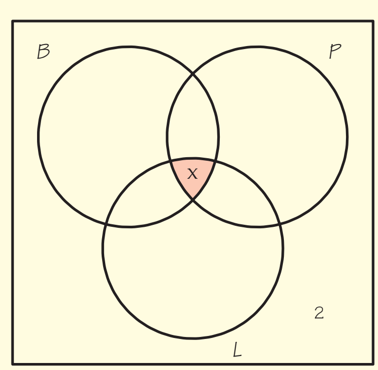
It is helpful to use the convention that the number we put into each region is the number or probability unique to that region. To do this it is often a good idea to label the intersection of all the groups with an unknown and work outwards. Do not try to label the total for regions joined together.

Worked example 22.13

In a class of 32, 19 have a bicycle, 21 have a mobile phone and 16 have a laptop computer. 11 have both a bike and a phone, 12 have both a phone and a laptop and 6 have both a bike and a laptop. 2 have none of these objects.

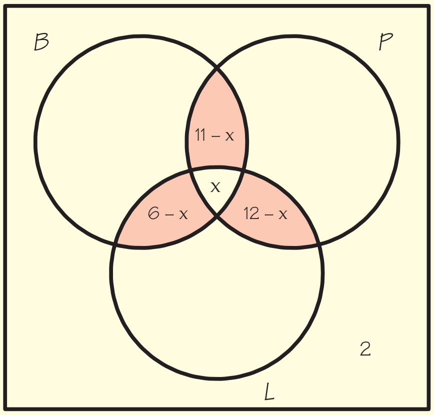
How many have a bike, phone and a laptop?

Draw a Venn diagram showing three overlapping groups, and label the size of the central region as x. We know that 2 people are outside these regions

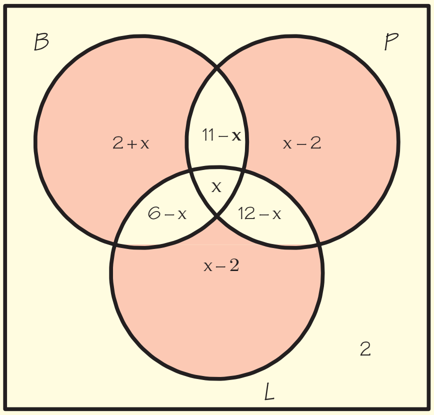


continued . . .

Work outwards. For example the number who have a bicycle and a phone but not a laptop will be $11 - x$



Continue working outwards. For example, the total of all the bicycle regions must be 19, so the remaining section is $19-(11-x)-(6-x)-x$ which is $2+x$



Use the fact that there are 32 people in the class to form an equation

$$(2+x)+(11-x)+(6-x)+x+(x-2)+(12-x)+(x-2)+2=32$$

$\Leftrightarrow 29+x=32$

$\Leftrightarrow x=3$

Therefore three people have a bicycle, a phone and a laptop.

Venn diagrams are particularly useful when thinking about conditional probability. We can use the given information to exclude parts of the Venn diagram which are not relevant.

Worked example 22.14

Daniel has 18 toys. 12 are made of plastic and 13 are red. 2 are neither red nor plastic.

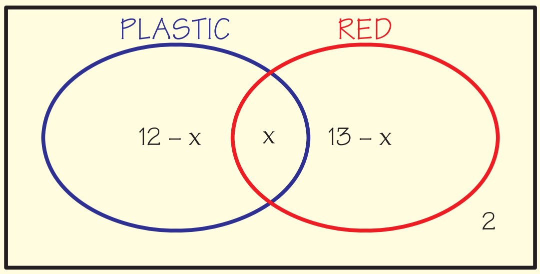
Daniel chooses a toy at random.

(a) Find the probability that it is a red plastic toy.

(b) If it is a red toy, find the probability that it is plastic.

We need to find the size of the intersection. To do this put the given information into a Venn diagram

(a)



We know that there are 18 toys

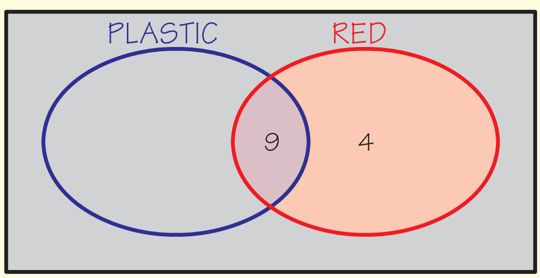
$(12 - x) + x + (13 - x) + 2 = 18 \Leftrightarrow 27 - x = 18$

$\Leftrightarrow x = 9$

$\therefore \text{P}(\text{plastic and red}) = \frac{9}{18} = \frac{1}{2}$

We can focus on the red toys

(b)



9 out of 13 red toys are plastic

$\therefore \text{P}(\text{plastic} \mid \text{red}) = \frac{9}{13}$

As well as frequencies, Venn diagrams can also represent probabilities. We can still apply the same methods.

Worked example 22.15

Events A and B are such that $P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cup B) = 0.9$. Find $P(B' \mid A')$.

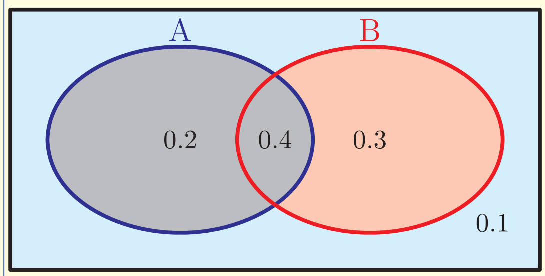
Find $P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore 0.9 = 0.6 + 0.7 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.4$

Draw a Venn diagram



The probability of not being in A is 0.4. Out of this, the probability of not being in B is 0.1

$\therefore P(B' \mid A') = \frac{0.1}{0.4} = \frac{1}{4}$

Some people find this is a useful way of thinking about the formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. Are visual arguments clearer than mathematical arguments? If so, why?

Exercise 22G

1. Out of 145 students in a college, 34 play football, 18 play badminton, and 5 play both sports.
   (a) Draw a Venn diagram showing this information.
   (b) How many students play neither sport?
   (c) What is the probability that a randomly chosen student plays badminton?
   (d) If we know that the chosen student plays football, what is the probability that they also play badminton?

2. Out of 145 students in a college, 58 study mathematics, 47 study economics and 72 study neither of the two subjects.
   (a) Draw a Venn diagram to show this information.

(b) How many students study both subjects?

(c) A student tells you that he studies mathematics. What is the probability that he studies both mathematics and economics?

3. Denise conducts a survey about food preferences in the college. She asks students which of the three meals (spaghetti bolognese, chilli con carne, and vegetable curry) they would eat. She finds out that, out of the 145 students:

43 would eat spaghetti bolognese

80 would eat vegetable curry

20 would eat both the bolognese and the curry

24 would eat both curry and chilli

35 would eat both chilli and bolognese

12 would eat all three meals

10 would not eat any of the three meals

(a) Draw a Venn diagram showing this information.

(b) How many students would eat only bolognese?

(c) How many students would eat chilli?

(d) What is the probability that a randomly selected student would eat only one of the three meals?

(e) Given that a student would eat only one of the three meals, what is the probability that they would eat curry?

(f) Find the probability that a randomly selected student would eat at least two of the three meals.

4. The probability that a girl has dark hair is 0.7, the probability that she has blue eyes is 0.4 and the probability that she has both dark hair and blue eyes is 0.2.

(a) Draw a Venn diagram showing this information.

(b) Find the probability that a girl has neither dark hair nor blue eyes.

(c) Given that a girl has dark hair, find the probability that she also has blue eyes.

(d) Given that a girl does not have dark hair, find the probability that she has blue eyes.

(e) Are the characteristics of having dark hair and having blue eyes independent? Explain your answer. *[11 marks]*

5. The probability that it rains on any given day is 0.45, and the probability that it is cold is 0.6. The probability that it is neither cold nor raining is 0.25.

(a) Find the probability that it is both cold and raining.

(b) Draw a Venn diagram showing this information.

(c) Given that it is raining, find the probability that it is not cold.

(d) Given that it is not cold, find the probability that it is raining.

(e) Are the events 'it's raining' and 'it's cold' independent? Explain your answer and show any supporting calculations. *[12 marks]*

22H Bayes' theorem

A test for a rare medical disease is 99% accurate in the sense that:

P (positive result | you have the disease) = 0.99 and

P (negative result | you don't have the disease) = 0.99

If you take the test and a positive result comes back, how likely is it that you have the disease? Most peoples' instinctive reaction to this is 99%, but that is not necessarily even close to the right answer. The problem is that the probability you are interested in is not P (positive result | you have the disease) but P (you have the disease | positive result). These two probabilities can be very different, and **Bayes' theorem** is a formula for relating the two.

Notice that the two given probabilities do not both have to be the same (in fact, they can be very different), so you have to be careful what you mean when you say that a test is 99% accurate. The two cases when the test gives a wrong result are referred to as false positives and false negatives, and are related to errors is hypothesis testing, which you will study if you do Statistics Option 7.

The starting point is the formula for conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

We can replace $P(A \cap B)$ with $P(A)P(B \mid A)$ to get Bayes' theorem:

KEY POINT 22.9

Bayes' theorem:

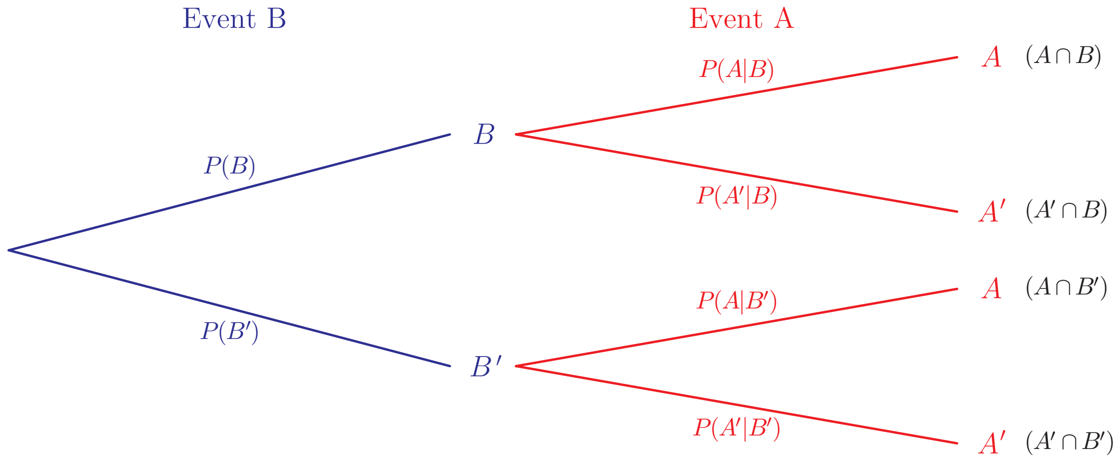
$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$$

EXAM HINT

Note that for P(B | A) it is:

$$P(B \mid A) = \frac{P(B)\,P(A \mid B)}{P(A)}$$

Often the hardest thing to calculate in Bayes' theorem turns out to be the denominator. This is often done by drawing out a tree diagram:



From the tree diagram we can see that there are two ways to get A, either after B has happened or after B has not happened. This can be expressed as $\mathrm{P}(A) = \mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')$, leading to the formula quoted in your Formula booklet:

KEY POINT 22.10

Bayes' theorem

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$$

Applying Bayes' theorem to the disease detection example above, if we denote A = positive result and B = you have the disease, then the required probability is $\mathrm{P}(B \mid A)$ and the information given tells us that

$\mathrm{P}(A \mid B) = 0.99$ and $\mathrm{P}(A' \mid B') = 0.99$, implying that

$\mathrm{P}(A \mid B') = 0.01$.

Hence:

$$\mathrm{P}(\text{you have the disease} \mid \text{positive result}) = \frac{0.99\ \mathrm{P}(B)}{0.99\ \mathrm{P}(B) + 0.01\ (1 - \mathrm{P}(B))}$$

We do not have sufficient information to calculate this probability, the answer depends on $\mathrm{P}(B)$ which represents the incidence of the disease in the population. It turns out that if $\mathrm{P}(B)$ is very small then so is the required probability. For example, if $\mathrm{P}(B) = \frac{1}{100}$ (so 1% of the population have the disease), the required probability is 50%, but if $\mathrm{P}(B) = \frac{1}{10000}$ it is under 1%. We can make sense of this by arguing 'the disease

The answer to this question may not be what you expected, but it was derived logically using mathematical formulae. When intuition and logic conflict, how do you know which one to trust? Mathematicians have responded to this by developing a rigorous system of proof so that they don't need to rely on intuition.

is so rare that a positive test is far more likely to be a faulty test result of a healthy person than an accurate result of a diseased person'.

The next example illustrates the use of Bayes' theorem when all the required information is given.

Worked example 22.16

Marika travels to school by bus on 70% of school days and by car on the other 30%. The probability that she is late is 0.05 if she travels by bus and 0.12 if she travels by car. On a particular day, Marika is late for school. What is the probability that she travelled by bus?

Write given information and required probability using mathematical notation

Let L = late for school and B = travels by bus

Then:

$P(B) = 0.7,\ P(L|B) = 0.05,\ P(L|B') = 0.12$

To find: $P(B|L)$

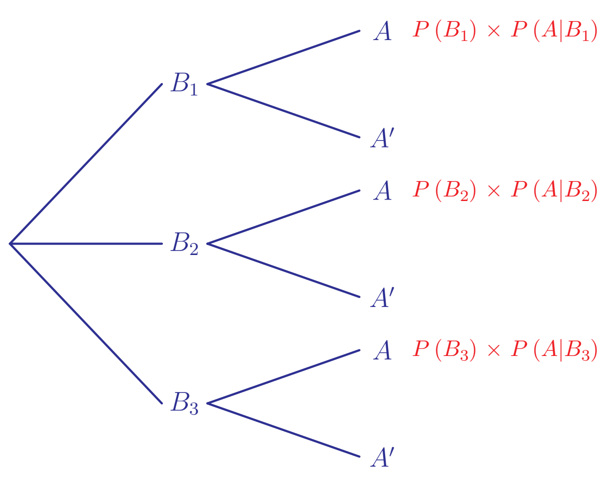
Need to 'reverse the conditional probability' so use Bayes' theorem

$$P(B|L) = \frac{P(B)P(L|B)}{P(B)P(L|B) + P(B')P(L|B')}$$

$$= \frac{0.7 \times 0.05}{0.7 \times 0.05 + 0.3 \times 0.12}$$

$$= 0.493$$

It is possible to extend Bayes' theorem to situations which have more options than simply B occurs or B does not occur. Suppose that there were three possible outcomes for the first event, called B_1, B_2 and B_3:



From this we can see that the total probability of A occurring can be expressed as:

$$P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)$$

Substituting this into Key point 22.9 gives the second version of Bayes' theorem quoted in the Formula booklet:

KEY POINT 22.11

Bayes' theorem

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + \ldots + P(B_n)P(A \mid B_n)}$$

Importantly, outcomes B_1, B_2 up to and including B_n must cover all the possible outcomes.

Worked example 22.17

The Adelaide Eagles want to be sponsored by the International Baccalaureate©. If they come first in the league there is a 90% chance that they will be sponsored. If they come second there is a 20% chance that they will be sponsored and if they come lower than second there is a 5% chance that they will be sponsored. There is a 30% chance that they will come first in the league and a 20% chance that they will come second. At the end of the season they are sponsored by the International Baccalaureate©. What is the probability that they came first in the league?

Decide what conditional probability is being asked for and then write out Bayes' theorem in terms of the events mentioned

Let Sp be the event 'being sponsored'

$$P(1^{st} \mid Sp) = \frac{P(Sp \mid 1^{st})P(1^{st})}{P(1^{st})P(Sp \mid 1^{st}) + P(2^{nd})P(Sp \mid 2^{nd}) + P(< 2^{nd})P(Sp \mid < 2^{nd})}$$

$$= \frac{0.9 \times 0.3}{0.3 \times 0.9 + 0.2 \times 0.2 + 0.5 \times 0.05}$$

$$= 0.806$$

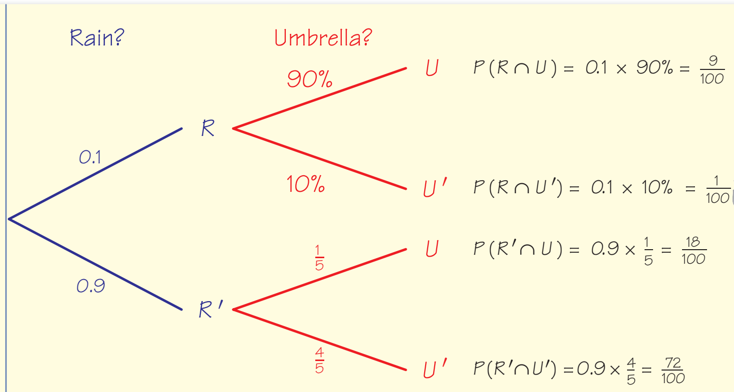
The formula for Bayes' theorem is quite complicated, and you may be worried about identifying all the probabilities correctly. The following example shows that you can also answer these questions without using the formula, you just need the ideas about tree diagrams and conditional probability (this is essentially repeating what we did in deriving the formula).

This illustrates one of the functions of proof in mathematics: To suggest a method for tackling related problems. Does this change your opinion about the usefulness of proof?

Worked example 22.18

If it is raining in the morning there is a 90% chance that I will bring my umbrella. If it is not raining in the morning there is only a $\frac{1}{5}$ chance of me taking my umbrella. On any given morning the probability of rain is 0.1. If you see me with an umbrella, what is the probability that it was raining that morning?

First draw a tree diagram



Write what is required in probability notation: we realise that it is a conditional probability, so use the correct formula

We want $P(\text{rain} \mid \text{umbrella}) = \dfrac{P(\text{rain} \cap \text{umbrella})}{P(\text{umbrella})}$

Use the tree diagram to find relevant probabilities

$$P(\text{rain} \cap \text{umbrella}) = \frac{9}{100}$$

$$P(\text{umbrella}) = \frac{9}{100} + \frac{18}{100} = \frac{27}{100}$$

Use the conditional probability formula

$$P(\text{rain} \mid \text{umbrella}) = \frac{P(\text{rain} \cap \text{umbrella})}{P(\text{umbrella})}$$

$$\frac{\frac{9}{100}}{\frac{27}{100}} = \frac{1}{3}$$

Exercise 22H

1. Box A contains 6 red balls and 4 green balls. Box B contains 5 red balls and 3 green balls. A standard fair cubical die is thrown. If a 6 is scored, a ball is selected from box A; otherwise a ball is selected from box B.

(a) Calculate the probability that the ball selected was red.

(b) Given that the ball selected was red, calculate the probability that it came from box B. *[7 marks]*

2. Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 7.30 a.m. train on Monday is $\frac{2}{3}$. The probability that he catches the 7.30 a.m. train on any other weekday is 90%. A weekday is chosen at random.

(a) Find the probability that he catches the 7.30 a.m train on that day.

(b) Given that he catches the 7.30 a.m. train on that day, find the probability that the chosen day is Monday. *[7 marks]*

3. Bag 1 contains 6 red cubes and 10 blue cubes. Bag 2 contains 7 red cubes and 3 blue cubes.

Two cubes are drawn at random, the first from bag 1 and the second from bag 2.

(a) Find the probability that the cubes are of the same colour.

(b) Given that the cubes selected are of different colours, find the probability that the red cube was selected from bag 1. *[8 marks]*

4. On any day in April there is a $\frac{2}{3}$ chance of rain in the morning. If it is raining there is a $\frac{4}{5}$ chance I will remember my umbrella, but if it is not raining there is only a $\frac{2}{5}$ chance of remembering my umbrella.

(a) On a random day in April, what is the probability I have my umbrella?

(b) Given that I have an umbrella on a day in April, what is the probability that it was raining? *[6 marks]*

5. The probability that a man leaves his umbrella in any shop he visits is $\frac{1}{5}$. After visiting two shops in succession, he finds he has left his umbrella in one of them. What is the probability that he left his umbrella in the second shop? *[4 marks]*

6. Only two international airlines fly daily into an airport. Pi Air has 40 flights a day and Lambda Air has 25 flights a day. Passengers flying with Pi Air have a $\frac{1}{10}$ probability of losing their luggage and passengers flying with Lambda Air have a $\frac{1}{4}$ probability of losing their luggage. Someone complains that their luggage has been lost. Find the probability that they travelled with Pi Air. *[6 marks]*

7. A girl walks to school every day. If it is not raining, the probability that she is late is $\frac{1}{5}$. If it is raining, the probability that she is late is $\frac{2}{3}$. The probability that it rains during her walk to school on a particular day is $\frac{1}{4}$. On one particular day the girl is late. Find the probability that it was raining on that day. *[7 marks]*

8. If $\text{P}(A) = 0.3$, $\text{P}(B \mid A') = 0.4$ and $\text{P}(A \mid B) = \frac{3}{17}$ find $\text{P}(B \mid A)$. *[6 marks]*

9. Lisa enters a chess tournament in which the result of every match is either win, lose or draw. The probability that she wins the tournament if she wins her first match is 60%. The probability that she wins the tournament if she draws her first match is 50%. The probability that she wins the tournament if she loses her first match is 10%. There is a 50% chance that she wins her first match and a 30% chance that she draws her first match. Given that she wins the tournament, find the probability that she drew her first match. *[6 marks]*

10. When Omar goes to school he walks $\frac{1}{4}$ of the time, catches the bus $\frac{1}{3}$ of the time and cycles the remaining times. If he walks there is a 50% chance of being late. If he catches the bus there is a 25% chance of being late and if he cycles there is a 10% chance of being late. If he is late, what was the probability that he caught the bus? *[6 marks]*

11. Two events A and B are such that $\text{P}(B \mid A) = \frac{3}{5}$ and $\text{P}(B' \cap A) = \frac{1}{3}$. Find $\text{P}(A \cap B)$. *[6 marks]*

12. A new blood test has been devised for early detection of a disease. Studies show that the probability that the blood test correctly identifies someone with this disease is 0.95, and the probability that the blood test correctly identifies someone without that disease is 0.99. The incidence of this disease in the general population is 0.0003.

The result of the blood test on one patient indicates that he has the disease. What is the probability that this patient has the disease? *[8 marks]*

13. You have two coins: one is a normal fair coin with heads on one side and tails on the other. The second coin has heads on both sides. You randomly pick a coin and flip it. The result comes up heads. What is the probability that you chose the fair coin? *[7 marks]*

Summary

- Probability is a measure of how likely an event is, varying from 0 for impossible events up to 1 for certain events.
- The **sample space** is a list of all the possible outcomes.
- The probability of an event A not happening is the **complement** of A (A') and $P(A) + P(A') = 1$.
- Probability can be estimated by looking at previous data (experimental probability) or it can be predicted by finding what proportion of the sample space contains the event (theoretical probability).
- If we have to look at a function of two events, such as their sum or their product, a convenient way of illustrating this is in a probability grid diagram.
- **Venn diagrams** are a useful tool for illustrating how different events can be combined; for representing the information when there are many overlapping groups; and for conditional probability. To calculate probabilities from a Venn diagram, label the intersection of all the groups with an unknown and work outwards.
- In combined probability, the probability of events A or B or both occurring is known as the **union** of A and B: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; in the case when the events are mutually exclusive this becomes: $P(A \cup B) = P(A) + P(B)$.
- The probability of events A and B both occurring is the **intersection** of A and B: $P(A \cap B)$.
- The probability of an event A happening given that an event B has already happened is known as **conditional probability** and is denoted by $P(A|B) = \frac{P(A \cap B)}{P(B)}$ where $P(A|B)$ means the probability of A given B.
- When two events, A and B, do not affect one another, they are said to be **independent events**:
$$P(A \cap B) = P(A)P(B)$$
- A sequence of events can be illustrated by a **tree diagram**. To calculate the probability following along a particular path of a tree diagram we use $P(A \cap B) = P(A)P(B | A)$, (for conditional probability) and $P(A \cap B) = P(A)P(B)$ and for independent events.

- **Bayes' theorem** is a formula for relating conditional probabilities given that $P(A|B)$ could be very different from $P(B|A)$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$ (when there are only two outcomes, e.g. event B occurs or does not occur)

or $$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$$ (where there are more than two outcomes, e.g. $B_1, B_2, \ldots B_n$.)

- The same outcome provided by Bayes' theorem can be achieved using a tree diagram, which can often be the far less complicated method.

- Counting principles, learned in chapter 1, can be helpful when calculating probabilities: $P(A) = \frac{\text{number of ways } A \text{ occurs}}{\text{total number of ways}}$.

Introductory problem revisited

A woman gives birth to non-identical twins. One of them is a girl. What is the probability that the other one is a girl?

There are four equally likely possibilities for the how twins can be born:

First child	Second child
Boy	Boy
Boy	Girl
Girl	Boy
Girl	Girl

If you find this result intriguing, you may like the famous Monty Hall problem.

If you gave the answer $\frac{1}{2}$ to this question, would you be making a mathematical mistake or an error of interpretation? Are they the same thing?

If we are told that one of them is a girl we can exclude the first situation. This leaves three equally likely situations in which only one is two girls, therefore the probability is $\frac{1}{3}$.
We end with a word of warning: it is very tempting to argue that the probability should be $\frac{1}{2}$, as the probability of the second child being a girl is independent of the gender of the first child. This would indeed be the case if the question had said that the *first child* is a girl, rather than that *one of the children* is a girl. Our intuition about probabilities is often flawed, which is why it is important to have precise mathematical methods and use language accurately.

Mixed examination practice 22

Short questions

1. A drawer contains 6 red socks, 4 black socks and 8 white socks. Two socks are picked at random. What is the probability that two socks of the same colour are drawn? *[5 marks]*

2. In a bilingual school there is a class of 21 pupils. In this class, 15 of the pupils speak Spanish as their first language and 12 of these 15 pupils are Argentine. The other 6 pupils in the class speak English as their first language and 3 of these 6 pupils are Argentine.

A pupil is selected at random from the class and is found to be Argentine. Find the probability that the pupil speaks Spanish as his/her first language. *[4 marks]*

(IB Organization 1999)

3. The probability that it rains during a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25°C is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25°C on a particular summer's day, find the probability that it rained on that day. *[6 marks]*

4. A set of five alphabet blocks bearing the letters A, C, H, R and T are dropped in a line at random.

(a) What is the probability that they spell out the word CHART?

(b) What is the probability that the word HAT is formed by three consecutive letters? *[6 marks]*

5. Given that $(A \cup B)' = \varnothing$, $\mathrm{P}(A' \mid B) = \frac{1}{5}$ and $\mathrm{P}(A) = \frac{14}{15}$, find $\mathrm{P}(B)$. *[5 marks]*

Long questions

1. (a) A large bag of sweets contains 8 red and 12 yellow sweets. Two sweets are chosen at random from the bag without replacement. Find the probability that 2 red sweets are chosen.

(b) A small bag contains 4 red and n yellow sweets. Two sweets are chosen without replacement from this bag. If the probability that two red sweets are chosen is $\frac{2}{15}$, show that $n = 6$.

Ayesha has one large bag and two small bags of sweets. She selects a bag at random and then picks two sweets without replacement.

(c) Calculate the probability that two red sweets are chosen.

(d) Given that two red sweets are chosen, find the probability that Ayesha had selected the large bag. *[15 marks]*

2. (a) If $P(X)$ represents a probability, state the range of $P(X)$.

(b) Express $P(A) - P(A \cap B)$ in terms of $P(A)$ and $P(B \mid A)$.

(c) (i) Show that:

$$P(A \cup B) - P(A \cap B) = P(A)\big(1 - P(B \mid A)\big) + P(B)\big(1 - P(A \mid B)\big).$$

(ii) Hence show that $P(A \cup B) \geq P(A \cap B)$. *[9 marks]*

3. The probability that a student plays badminton is 0.3.The probability that a student plays neither football nor badminton is 0.5, and the probability that a student plays both sports is x.

(a) Draw a Venn diagram showing this information.

(b) Find the probability that a student plays football, but not badminton.

Given that a student plays football, the probability that they also play badminton is 0.5.

(c) Find the probability that a student plays both badminton and football.

(d) Hence complete your Venn diagram. What is the probability that a student plays only badminton?

(e) Given that a student plays only one sport, what is the probability that they play badminton? *[13 marks]*

23 Discrete probability distributions

In this chapter you will learn:

- what a random variable is and how to list its possible values
- how to predict the average value and spread of a random variable
- how to calculate the probability of getting a given number of successes over a fixed number of trials (Binomial distribution)
- how to calculate probabilities for events which occur at a fixed rate (Poisson distribution).

Introductory problem

A casino offers a game where a coin is tossed repeatedly. If the first head occurs on the first throw you get £2, if the first head occurs on the second throw you get £4, if the first head is on the third throw you get £8, and so on with the prize doubling each time. How much should the casino charge for this game if they want to make a profit?

In statistics we often find the mean or standard deviation of data we have already collected. However, in real life it is often useful to be able to predict these quantities in advance. Even though it is impossible in a random situation to predict the outcome of a single event, such as one roll of a die, it turns out that if you look at enough events, the average can be predicted quite precisely.

23A Random variables

A **random variable** is a quantity whose value depends upon chance; for example, the outcome when a die is rolled. If the probabilities associated with each possible value are known, useful mathematical calculations can be made. A random variable is conventionally represented by a capital letter and the values which the random variable can take are represented by the equivalent small letter. We may use X to represent the random variable outcome when a die is rolled and in one particular experiment you may find that $x = 2$.

The list of all values in the sample space of a random variable, together with their corresponding probabilities, is called the **probability distribution** or **probability mass function** of the variable; this information is often best displayed in a table.

The idea of being able to predict averages but not individual events is central to many areas of knowledge. Economists cannot predict what an individual will do when interest rates increase, but they can predict how it will affect the economy. In a waterfall a physicist knows that any given water molecule may actually be moving up, but on average the flow is definitely going to be downwards.

Worked example 23.1

Draw a table to show the probability mass function of the outcomes of a fair six-sided die.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probabilities cannot be just any numbers:

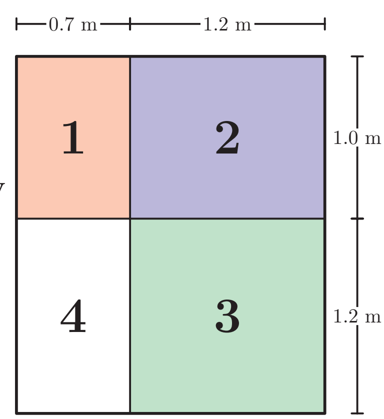
KEY POINT 23.1

The total of all the probabilities of a probability distribution must always equal 1.

This fact is useful if we do not have complete information about the probabilities.

Worked example 23.2

In a game at a fair, a ball is thrown at a rectangular target. The dimensions of the target (in metres) are as shown. The probability of hitting each region is proportional to its area. The prize for hitting a region is the number of chocolates equal to the number shown in that region. Find the probability distribution of the number of chocolates won.



Show the possible values in the table

Express the fact that the probability is proportional to the area by writing $p = k \times \text{area}$

Let X = the number of chocolates won.

x	1	2	3	4
$P(X = x)$	$0.7k$	$1.2k$	$1.44k$	$0.84k$

→

continued . . .

Use the fact that the probabilities add up to 1

$0.7k + 1.2k + 1.44k + 0.84k = 1$
$\therefore k = 0.239$

We can now calculate all the probabilities
We are not asked for exact values, so round them to 3SF

x	1	2	3	4
$P(X = x)$	0.167	0.287	0.344	0.201

One of the most obvious questions to ask about a random variable is what value it is most likely to have. This value is called the **mode**. The random variable X in the above example has mode 3; the most likely number of chocolates you will win is three. A random variable may not have a mode (for example, the outcomes of a fair die are all equally likely) or it may have more than one mode. In particular, if the largest probability corresponds to two of the outcomes, the random variable is called **bimodal**.

Another question we could ask is, if we were to play the above game many times, on average how many chocolates would we expect to win? The answer is not necessarily the same as the most likely outcome. We will see how to answer this question in the next section.

Exercise 23A

In this exercise you will need to use ideas from chapter 22, particularly tree diagrams. For Question 2(c) you may want to look at chapter 7 on Geometric sequences.

1. For each of the following, draw out a table to represent the probability distribution of the random variable described:
   (a) A fair coin is thrown four times. The random variable W is the number of tails obtained.
   (b) Two fair dice are thrown. The random variable D is the difference between the larger and the smaller score, or zero if they are the same.
   (c) A fair die is thrown once. The random variable X is calculated as half the result if the die shows an even number, or one higher than the result if the die shows an odd number.
   (d) A bag contains six red and three green counters. Two counters are drawn at random from the bag without replacement. G is the number of green counters remaining in the bag.

(e) Karl picks a card at random from a standard pack of 52 cards. If he draws a diamond, he stops; otherwise, he replaces the card and continues to draw cards at random, with replacement, until he has either drawn a diamond or has drawn a total of 4 cards. The random variable C is the total number of cards drawn.

(f) Two fair four-sided spinners, each labelled 1, 2, 3 and 4, are spun. The random variable X is the product of the two values shown.

2. Find the missing value k:

(a) (i)

x	3	7	9	11
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	k

(ii)

x	5	6	7	10
$\mathrm{P}(X = x)$	0.2	0.3	k	0.5

(b) (i) $\mathrm{P}(Y = y) = ky$ for $y = 1, 2, 3, 4$

(ii) $\mathrm{P}(X = x) = \frac{k}{x}$ for $y = 1, 2, 3, 4$

(c) (i) $\mathrm{P}(X = x) = k(0.1)^x$ for $x \in \mathbb{N}$

(ii) $\mathrm{P}(R = r) = k(0.9)^r$ for $y \in \mathbb{N}$

3. In a game a player rolls a biased four-sided die. The probability of each possible score is shown below.

Score	1	2	3	4
Probability	$\frac{1}{3}$	$\frac{1}{4}$	k	$\frac{1}{5}$

Find the probability that the total score is four after two rolls. *[5 marks]*

23B Expectation, median and variance of a discrete random variable

The **expectation** of a random variable is a value which represents the mean result if the variable were to be repeatedly measured an infinite number of times. It is a representation of the 'average' value of the random variable.

KEY POINT 23.2

The expected value of a discrete random variable X is written $\mathrm{E}(X)$ and calculated as:

$$\mathrm{E}(X) = \sum_x x\mathrm{P}(X = x)$$

EXAM HINT

In the Formula booklet, μ is included to denote $\sum_x x\,\mathrm{P}(X = x)$.

Worked example 23.3

The random variable X has probability distribution as shown in the table below. Calculate $\mathrm{E}(X)$.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$

Apply the formula

$$\mathrm{E}(X) = 1\times\frac{1}{10} + 2\times\frac{1}{4} + 3\times\frac{1}{10} + 4\times\frac{1}{4} + 5\times\frac{1}{5} + 6\times\frac{1}{10}$$

$$= \frac{7}{2}$$

Just as the mean of a set of integers could be fractional, so the expectation of a random variable need not be a value which the variable can itself take.

To find the median of a discrete random variable we use the defining property of the median – that half of the data should fall below it. In the context of a probability distribution this means that:

Median, m, is the smallest value of X that satisfies $\mathrm{P}(X \le m)$ is more than $\frac{1}{2}$. If there is a value m such that $\mathrm{P}(X \le m) = \frac{1}{2}$ then the median is the mean of this value and the next largest value of X.

Probabilities of the form $\mathrm{P}(X \le x)$ which give the probability of being less than or equal to a certain value are called **cumulative probabilities**.

Worked example 23.4

Find the median of the probability distribution below:

x	1	3	6	8
$P(X = x)$	0.2	0.4	0.3	0.1

To find the median evaluate the probability of being below each value until you get above 0.5

$P(X \le 1) = 0.2$
$P(X \le 3) = 0.6$
Therefore the median is 3.

In the above example if the distribution had been

x	1	3	6	8
$P(X = x)$	0.2	0.3	0.4	0.1

then $P(X \le 3)$ is exactly 0.5. The median is the average of 3 and 6, so it is 4.5.

As well as knowing the expectation and median, we may also be interested in how far away from the average we can expect an outcome to be. The variance of a random variable is a value representing the degree of variation that would be seen if the variable were to be repeatedly measured an infinite number of times. It is related to how spread out the variable is.

Standard deviation is a much more meaningful representation of the spread of the variable. So why do we bother with variance at all? The answer is purely to do with mathematical elegance. If you do the statistics option (Topic option 7) you will see that the algebra of variance is far neater than the algebra of standard deviations.

KEY POINT 23.3

The variance of a random variable X is written Var (X) and is calculated as

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

where $\mathrm{E}(X^2) = \sum x^2 \mathrm{P}(X = x)$

(in the Formula booklet this includes an interim notation: $\mathrm{E}(X - \mu)^2$

This is the same idea as the variance of the set of data from Section 21B.

This formula is often quoted as 'the mean of the squares minus the square of the mean'.

Worked example 23.5

Calculate Var(X) for the probability distribution in Worked example 23.3.

Find the expectation

From above, $E(X) = 3.5$

Apply the values from the distribution

$$E(X^2) = 1^2 \times \frac{1}{10} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{10} + 4^2 \times \frac{1}{4} + 5^2 \times \frac{1}{5} + 6^2 \times \frac{1}{10}$$
$$= 14.6$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$= 14.6 - 12.25 = 2.35$$

Exercise 23B

1. Calculate the expectation, median and variance of each of the following random variables:

(a) (i)

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

(ii)

w	8	9	10	11
$P(W = w)$	0.4	0.3	0.2	0.1

(b) (i) $P(X = x) = \frac{x^2}{14}$, $x = 1, 2, 3$

(ii) $P(X = x) = \frac{1}{x}$, $x = 2, 3, 6$

2. A discrete random variable X is given by

$$P(X = x) = k(x + 1) \text{ for } x = 2, 3, 4, 5, 6.$$

(a) Show that $k = 0.04$.

(b) Find $E(X)$. *[5 marks]*

3. The discrete random variable V has the probability distribution shown below and $E(V) = 6.1$. Find the value of k and the median of V.

v	1	2	5	8	k
$P(V = v)$	0.2	0.3	0.1	0.1	0.3

[6 marks]

4. A discrete random variable X has its probability mass function given by

$$P(X = x) = k(x+3), \text{ where } x \text{ is } 0, 1, 2, 3.$$

(a) Show that $k = \dfrac{1}{18}$.

(b) Find the exact value of $E(X)$. *[6 marks]*

5. The probability distribution of a discrete random variable X is defined by:

$$P(X = x) = kx(4 - x), x = 1, 2, 3$$

(a) Find the value of x.

(b) Find $E(X)$. *[6 marks]*

6. A fair six-sided die, with sides numbered 1, 1, 2, 2, 2, 5 is thrown. Find the mean and variance of the score. *[6 marks]*

7. The table below shows the probability distribution of a discrete random variable X.

x	0	1	2	3
$P(X = x)$	0.1	p	q	0.2

(a) Given that $E(X) = 1.5$, find the values of p and q.

(b) Calculate $Var(X)$. *[9 marks]*

8. A biased die with four faces is used in a game. A player pays 5 counters to roll the die. The table below shows the possible scores on the die, the probability of each score and the number of counters the player wins for each score.

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{20}$
Number of counters player receives	4	5	15	n

Find the value of n in order for the player to get an expected return of 3.25 counters per roll. *[5 marks]*

9. In a game a player pays an entrance fee of $\$n$. He then selects one number from 1, 2, 3 or 4 and rolls three standard dice.

If his chosen number appears on all three dice he wins four times his entrance fee.

If his number appears on exactly two of the dice he wins three times the entrance fee.

If his number appears on exactly one die he wins \$1.

If his number does not appear on any of the dice he wins nothing.

(a) Copy and complete the probability table below.

Profit (\$)	$-n$		$2n$	$3n$
Probability		$\frac{27}{64}$		

(b) The game organiser wants to make a profit over many plays of the game. Given that he must charge a whole number of cents, what is the minimum amount the organiser must charge? *[10 marks]*

23C The binomial distribution

Some discrete probability distributions are met so often that they have been given names and formal notation. One of the most important of these is the **binomial distribution**. There are several others, some of which you will meet in this chapter and some if you study the statistics option (Topic option 7).

A binomial distribution occurs in situations where you have a set number of 'experiments' (or 'trials') each of which have two possible outcomes. The number of trials is usually denoted n. One outcome is conventionally called a 'success' and the other a 'failure'. The probability of success is denoted p. If the probability of success in a trial is constant, and trials are conducted independently of each other, then the number of successes can be modelled using the binomial distribution.

The symbol ~ is used to denote the concept 'follows this distribution', and one or two letter abbreviations are used for the standard distributions. So if a random variable X follows the binomial distribution with n trials and probability of success p, we would write $X \sim \mathrm{B}(n, p)$.

So what is this distribution? Let us consider a specific example: suppose a die is rolled four times, what is the probability of getting exactly two fives? There are four trials so $n = 4$ and if we label a five as a success then $p = \frac{1}{6}$. The probability of a failure is therefore $\frac{5}{6}$.

One way of getting two fives is if the first two times we get a five and the last two times we get something else. The probability of this happening is $\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$. But this is not the only way

Counting the number of possible selections was discussed in chapter 1.

in which two fives can occur. The two fives may be first and third or second and fourth. In fact, we have to consider all the ways in which we pick two trials out of the four for the 5 to occur. This can happen in $\binom{4}{2}$ ways. Each of them has the same probability as the first case. If X is the random variable 'number of 5s thrown when four dice are rolled' then we can say that:

$$P(X=2)=\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

The useful thing about identifying a binomial distribution is that you can then apply standard results without having to go through this argument every time. In particular, the expectation and variance of the binomial distribution can just be quoted using the formulae below. The proofs of these are beyond what is expected in the International Baccalaureate®, but if you are interested they are on Fill-in proof 25 'Expectation and variance of the binomial distribution' on the CD-ROM.

KEY POINT 23.4

Standard results of the binomial distribution

Statement of distribution	$X \sim B(n, p)$
Probability formula	$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$
Expectation (E(X))	np
Variance (Var(X))	$np(1-p)$

(Note: in the Formula booklet, the expectaion is referred to as the mean)

Worked example 23.6

Rohir has a 30% chance of correctly answering a multiple-choice question. In a test there are ten questions.

(a) What is the probability that Rohir gets exactly four of them correct? Give your answer to five significant figures.

(b) What is the probability that Rohir gets at least one correct in the first five questions?

(c) Suggest which requirements for a binomial distribution might not be satisfied in this situation?

continued . . .

Define the random variable if not already defined in the question

(a) Let X be the number of correct answers in the first ten questions

Give the probability distribution, checking that the conditions are met

$X \sim B(10, 0.3)$

Express the formula for the probability required, and calculate the answer

$P(X=4)=\binom{10}{4}(0.3)^4(0.7)^6=0.20012$ (5SF)

Define the random variable if not already defined in the question

(b) Let Y be the number of correct answers in the first five questions

Give the probability distribution

$Y \sim B(5, 0.3)$

Write down the probability required

We are interested in $X \geq 1$, which means that $X = 1, 2, 3, 4$ or 5. Remember that a quicker way to do the calculation is to find $1-P(X<1)$

$P(X \geq 1)=1-P(X=0)$

Express the formula for the probability required, and calculate the answer

$=1-\binom{5}{0}0.3^0\,0.7^5$

$=0.832$ (3SF)

Consider the requirements for the distribution

(c) Binomial requires:

- two outcomes at each trial
- constant probability of success in each trial
- trial results independent of each other

Identify a requirement which is failed in this context: there are two outcomes, and trials are independent (answering one question does not make it easier or harder to answer another)

All questions are not of the same difficulty, so there might not be a constant probability of success.

If you need to find a probability of a range of successes, you could in theory add up the probabilities of individual outcomes. This can be very time consuming, so your calculator has a function giving the probability of getting up to and including any number of successes.

EXAM HINT

Most GDCs can calculate binomial probabilities automatically given n and p, see Calculator sheet 13 on the CD-ROM. But you may also be tested on applying the formula, which is given in the Formula booklet.

Worked example 23.7

Random variable X has distribution B(15, 0.6). Find $P(5 < X \leq 10)$.

The calculator can give us probabilities of the form $P(X \leq k)$

$X \sim B(15, 0.6)$

$P(5 < X \leq 10) = P(X \leq 10) - P(X \leq 5)$

$= 0.7827 - 0.0338$

$= 0.749$ (3SF) (from GDC)

EXAM HINT

Even when you are using a calculator to find probabilities, you should still use correct mathematical notation (not calculator notation) in your answer. You do not need to explain how you did things on the calculator – just state the distribution you used, the probabilities calculated, and give the answer (usually to 3 significant figures).

Exercise 23C

Remember to round your answer to three significant figures when using the calculator.

1. The random variable X has a binomial distribution with $n = 8$ and $p = 0.2$. Calculate:

(a) (i) $P(X = 3)$ (ii) $P(X = 4)$

(b) (i) $P(X \leq 3)$ (ii) $P(X \leq 2)$

(c) (i) $P(X > 3)$ (ii) $P(X > 4)$

(d) (i) $P(X < 5)$ (ii) $P(X < 3)$

(e) (i) $P(X \geq 3)$ (ii) $P(X \geq 1)$

(f) (i) $P(3 < X \leq 6)$ (ii) $P(1 \leq X < 4)$

2. Given that $Y \sim B(5,\ 0.5)$, find the exact value of:

(a) (i) $P(Y = 1)$ (ii) $P(Y = 0)$

(b) (i) $P(Y \geq 1)$ (ii) $P(Y \leq 1)$

(c) (i) $P(Y > 4)$ (ii) $P(Y \leq 3)$

3. Find the mean and standard deviation of each of the following variables:

(a) (i) $Y \sim B\left(100,\ \frac{1}{10}\right)$ (ii) $X \sim B\left(16,\ \frac{1}{2}\right)$

(b) (i) $X \sim B(15,\ 0.3)$ (ii) $Y \sim B(20,\ 0.35)$

(c) (i) $Z \sim B\left(n-1,\ \frac{1}{n}\right)$ (ii) $X \sim B\left(n,\ \frac{2}{n}\right)$

4. (a) Jake beats Marco at chess in 70% of their games. Assuming that this probability is constant and that the results of games are independent of each other, what is the probability that Jake will beat Marco in at least 16 of their next 20 games?

(b) On a television channel, the news is shown at the same time each day; the probability that Salia watches the news on a given day is 0.35. Calculate the probability that on 5 consecutive days she watches the news on exactly 3 days.

(c) Sandy is playing a computer game and needs to accomplish a difficult task at least three times in five attempts in order to pass the level. There is a 1 in 2 chance that he accomplishes the task each time he tries, unaffected by how he has done before. What is the probability that he will pass to the next level?

5. 15% of students at a large school travel by bus. A random sample of 20 students is taken.

(a) Explain why the number of students in the sample who travel by bus is only approximately a binomial distribution.

(b) Use the binomial distribution to estimate the probability that exactly five of the students travel by bus. *[3 marks]*

6. A coin is biased so that when it is tossed the probability of obtaining heads is $\frac{2}{3}$. The coin is tossed 4050 times. Let X be the number of heads obtained. Find:

(a) the mean of X

(b) the standard deviation of X. *[3 marks]*

7. A biology test consists of eight multiple-choice questions. Each question has four answers, only one of which is correct. At least five correct answers are required to pass the test. Sheila does not know the answers to any of the questions, so answers each question at random.

(a) What is the probability that Sheila answers exactly five questions correctly?

(b) What is the expected number of correct answers Sheila will give?

(c) What is the standard deviation in the number of correct answers Sheila will give?

(d) What is the probability that Sheila manages to pass the test? *[7 marks]*

8. 0.8% of people in the country have a particular cold virus at any time. On a single day, a doctor sees 80 patients.

(a) What is the probability that exactly 2 of them have the virus?

(b) What is the probability that 3 or more of them have the virus?

(c) State an assumption you have made in these calculations. *[5 marks]*

9. Given that $Y \sim B(12,0.4)$:

(a) Find the expected mean of Y.

(b) Find the mode of Y. *[3 marks]*

10. On a fair die, which is more likely: rolling 3 sixes in 4 throws or rolling a five or a six in 5 out of 6 throws? *[6 marks]*

11. Over a one month period, Ava and Sven play a total of x games of tennis. The probability that Ava wins any game is 0.4. The result of each game played is independent of any other game played. Let X denote the number of games won by Ava over a one month period.

(a) Find an expression for $P(X = 2)$ in terms of n.

(b) If the probability that Ava wins two games is 0.121 correct to three decimal places, find the value of n.

[5 marks]

12. A coin is biased so that the probability of it showing tails is p. The coin is tossed n times. Let X be a random variable representing the number of tails. It is known that the mean of X is 19.5 and the variance is 6.825. Find the values of n and p. *[5 marks]*

13. A die is biased so that the probability of rolling a 6 is p. If the probability of rolling 2 sixes in 12 throws is 0.283 (to three significant figures), find the possible values of p correct to two decimal places. *[5 marks]*

14. In an experiment, a trial is repeated n times. The trials are independent and the probability p of success in each trial is constant. Let X be the number of successes in the n trials. The mean of X is 12 and the standard deviation is 2. Find n and p. *[5 marks]*

15. X is a binomial random variable, where the number of trials is 4 and the probability of success of each trial is p. Find the possible values of p if $P(X = 3) = 0.3087$. *[5 marks]*

16. X is a binomial random variable, where the number of trials is 4 and the probability of success of each trial is p. Find the possible values of p if $P(X = 2) = \frac{96}{625}$. *[6 marks]*

Question 10 is the problem which was posed to Pierre de Fermat in 1654 by a professional gambler who could not understand why he was losing. It inspired Fermat (with the assistance of Pascal) to set up probability as a rigorous mathematical discipline.

23D The Poisson distribution

When you are waiting for a bus there are at any given moment two possible outcomes – it either arrives or it does not. We can try modelling this situation using a binomial distribution, but it is not clear what an individual trial is. Instead we have a rate of success – the number of buses that arrive in a fixed time period.

EXAM HINT

If a question mentions average rate of success, or events occurring at a constant rate, you should use Poisson distribution. If you can identify a fixed number of trials then binomial distribution is required.

There are many situations in which we know the rate of events within a given space or time, in contexts ranging from commercial (such as the number of calls through a telephone exchange per minute) to biological (such as the number of clover plants seen per square metre in a pasture). Where the events occur singly (one at a time) and can be considered independent of each other (so that the probability of each event is not affected by what has already happened), the number of events in a fixed space or time interval can be modelled using **Poisson distribution**. This distribution is fully defined once we know the rate of success, which is conventionally called m.

KEY POINT 23.5

Standard results of the Poisson distribution

Statement of distribution	$X \sim \mathrm{Po}(m)$
Probability formula	$\mathrm{P}(X = x) = \dfrac{\mathrm{e}^{-m} m^x}{x!}$ *for* $x = 0, 1, 2, \ldots$
Expectation E(X)	m
Variance Var (X)	m

(Note: in the Formula booklet, E(X) is called the mean)

Worked example 23.8

Recordable accidents occur in a factory at an average rate of 7 every year, independently of each other. Find the probability that in a given year exactly 3 recordable accidents occurred.

Define the random variable: Let X be the number of accidents in a year

Give the probability distribution: $X \sim \mathrm{Po}(7)$

Write down the probability required, and calculate the answer:

$$\mathrm{P}(X = 3) = \frac{e^{-7} 7^3}{3!}$$

$$= 0.521 \text{ (3SF)}$$

The Poisson distribution is scaleable. If the number of butterflies seen on a flower in 10 minutes follows a Poisson distribution with mean (expectation) m, then the number of butterflies seen on a flower in 20 minutes follows a Poisson distribution with mean $2m$; the number of butterflies seen on a flower in 5 minutes follows a Poisson distribution with mean $\frac{m}{2}$.

EXAM HINT

See Calculator sheet 13 on the CD-ROM. Your GDC can calculate Poisson probabilities and cumulative probabilities, but you may be explicitly asked to use the formula. Remember to round your answers to 3SF.

Worked example 23.9

If there are an average of 12 buses per hour arriving at a bus stop, find the probability that there are more than 6 buses in 30 minutes.

Define the random variable	Let X be the number of buses in 30 minutes
Give the probability distribution	$X \sim Po(6)$
Write down the probability required. To use the calculator we must relate it to $P(X \leq k)$	$P(X > 6) = 1 - P(X \leq 6)$ $= 0.161$ (3SF) from GDC

Exercise 23D

1. State the distribution of the variable in each of the following situations:

   (a) Cars pass under a motorway bridge at an average rate of 6 per 10 second period.

   (i) the number of cars passing under the bridge in 1 minute

   (ii) the number of cars passing under the bridge in 15 seconds

   (b) Leaks occur in water pipes at an average rate of 12 per kilometre.

   (i) the number of leaks in 200 m

   (ii) the number of leaks in 10 km

(c) A widget machine manufactures on average 96 functional widgets out of 100.

(i) the number of faulty widgets in a sample of 10

(ii) the number of functioning widgets in sample of 20

(d) 12 worms are found on average in a 1 m^2 area of a garden.

(i) the number of worms found in a 0.3 m^2 area

(ii) the number of worms found in a 2 m by 2 m area

2. Calculate the following probabilities:

(a) If $X \sim \text{Po}(2)$

(i) $\text{P}(X = 3)$ (ii) $\text{P}(X = 1)$

(b) If $Y \sim \text{Po}(1.4)$

(i) $\text{P}(Y \le 3)$ (ii) $\text{P}(Y \le 1)$

(c) If $Z \sim \text{Po}(7.9)$

(i) $\text{P}(Z < 6)$ (ii) $\text{P}(Z < 10)$

(d) If $X \sim \text{Po}(5.9)$

(i) $\text{P}(X \ge 3)$ (ii) $\text{P}(X > 1)$

(e) If $X \sim \text{Po}(11.4)$

(i) $\text{P}(8 < X < 11)$ (ii) $\text{P}(8 \le X \le 12)$

3. A random variable X follows a Poisson distribution with mean 1.7. Copy and complete the following table of probabilities, giving results to 3 significant figures:

x	0	1	2	3	4	>4
$\text{P}(X = x)$	0.183					

4. From a particular observatory, shooting stars are observed in the night sky at an average rate of one every 5 minutes. Assuming that this rate is constant and that shooting stars occur (and are observed) independently of each other, what is the probability that more than 20 are seen over a period of 1 hour? *[4 marks]*

5. When examining blood from a healthy individual under a microscope, a haematologist knows that on average he should see 4 white blood cells in each high power field. Find the probability that blood from a healthy individual will show:

(a) 7 white blood cells in a single high power field

(b) a total of 28 white blood cells in 6 high power fields, selected independently. *[5 marks]*

6. Salah is sowing flower seeds in his garden. He scatters seeds randomly so that the number of seeds falling on any particular region is a random variable with a Poisson distribution, with mean value proportional to the area. He will sow fifty thousand seeds over an area of 2 m^2.

(a) Calculate the expected number of seeds falling on a 1 cm^2 region.

(b) Calculate the probability that a given 1 cm^2 area receives no seeds. *[4 marks]*

7. A wire manufacturer is looking for flaws. Experience suggests that there are on average 1.8 flaws per metre in the wire.

(a) Determine the probability that there is exactly 1 flaw in 1 metre of the wire.

(b) Determine the probability that there is at least one flaw in 2 metres of the wire. *[5 marks]*

8. The random variable X has a Poisson distribution with mean 5. Calculate:

(a) $P(X \le 5)$

(b) $P(3 < X \le 5)$

(c) $P(X \ne 4)$

(d) $P(3 < X \le 5 \mid X \le 5)$ *[8 marks]*

9. Patients arrive at random at an emergency room in a hospital at the rate of 14 per hour throughout the day.

(a) Find the probability that exactly 4 patients will arrive at the emergency room between 18:00 and 18:15.

(b) Given that fewer than 15 patients arrive in one hour, find the probability that more than 12 arrive. *[6 marks]*

10. The number of eagles observed in a forest in one day follows a Poisson distribution with mean 1.4.

(a) Find the probability that more than three eagles will be observed on a given day.

(b) Given that at least one eagle is observed on a particular day, find the probability that exactly two eagles are seen that day. *[6 marks]*

11. The random variable X follows a Poisson distribution. Given that $P(X \ge 1) = 0.4$, find:

(a) the mean of the distribution

(b) $P(X > 2)$. *[5 marks]*

12. The random variable X is Poisson distributed with mean m and satisfies $P(X = 3) = P(X < 3)$.

(a) Find the value of m, correct to four decimal places.

(b) For this value of m evaluate $P(2 \le X < 4)$. *[6 marks]*

13. Let X be a random variable with a Poisson distribution, such that $P(X > 2) = 0.3$. Find $P(X < 2)$. *[5 marks]*

14. The number of emails you receive per day follows a Poisson distribution with mean 6. Let D be the number of emails received in one day and W the number of emails received in a week.

(a) Calculate $P(D = 6)$ and $P(W = 42)$.

(b) Find the probability that you receive 6 emails every day in a seven-day week.

(c) Explain why this is not the same as $P(W = 42)$. *[8 marks]*

15. The number of mistakes a teacher makes whilst marking homework has a Poisson distribution with a mean of 1.6 errors per piece of homework.

(a) Find the probability that there are at least two marking errors in a randomly chosen piece of homework.

(b) Find the most likely number of marking errors occurring in a piece of homework. Justify your answer.

(c) Find the probability that in a class of 12 pupils fewer than half of them have errors in their marking. *[9 marks]*

16. A car company has two limousines that it hires out by the day. The number of requests per day has a Poisson distribution with mean 1.3 requests per day.

(a) Find the probability that neither limousine is hired.

(b) Find the probability that some requests have to be denied.

(c) If each limousine is to be equally used, on how many days in a period of 365 days would you expect a particular limousine to be in use? *[8 marks]*

17. A shop has 4 copies of the book 'Ballroom Dancing' delivered each week. The demand for the book follows a Poisson distribution with mean 3.2 requests per week.

(a) Calculate the probability that the shop cannot meet the demand in a given week.

(b) Find the most probable number of books sold in one week.

(c) Find the expected number of books sold in one week.

(d) Determine the smallest number of copies of the book that should be ordered each week to ensure that the demand is met with a probability of at least 98%. *[8 marks]*

18. The random variable X follows Poisson distribution with mean λ. If $\mathrm{P}(X=2)=\mathrm{P}(X=0)+\mathrm{P}(X=1)$, find the exact value of λ. *[4 marks]*

19. The random variable X follows a Poisson distribution with mean λ.

(a) Show that $\mathrm{P}(Y=y+2)=\dfrac{\lambda^2}{(y+1)(y+2)}\mathrm{P}(Y=y)$.

(b) Given that $\lambda=6\sqrt{2}$, find the value of y such that $\mathrm{P}(Y=y+2)=\mathrm{P}(Y=y)$. *[4 marks]*

Summary

- A **random variable** is a quantity whose value depends on chance. A list of all possible outcomes and their associated probabilities is called a **probability distribution** or **probability mass function**.
- The total of all the probabilities of a probability distribution must always equal 1.
- Even though the outcome of any one observation of a random variable is impossible to predict with any certainty, the **expectation** (of the mean) and variance of observations can be predicted quite accurately, using:

$$\mathrm{E}(X)=\sum_x x\mathrm{P}(X=x)$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2$$

- If there is a fixed number of trials (each with two possible outcomes) with constant and independent probability of success in each trial then the number of successes follows a **Binomial distribution**: $X \sim \mathrm{B}(n,p)$.
- If events occur singly, independently and at a constant rate, then the number of events in a given period follows a **Poisson distribution**: $X \sim \mathrm{Po}(m)$, where m is the **rate of success**.
- Once the distribution has been identified then probabilities and statistics for the distribution can be immediately quoted:

Distribution	Notation	$\mathrm{P}(X=x)$	$\mathrm{E}(X)$	$\mathrm{Var}(X)$
Binomial	$X \sim \mathrm{B}(n,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Poisson	$X \sim \mathrm{Po}(m)$	$\dfrac{\mathrm{e}^{-m}m^x}{x!}$	m	m

- Probabilities in the form $P(X \le x)$ give the probability of being less than or equal to a certain value and are called **cumulative probabilities**.
- You can use your GDC to calculate probabilities in the Binomial and Poisson distributions. Make sure you use the correct setting depending on whether the probability is cumulative or not.

Introductory problem revisited

A casino offers a game where a coin is tossed repeatedly. If the first head occurs on the first throw you get £2, if the first head occurs on the second throw you get £4, if the first head is on the third throw you get £8, and so on with the prize doubling each time. How much should the casino charge for this game if they want to make a profit?

The probability of getting heads on the first throw is $\frac{1}{2}$, so $P(\text{win £}2) = \frac{1}{2}$. The probability of the first head being on the second throw is $P(\text{tails}) \times P(\text{heads}) = \frac{1}{4}$, so $P(\text{win £}4) = \frac{1}{4}$; and so on.

If X is the random variable 'amount of money won', the probability distribution is:

X	2	4	8	…
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	…

$E(X)$ is therefore $2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \ldots$ which is $1 + 1 + 1 \ldots$

This sum continues for ever, therefore $E(X) = \infty$. The expected payout over a long period of time is infinite, the casino could not charge sufficient money to cover the expected payout.

Even if you were offered the opportunity to play this game, you should think twice. The result assumes that you can play the game infinitely many times, and in reality this is not the case. It is an example of a famous fallacy called 'Gambler's Ruin'.

Mixed examination practice 23

Short questions

1. A factory making bottles knows that on average, 1.5% of its bottles are defective. Find the probability that, in a randomly selected sample of 20 bottles, at least 1 bottle is defective. *[4 marks]*

2. A biased die with four faces is used in a game. A player pays 10 counters to roll the die and receives a number of counters equal to the value shown on the die. The table below shows the different values on the die and the probability of each.

Value	1	5	10	N
Probability	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$

Find the value represented by N, given that the player has an expected loss of 1 counter each time he plays the game. *[5 marks]*

3. When a boy bats at baseball, the probability that he hits the ball is 0.4. In practice he gets pitched 12 balls. Let X denote the total number of balls he hits. Assuming that his hits are independent, find:

(a) $\mathrm{E}(X)$

(b) $\mathrm{P}(X \leq \mathrm{Var}(X))$ *[5 marks]*

4. A receptionist at a hotel answers on average 35 phone calls a day.

(a) Find the probability that on a particular day she will answer more than 40 phone calls.

(b) Find the probability that she will answer more than 35 phone calls every day during a five-day week. *[5 marks]*

5. When Robyn shoots an arrow at a target, the probability that she hits the target is 0.6. In a competition she has eight attempts to hit the target. If she gets at least seven hits on target she will qualify for the next round.

(a) Find the probability that she hits the target exactly four times.

(b) Find the probability that she fails to qualify for the next round.

(c) Find the probability that she hits the target for the first time on her third attempt. *[6 marks]*

6. During the month of August in Bangalore, India, there are on average 11 rainy days.

(a) Find the probability that there are fewer than seven rainy days during the month of August in a particular year.

(b) Find the probability that in ten consecutive years, exactly five have fewer than seven rainy days in August. *[5 marks]*

7. A company producing light bulbs knows that the probability that a new light bulb is defective is 0.5%.

(a) Find the probability that a pack of 6 light bulbs contains at least one defective one.

(b) Mario buys 20 packs of six light bulbs. Find the probability that more than 4 of the boxes contain at least one defective light bulb. *[6 marks]*

8. (a) Given that $X \sim \text{Po}(m)$ and $\text{P}(X = 0) = 0.305$ find the value of m.

(b) $Y \sim \text{Po}(k)$. Find the possible values of k such that $\text{P}(X = 1) = 0.2$.

(c) If $W \sim \text{Po}(\lambda)$ and $\text{P}(W = w + 1) = \text{P}(W = w)$ express w in terms of λ. *[8 marks]*

9. When a fair die is rolled n times, the probability of getting at most two sixes is 0.532 correct to three significant figures.

(a) Find the value of n.

(b) Find the probability of getting exactly two sixes. *[7 marks]*

10. Sonja rolls a single die until she has seen a six twice. Find the probability that she needs more than 5 rolls to do this. *[6 marks]*

Long questions

1. A bag contains a very large number of ribbons. One quarter of the ribbons are yellow and the rest are blue. Ten ribbons are selected at random from the bag.

(a) Find the expected number of yellow ribbons selected.

(b) Find the probability that exactly six of these ribbons are yellow.

(c) Find the probability that at least two of these ribbons are yellow.

(d) Find the most likely number of yellow ribbons selected.

(e) What assumption have you made about the probability of selecting a yellow ribbon? *[11 marks]*

2. A geyser erupts randomly. The eruptions at any given time are independent of one another and can be modelled using a Poisson distribution with mean 20 per day.

(a) Determine the probability that there will be exactly one eruption between 10 a.m and 11 a.m.

(b) Determine the probability that there are more than 22 eruptions during one day.

(c) Determine the probability that there are no eruptions in the 30 minutes Naomi spends watching the geyser.

(d) Find the probability that the first eruption of a day occurs between 3 a.m and 4 a.m.

(e) If each eruption produces 12 000 litres of water, find the expected volume of water produced in a week.

(f) Determine the probability that there will be at least one eruption in each of at least six out of the eight hours the geyser is open for public viewing.

(g) Given that there is at least one eruption in an hour, find the probability that there is exactly one eruption in an hour. *[23 marks]*

3. The probability that a student forgets to do their homework is 5%, independent of other students. If at least one student forgets to do homework, the whole class has to do a test.

(a) There are 12 students in a class. Find the probability that the class will have to do a test.

(b) For a class with n students, write down an expression for the probability that the class will have to do a test.

(c) Hence find the smallest number of students in the class such that the probability that the class will have to do a test is at least 80%. *[12 marks]*

4. Two women, Anna and Brigid, play a game in which they take it in turns to throw an unbiased six-sided die. The first woman to throw a '6' wins the game. Anna is the first to throw.

(a) Find the probability that:

(i) Brigid wins on her first throw

(ii) Anna wins on her second throw

(iii) Anna wins on her nth throw.

(b) Let p be the probability that Anna wins the game. Show that $p = \frac{1}{6} + \frac{25}{36}p$.

(c) Find the probability that Brigid wins the game.

(d) Suppose that the game is played six times. Find the probability that Anna wins more games than Brigid. *[17 marks]*

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5. On a particular road, serious accidents occur at an average rate of two per week and can be modelled using a Poisson distribution.

(a) (i) What is the probability of at least eight serious accidents occurring during a particular four-week period?

(ii) Assume that a year consists of thirteen periods of four weeks. Find the probability that in a particular year, there are more than nine four-week periods in which at least eight serious accidents occur.

(b) Given that the probability of at least one serious accident occurring in a period of n weeks is greater than 0.99, find the least possible value of n, $n \in \mathbb{Z}^+$. *[18 marks]*

(*© IB Organization 2001*)

6. Two children, Aleric and Bala, each throw two fair cubical dice simultaneously. The score for each child is the sum of the two numbers shown on their respective dice.

(a) (i) Calculate the probability that Aleric obtains a score of 9.

(ii) Calculate the probability that Aleric and Bala each obtain a score of 9.

(b) (i) Calculate the probability that Aleric and Bala obtain the same score.

(ii) Deduce the probability that Aleric's score exceeds Bala's score.

(c) Let X denote the largest number shown on the four dice.

(i) Show that $\text{P}(X \leq x) = \left(\frac{x}{6}\right)^4$, for $x = 1, 2, \ldots, 6$.

(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
$\text{P}(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$				$\frac{671}{1296}$

(iii) Calculate $\text{E}(X)$. *[13 marks]*

(*© IB Organization 2007*)

7. Adele is a senior typist and makes an average of 2.5 mistakes per letter. Bozena is a trainee typist and makes an average of 4.1 mistakes per letter. Assume that the number of mistakes made by any typist follows a Poisson distribution.

(a) Calculate the probability that on a particular letter:

(i) Adele makes exactly three mistakes

(ii) Bozena makes exactly three mistakes.

(b) Adele types 80% of all the letters.

(i) Find the probability that a randomly chosen letter contains exactly three mistakes.

(ii) Given that a letter contains exactly three mistakes, find the probability that it was typed by Adele.

(c) Adele and Bozena type one letter each. Given that the two letters contain a total of three mistakes, find the probability that Adele made more mistakes than Bozena. *[16 marks]*

24 Continuous distributions

In this chapter you will learn:

- how to describe probabilities of continuous variables
- how to calculate expected statistics of continuous variables
- about an extremely important distribution called the normal distribution
- how to work backwards from probabilities to estimate information about the data.

Introductory problem

The height of many trees in a forest is measured and they have a mean of 7 m and a standard deviation of 1.5 m. Estimate the proportion of trees above 10 m tall.

In chapter 23 we saw that being able to describe random variables allowed us to make predictions about their properties. However, a major limitation was that those methods only applied to discrete variables. In reality, many variables we are interested in, such as height, weight and time, are continuous variables. In this chapter we shall extend the methods of chapter 23 to work with continuous variables. We will also meet the incredibly important 'normal distribution', which is used to model a large number of continuous variables in the physical world.

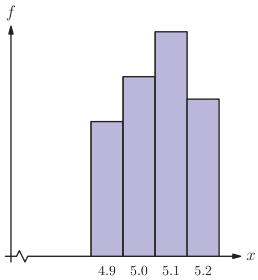
24A Continuous random variables

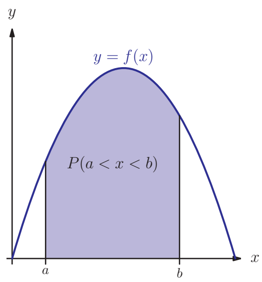
Consider the following data for masses of 5 kg bags of rice.

Mass / kg	Frequency
4.9	12
5.0	16
5.1	20
5.2	14

Not all of the data labelled 5.1 kg has a mass of *exactly* 5.1 kg. A bag with mass 5.1358 kg or 5.0879546 kg would be counted in this category. It would be impossible to list all the different possible actual masses, and it would be impossible to measure the mass absolutely accurately. When we collect continuous data we *have* to put it into groups. This means that we cannot talk about the probability of a single value of a **continuous random variable (crv)**. We can only talk about the probability of the crv being in a specified range. A useful way of representing this is by the area under a graph. This also has the

Which is more useful – knowing that the mass of a bag of rice is 5.1 kg to 2 significant figures or knowing that the mass is 5.0879546 kg? Is knowing the exact answer always better?





Does representing probability by area – that is, providing a visual interpretation – give us any new knowledge? Is it helpful?

property that there is no area above a single value but we can find the area above any range.

The area under a graph can be found by integrating. The function which we have to integrate is called the **probability density function (pdf)**, and it is often denoted $f(x)$. The defining feature of $f(x)$ is that the area between two x values is the probability of the crv falling between those two values.

KEY POINT 24.1

The probability of the crv falling between values a and b is:

$$P(a < x < b) = \int_a^b f(x)\mathrm{d}x$$

EXAM HINT

For a continuous variable it does not matter whether you use strict inequalities ($a < x < b$) or inclusive inequalities ($a \le x \le b$).

As with discrete probabilities, the total probability over all cases must equal 1. Also, no probability can ever be negative. This provides two requirements for a function to be a probability density function.

KEY POINT 24.2

For a probability density function $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1,\ f(x) \ge 0$$

EXAM HINT

The limits $-\infty$ and ∞ represent the fact that, in theory, a continuous random variable can take any real value. In practice, the limits of the integral are set to the lowest and the highest value the variable can take.

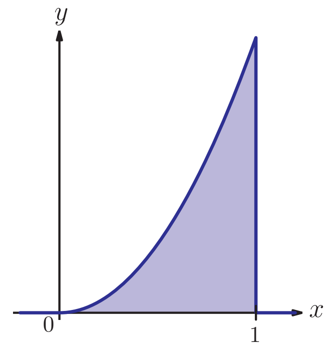
A probability density function will often have different rules over different domains. If a probability density function is only defined over a particular domain you may assume that it is zero everywhere else.

Worked example 24.1

A continuous random variable has a pdf:

$$f(x)=\begin{cases} kx^2 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of k.
(b) Find the probability of x being between 0.2 and 0.6.



Total area is 1. Area is only found between 0 and 1

(a) $1=\int_0^1 kx^2\,dx$

$=\left[\frac{kx^3}{3}\right]_0^1$

$=\frac{k}{3}$

$\Leftrightarrow k=3$

Probability X lies in [a, b] is $\int_a^b f(x)\,dx$

(b) $P(0.2<X<0.6)=\int_{0.2}^{0.6} 3x^2\,dx$

$=\left[x^3\right]_{0.2}^{0.6}$

$=0.208$

Exercise 24A

1. For each of these distributions, find the possible values of the unknown parameter k:

(a) (i) $f(x)=kx^3,\ 2<x<3$ (ii) $f(x)=k\sqrt{x},\ 1<x<4$

(b) (i) $f(x)=x^2+k,\ -1<x\le 2$ (ii) $f(x)=3x+k,\ -2\le x<3$

(c) (i) $f(x)=e^{kx},\ 0<x<2$ (ii) $f(x)=\sin kx,\ 0<x<\pi$

(d) (i) $f(x)=\frac{1}{(x+k)^2},\ 0\le x\le 1$ (ii) $f(x)=\frac{1}{x+k},\ 0\le x\le 1$

(e) (i) $f(x)=x^3,\ 0<x<k$ (ii) $f(x)=2x-1,\ 1<x<k$

(f) (i) $f(x)=\frac{1}{1+x},\ k<x<k+1$ (ii) $f(x)=x^2,\ k<x<2k$

(g) (i) $f(x)=kx^2,\ 0<x<k$ (ii) $f(x)=x+k,\ 0<x<k$

(h) (i) $f(x) = ke^{-x^2}$, $3 < x < 8$ (ii) $f(x) = k\sin\sqrt{x}$, $\pi < x < \pi^2$

(i) (i) $f(x) = \frac{1}{x^2}$, $1 < x < k$ (ii) $f(x) = \frac{1}{2\sqrt{x}}$, $k < x < 1$

2. (a) If $f(x) = \begin{cases} 2-2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(i) Find $P(0.3 < X < 0.9)$. (ii) Find $P(0 < X < 0.5)$.

(b) If $f(x) = \begin{cases} \cos x & 0 < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

(i) Find $P\left(\frac{\pi}{4} < X \le \frac{\pi}{3}\right)$. (ii) Find $P\left(0 \le X < \frac{\pi}{6}\right)$.

(c) If $f(x) = \begin{cases} \frac{1}{x\ln 10} & 1 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

(i) Find $P(X > 5)$. (ii) Find $P(X \le 3)$.

3. (a) If $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(i) Find a if $P(X < a) = 0.4$. (ii) Find b if $P(X < b) = 0.9$.

(b) If $f(x) = \begin{cases} \frac{x}{8} & 0 < x < 8 \\ 0 & \text{otherwise} \end{cases}$

(i) Find a if $P(X > a) = 0.9$. (ii) Find b if $P(X > b) = 0.5$.

(c) If $f(x) = \begin{cases} \frac{x}{16} & 2 < x < 6 \\ 0 & \text{otherwise} \end{cases}$

(i) Find a if $P(2 + a < X < 6 - a) = 0.8$.
(ii) Find b if $(b < X < b + 1) = 0.25$.

4. A model predicts that the angle, G, by which an alpha particle is deflected by a nucleus is modelled by:

$$f(g) = \begin{cases} kg^2 & 0 < g < \pi \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant k.

(b) 10 000 alpha particles are fired at a nucleus. If the model is correct, estimate the number of alpha particles deflected by less than $\frac{\pi}{3}$. *[6 marks]*

5. A random variable Y has distribution:

$$f(y) = \begin{cases} 3e^{-3x} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the exact value of $P(Y > 2)$. *[4 marks]*

6. If the continuous random variable X has a probability density

$$f(x) = \begin{cases} \sec^2 x & 0 < x < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

find the interquartile range of X. *[6 marks]*

7. If $f(x) = \begin{cases} \frac{1}{x} & 1 < x < e \\ 0 & \text{otherwise} \end{cases}$

(a) Find b in terms of k if $P(b < X < b^2) = k$.

(b) Find a in terms of k if $P(2 - a < X \leq 2 + a) = k$. *[7 marks]*

8. If $f(x) = \begin{cases} e^x & k < x < 2k \\ 0 & \text{otherwise} \end{cases}$

find $P\left(X > \frac{3k}{2}\right)$. *[7 marks]*

24B Expectation and variance of continuous random variables

The expressions for expectation and variance of continuous random variables all involve integration.

KEY POINT 24.3

Expectation and variance of continuous random variables:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Note: The formulae in the Formula booklet look different but equate to the same thing.

You may have noticed that the expressions for E(X) and Var(X) look similar to those for discrete random variables, but with integration signs instead of summation signs. This is because there is a link between sums and integrals, which you met in chapter 17. You will explore this further if you study chapter 28 in the Calculus option (Topic option 9).

Worked example 24.2

If a continuous random variable has pdf

$$f(x)=\begin{cases}\frac{3}{4}x(2-x) & 0<x<2,\\ 0 & \text{otherwise}\end{cases}$$

find E(X) and the standard deviation of X.

We can do the integration on the calculator. If you need reminding how, See Calcultor skills sheet 10 on the CD-ROM

$$E(X)=\int_0^2 x\times\frac{3}{4}x(2-x)\,dx$$
$$=\frac{3}{4}\int_0^2 x^2(2-x)\,dx$$
$$=1 \text{ (from GDC)}$$

To find standard deviation we must first find Var(X) which requires us to find E(X^2)

$$E(X^2)=\int_0^2 x^2\times\frac{3}{4}x(2-x)\,dx$$
$$=\frac{3}{4}\int_0^2 x^3(2-x)\,dx$$
$$=1.2 \text{ (from GDC)}$$
$$\therefore \text{Var}(X)=E(X^2)-E(X)^2$$
$$=1.2-1^2$$
$$=0.2$$
$$\text{standard deviation}=\sqrt{0.2}=0.447$$

EXAM HINT

The expected mean appears in examination questions more often than the median or mode.

It is also possible to find the median and mode for a continuous distribution.

The defining feature of the median is that half of the data should be below this value and half above. The mode is the most likely value. We can interpret this in terms of probability.

KEY POINT 24.4

The median, m, satisfies

$$\int_{-\infty}^{m} f(x)\,dx=\frac{1}{2}$$

The mode is the value of x at the maximum value of $f(x)$.

The maximum value of $f(x)$ is not necessarily where $\frac{df}{dx}=0$, see Section 16H.

Worked example 24.3

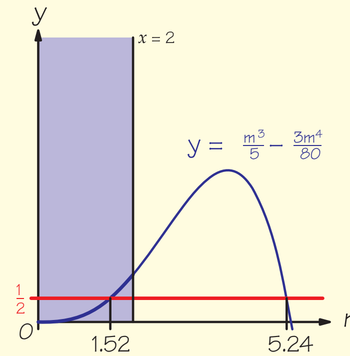
If $f(x)=\begin{cases}\frac{3}{20}(4x^2-x^3) & 0<x<2\\ 0 & \text{otherwise}\end{cases}$ find the median and mode of X.

Probability of being below the median is $\frac{1}{2}$

$$\int_0^m \frac{3}{20}(4x^2-x^3)\,dx=\frac{1}{2}$$

$$\Leftrightarrow \frac{m^3}{5}-\frac{3m^4}{80}=\frac{1}{2}$$

This is a quartic equation without any easy substitution. Time to use the calculator



From GDC: m = 1.52 or 5.24

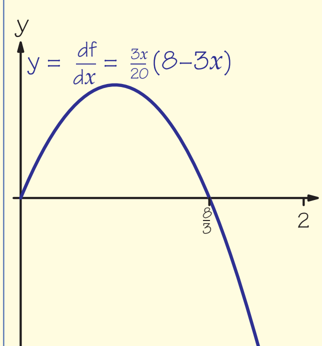
However $0<m<2$ therefore median = 1.52

For the mode check for a local maximum

$$\frac{df}{dx}=\frac{6x}{5}-\frac{9x^2}{20}=0$$

$$=\frac{3x}{20}(8-3x)$$

$$\Leftrightarrow x=0 \text{ or } x=\frac{8}{3}$$



From the graph of f(x) it is clear that $x=\frac{8}{3}$ corresponds to a maximum, so the mode is $\frac{8}{3}$

Exercise 24B

1. Find E(X), the median of X, the mode of X and Var(X) if X has the given probability density function:

(a) (i) $f(x)=2-2x \quad 0<x<1$ (ii) $f(x)=\frac{x}{8} \quad 0<x<8$

(b) (i) $f(x)=\frac{1}{x\ln 10} \quad 1<x<10$ (ii) $f(x)=\frac{2}{x^2} \quad 1<x<2$

(c) (i) $f(x)=\cos x \quad 0<x<\frac{\pi}{2}$ (ii) $f(x)=e^x \quad 0<x<\ln 2$

(d) (i) $f(x)=\frac{3}{x^4} \quad x>1$ (ii) $f(x)=\frac{4}{x^5} \quad x>1$

2. (a) Given that E(X) = 1.1, find k if:

(i) $f(x)=\begin{cases}\frac{1}{x\ln k} & 1<x<k\\ 0 & \text{otherwise}\end{cases}$ (ii) $f(x)=\begin{cases}\frac{k}{x^k} & 1<x<\infty\\ 0 & \text{otherwise}\end{cases}$

(b) Given that E(X) = 3, find k if:

(i) $f(x)=\begin{cases}k & k<x<k+\frac{1}{k}\\ 0 & \text{otherwise}\end{cases}$

(ii) $f(x)=\begin{cases}\frac{1}{x+k} & 0<x<(e-1)\\ 0 & \text{otherwise}\end{cases}$

3. The continuous random variable X has pdf:

$$f(x)=\begin{cases}\frac{3}{20}(4x^2-x^3) & 0<x<2\\ 0 & \text{otherwise}\end{cases}$$

(a) Find the expected mean of X.

(b) Find the mode of X. *[6 marks]*

4. A continuous random variable B has pdf:

$$f(b)=\begin{cases}ab^2 & 3<b<10\\ 0 & \text{otherwise}\end{cases}$$

(a) Find the value of the constant a.

(b) Find E(B). *[7 marks]*

5. A function f is given by:

$$f(y)=\begin{cases}ke^{-kx} & y>0\\ 0 & \text{otherwise}\end{cases}$$

(a) Show that f is a probability density function

(b) A random variable Y has distribution given by $f(x)$. Find E(Y) in terms of k. *[10 marks]*

6. Y is a continuous random variable with probability density function:

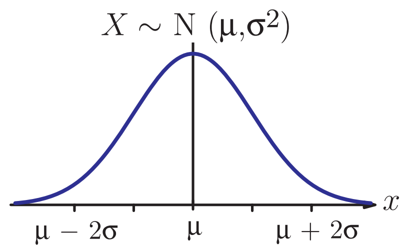
$$f(y)=\begin{cases} ay^2 & -k<y<k \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $a=\dfrac{3}{2k^3}$.

(b) Given that $\text{Var}(Y) = 5$ find the exact value of k. *[11 marks]*

7. Given that $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, x \in \mathbb{R}$ is a probability distribution find $\text{E}(X)$ and prove that $\text{Var}(X) = 1$. *[9 marks]*

24C The normal distribution



There are many situations where a variable is most likely to be close to its average value, and values further away from the average become increasingly unlikely. Many such situations can be modelled using the **normal distribution.**

All that is needed to describe this distribution is its mean (μ) and variance (σ^2). If a variable follows this distribution we use the notation $X \sim N(\mu, \sigma^2)$.

EXAM HINT

Be careful with the notation: σ^2 is the variance, so $X \sim N(10, 9)$ has standard deviation $\sigma = 3$.

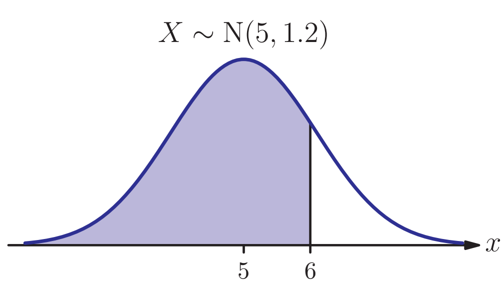
The probability density function (pdf) for the normal distribution is quite complicated:

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This function cannot be integrated in terms of other well-known functions, but your calculator can find approximate probabilities.

See Calculator sheet 14 on the CD-ROM.

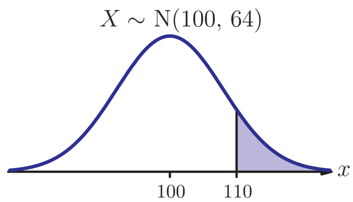
You may find it helpful to sketch a diagram to get a visual representation of the probability you are trying to find.



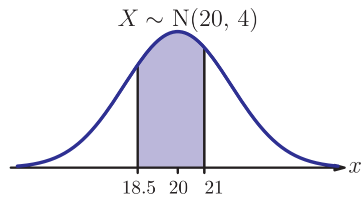
$P(X < 6)$

Historically, cumulative probabilities for the normal distribution were recorded in tables and these are still used if you don't have a graphical calculator. As there cannot be separate tables for every possible different μ and σ, all values needed to be converted into a Z-score described later.

The diagrams can also provide a useful check, to see whether you should expect the probability to be smaller or greater than 0.5.



P $(X > 110)$



P $(18.5 < X < 21)$

Worked example 24.4

The average height of people in a town is 170 cm with standard deviation of 10 cm. What is the probability that a randomly selected resident:

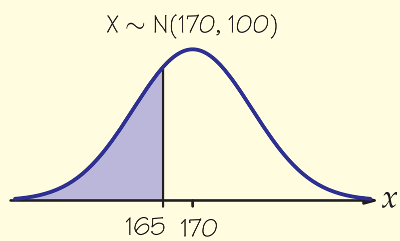
(a) is less than 165 cm tall?
(b) is between 180 cm and 190 cm tall?
(c) is over 176 cm tall?

State the distribution used

X is the crv 'height of a town resident' so
$X \sim N(170, 100)$

State the probability to be found and use the calculator

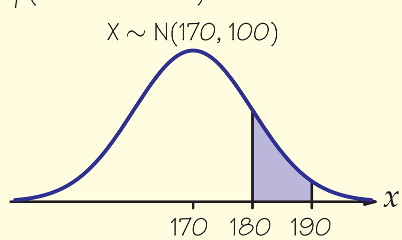
(a) $p(X < 165)$



$P(X < 165) = 0.309$ (3SF) (from GDC)

State the probability to be found and use the calculator

(b) $p(180 < X < 190)$



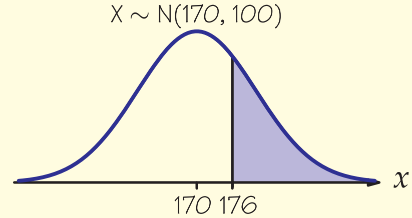
$P(180 < X < 190) = 0.136$ (3SF) (from GDC)

→

continued . . .

State the probability to be found and use the calculator

(c) $X \sim N(170, 100)$



$P(X > 176)$

$P(X > 176) = 0.274(3SF)$ (from GDC)

If a normally distributed random variable has mean 120, should a value of 150 be considered unusually large? The answer depends on how spread out the variable is, which is measured by its standard deviation. If the standard deviation is 30 then a value around 150 will be quite common; however, if the standard deviation were 5 then 150 would be very unusual.

It turns out that the probability of a normally distributed random variable being less than a given value ($P(X \le x)$, called the **cumulative probability**) depends only on the number of standard deviations x is from the mean. This is called the **Z-score**.

In the real world, there is always a possibility that a result may have occurred by random chance. Supplementary sheet 12 'Significant discoveries' explores how unlikely a result has to be before we accept it was not a fluke, which is often stated in terms of the z-score.

KEY POINT 24.5

For $X \sim N(\mu, \sigma^2)$, the Z-score measures the number of standard deviations of x above the mean.

$$z = \frac{x - \mu}{\sigma}$$

(a negative Z-score means x is below the mean)

Worked example 24.5

Given that $X \sim N(15,\ 6.25)$:

(a) How many standard deviations is $x = 16.1$ away from the mean?
(b) Find the value of X which is 1.2 standard deviations below the mean.

The number of standard deviations away from the mean is measured by the Z-score

(a) $z = \dfrac{x - \mu}{\sigma}$

continued . . .

6.25 is the variance

$\sigma = \sqrt{6.25} = 2.5$

$\therefore z = \frac{16.1 - 15}{2.5} = 0.44$

16.1 is 0.44 standard deviations away from the mean.

Values below the mean have a negative Z-score

(b) $z = -1.2$

$-1.2 = \frac{x - 15}{2.5}$

$\Rightarrow x - 15 = -3$

$\Rightarrow x = 12$

Before graphical calculators existed (which wasn't so long ago!) people used tables showing cumulative probabilities of the standard normal distribution. Because of their importance they were given special notation: $\Phi(z) = \mathrm{P}(Z \le z)$. Although you do not have to use this notation, you should understand what it means.

If we are given a random variable $X \sim \mathrm{N}(\mu, \sigma^2)$ we can create a new random variable Z which takes the values equal to the Z-scores of the values of X. In other words, for each x there is a corresponding $z = \frac{x - \mu}{\sigma}$. This is called the standardised value.

It turns out that, whatever the original mean and standard deviation of X, this new random variable always has normal distribution with mean 0 and variance 1, called the **standard normal distribution**: $Z \sim N(0, 1)$. This is an extremely important property of the normal distribution which needs to be used in situations when the mean and standard deviation of X are not known (see next section).

KEY POINT 24.6

The probabilities of X and Z are related by

$$\mathrm{P}(X \le x) = \mathrm{P}\left(Z \le \frac{x - \mu}{\sigma}\right)$$

Worked example 24.6

Let $X \sim \mathrm{N}(6, 0.5^2)$. Write the following in terms of probabilities of Z:

(a) $\mathrm{P}(X \le 6.1)$
(b) $\mathrm{P}(5 < X < 7)$
(c) $\mathrm{P}(X > 6.5)$

We are given that $x = 6.1$ so we can calculate z

(a) $\mathrm{P}(X \le 6.1) = \mathrm{P}\left(Z \le \frac{6.1 - 6}{0.5}\right) = \mathrm{P}(Z \le 0.2)$

continued . . .

The relationship between X and Z above is stated for probabilities of the form $P(X \le k)$, so convert to that form first

(b) $P(5<X<7)=P(X<7)-P(X<5)$

$=P\left(Z<\frac{7-6}{0.5}\right)-P\left(Z<\frac{5-6}{0.5}\right)$

$=P(Z<2)-P(Z<-2)=P(-2<Z<2)$

(c) $P(X>6.5)=1-P(X\le 6.5)$

$=1-P\left(Z\le\frac{6.5-6}{0.5}\right)=1-P(Z\le 1)$

$=P(Z>1)$

You can see from the examples above that you don't actually have to convert probabilities into the form $P(X\le k)$ every time; simply replace the x values by the corresponding z scores.

Exercise 24C

1. Find the following probabilities:

(a) If $X \sim N(20,100)$,

(i) $P(X\le 32)$ (ii) $P(X<12)$

(b) If $Y \sim N(4.8,1.44)$,

(i) $P(Y>5.1)$ (ii) $P(Y\ge 3.4)$

(c) If $R \sim N(17,2)$

(i) $P(16<R<20)$ (ii) $P(17.4<R<18.2)$

(d) If Q has a normal distribution with mean 12 and standard deviation 3:

(i) $P(Q>9.4)$ (ii) $P(Q<14)$

(e) If F has a normal distribution with mean 100 and standard deviation 25:

(i) $P(|F-100|<15)$ (ii) $P(|F-100|>10)$

2. Find the Z-score corresponding to the given value of X:

(a) (i) $X \sim N(12, 2^2), x=13$ (ii) $X \sim N(38, 7^2), x=45$

(b) (i) $X \sim N(20, 9), x=15$ (ii) $X \sim N(162, 25), x=160$

3. Given that $X \sim N(16,\ 2.5^2)$, write the following in terms of probabilities of the standard normal variable:

(a) (i) $P(X<20)$ (ii) $P(X<19.2)$

(b) (i) $P(X\ge 14.3)$ (ii) $P(X\ge 8.6)$

(c) (i) $P(12.5<X<16.5)$ (ii) $P(10.1\le X\le 15.5)$

4. It is found that the lifespan of a certain brand of laptop batteries follows normal distribution with mean 16 hours and standard deviation 5 hours. A particular battery has a lifespan of 10.2 hours.

(a) How many standard deviations below the mean is this?

(b) What is the probability that a randomly chosen laptop battery has a lifespan shorter than this? *[6 marks]*

5. When Ali competes in long-jump competitions, the lengths of his jumps are normally distributed with mean 5.2 m and standard deviation 0.7 m.

(a) What is the probability that Ali will record a jump between 5 m and 5.5 m?

(b) Ali needs to jump 6 m to qualify for the school team.

(i) What is the probability that he will qualify with a single jump?

(ii) If he is allowed three jumps, what is the probability that he will qualify for the school team? *[7 marks]*

6. Weights of a species of cat have a normal distribution with mean 16 kg and variance 16 kg². In a sample of 2000 such cats, estimate the number which will have a weight above 13 kg. *[6 marks]*

You saw in chapter 22 that $\cap$ means intersection.

7. If $D \sim \mathrm{N}(250, 400)$, find:

(a) $\mathrm{P}(D > 265 \cap D < 280)$

(b) $\mathrm{P}(D > 265 \mid D < 280)$

(c) $\mathrm{P}(D < 242 \cap D > 256)$ *[6 marks]*

8. If $Q \sim \mathrm{N}(4, 160)$, find:

(a) $\mathrm{P}(5 < |Q|)$

(b) $\mathrm{P}(Q > 5 \mid 5 < |Q|)$ *[6 marks]*

9. The weights of apples are normally distributed with mean weight 150 g and standard deviation 25 g. Supermarkets classify apples as 'medium' if they are between 120 g and 170 g.

(a) What proportion of apples are medium?

(b) In a bag of 10 apples what is the probability that there are at least 8 medium apples? *[6 marks]*

10. The wingspans of a species of pigeon are normally distributed with mean length 60 cm and standard deviation 6 cm. A pigeon is chosen at random.

(a) Find the probability that its wingspan is greater than 50 cm.

(b) Given that its length is greater than 50 cm, find the probability that a wingspan is greater than 55 cm. *[6 marks]*

11. Grains of sand are believed to have a normal distribution with mean 2 mm and variance 0.25 mm^2.

(a) Find the probability that a randomly chosen grain of sand is larger than 1.5 mm.

(b) The sand is passed through a filter which blocks grains wider than 2.5 mm. The sand that passes through the filter is examined. What is the probability that a randomly chosen grain of filtered sand is larger than 1.5 mm? *[6 marks]*

12. The amount of paracetamol per tablet is believed to be normally distributed with mean 500 mg and standard deviation 160 mg. A dose of less than 300 mg is ineffective in dealing with toothache. In a trial of 20 people treated for toothache with a single tablet, what is the probability that 2 or more of them have less than the effective dose? *[6 marks]*

13. A variable has a normal distribution with a mean that is 7 times its standard deviation. What is the probability of the variable taking a value less than 5 times the standard deviation? *[6 marks]*

14. If $X \sim \mathrm{N}(\mu, \sigma^2)$ and $\mathrm{P}(X \le x) = k$ find $\mathrm{P}(X \le 2\mu - x)$ in terms of k. *[5 marks]*

24D Inverse normal distribution

In Section C we saw how to find probabilities when we knew information about the variable. In real life it is often useful to work backwards from probabilities to estimate information about the data. This requires the **inverse normal distribution**.

KEY POINT 24.7

For a given value of probability p the inverse normal distribution gives the value of x such that $\mathrm{P}(X \le x) = p$.

EXAM HINT

Remember that p must be the *cumulative* probability.

EXAM HINT

Some calculators can find the value of x such that $\mathrm{P}(X > x) = p$, as well as $\mathrm{P}(X \le x) = p$.

You will need to use your GDC to work out the inverse normal distribution (see Calculator skills sheet 14, the section on 'Finding the boundary' on the CD-ROM). To work out $P(X > x)$ you might need to do $1 - P(X \le x)$. Note that many textbooks use the $\Phi(z)$ notation mentioned in the previous section to write inverse normal distribution: If $P(X \le x) = p$, then $\Phi^{-1}(p) = z = \frac{x - \mu}{\sigma}$.

Worked example 24.7

The size of men's feet is thought to be normally distributed with mean 22 cm and variance 25 cm^2. A shoe manufacturer wants only 5% of men to be unable to find shoes large enough for them. How big should their largest shoe be?

Convert question into mathematical terms

If X is the crv 'length of a man's foot' then $X \sim N(22, 25)$

We want to find the value of x such that

$$P(X > x) = 0.05$$

Use inverse normal distribution. We may have to convert into a probability of the form $P(X \le x)$

$P(X \le x) = 1 - P(X > x) = 0.95$

$\Rightarrow x = 30.2$ cm (from GDC)

So their largest shoe must fit a foot 30.2 cm long.

EXAM HINT

This will involve solving equations, and sometimes simultaneous equations. As the numbers are usually not 'nice' you may want to use your calculator.

One of the main applications of statistics is to determine parameters of the population given information about the data. But how can we use the normal distribution calculations if the mean or the standard deviation is unknown? This is where the standard normal distribution comes in useful; we can replace all the X values by their Z-scores, as they follow a known distribution, N (0, 1).

Worked example 24.8

The masses of gerbils are thought to be normally distributed. If 30% of gerbils have a mass of more than 65 g and 20% have a mass of less than 40 g, estimate the mean and the variance of the mass of a gerbil.

continued . . .

Convert the information into mathematical terms

If X is the crv 'mass of a gerbil' then $X \sim N(\mu, \sigma^2)$

$$P(X > 65) = 0.3$$

$$P(X < 40) = 0.2 \qquad (1)$$

If you need all the probabilities to be in the form $P(X \le k)$, convert the first one

$$P(X \le 65) = 0.7 \qquad (2)$$

Use inverse normal distribution for Z ($Z \sim N(0, 1)$) and relate it to the given X values

from (1) $P(Z < z) = 0.2 \Rightarrow z = \dfrac{40 - \mu}{\sigma} = -0.842$

from (2) $P(Z \le z) = 0.7 \Rightarrow z = \dfrac{65 - \mu}{\sigma} = 0.524$

(from GDC)

Solve simultaneous equations

$$40 - \mu = -0.842\sigma \qquad (3)$$

$$65 - \mu = 0.524\sigma \qquad (4)$$

(4) – (3) $\quad 25 = 1.366\sigma$

$\Rightarrow \sigma = 18.3\,g$

$\therefore \mu = 55.4\,g$

Exercise 24D

1. (a) If $X \sim N(14, 49)$, find x if:

(i) $P(X < x) = 0.8$ (ii) $P(X < x) = 0.46$

(b) If $X \sim N(36.5, 10)$, find x if:

(i) $P(X > x) = 0.9$ (ii) $P(X > x) = 0.4$

(c) If $X \sim N(0, 12)$, find x if:

(i) $P(|X| < 0.5)$ (ii) $P(|X| < 0.8)$

2. (a) If $X \sim N(\mu, 4)$, find μ if

(i) $P(X > 4) = 0.8$ (ii) $P(X > 9) = 0.2$

(b) If $X \sim N(8, \sigma^2)$ find σ if

(i) $P(X \le 19) = 0.6$ (ii) $P(X \le 0) = 0.3$

3. If $X \sim N(\mu, \sigma^2)$, find μ and σ if:

(a) (i) $P(X > 7) = 0.8$ and $P(X < 6) = 0.1$

(ii) $P(X > 150) = 0.3$ and $P(X < 120) = 0.4$

(b) (i) $P(X > 0.1) = 0.4$ and $P(X \ge 0.6) = 0.25$

(ii) $P(X > 700) = 0.8$ and $P(X \ge 400) = 0.99$

4. IQ tests are designed to have a mean of 100 and a standard deviation of 20. What IQ score is needed to be in the top 2% of IQ scores? *[5 marks]*

5. Rabbits' masses are normally distributed with an average mass of 2.6 kg and a variance of 1.44 kg^2. A vet decides that the top 20% of rabbits are obese. What is the minimum mass for an obese rabbit? *[5 marks]*

6. A manufacturer knows that his machines produce bolts whose diameters follow a normal distribution with standard deviation 0.02 cm. He takes a random sample of bolts and finds that 6% of them have diameter greater than 2 cm. Find the mean diameter of the bolts. *[6 marks]*

7. (a) 30% of sand from Playa Gauss falls through a sieve with gaps of 1 mm, but 90% passes through a sieve with gaps of 2 mm. Assuming that a grain of sand's diameter is normally distributed, estimate the mean and standard deviation of the sand grains.

(b) 80% of sand from Playa Fermat falls through a sieve with gaps of 2 mm. 40% of this filtered sand passes through a sieve with gaps of 1 mm. Assuming that a grain of sand's diameter is normally distributed, estimate the mean and standard deviation of the sand grains. *[7 marks]*

8. The actual voltage of a brand of 9 V battery is thought to be normally distributed with standard deviation 0.8 V and mean $(9.2 - t)$ V where t is the time in hours that the battery has been used. When a battery's voltage drops below 7 V it can no longer power a lamp. A batch of batteries is found and only 10% can power the lamp. Assuming that the model is correct and that they were all used for the same amount of time, estimate for how long the batteries have been used. *[7 marks]*

9. The times taken for students to complete a test are normally distributed with a mean of 32 minutes and standard deviation of 6 minutes.

(a) Find the probability that a randomly chosen student completes the test in less than 35 minutes.

(b) 90% of students complete the test in less than t minutes. Find the value of t.

(c) A random sample of 8 students had their time for the test recorded. Find the probability that exactly 2 of these students completed the test in less than 30 minutes. *[7 marks]*

10. An old textbook says that the range of data can be estimated as 6 times the standard deviation. If the data is normally distributed what percentage of the data is within this range? *[6 marks]*

11. A scientist noticed that 36% of temperature measurements were at least 4°C lower than the mean. Assuming that the measurements follow a normal distribution, estimate the standard deviation. *[5 marks]*

12. For a normal distribution find the ratios:

(a) $\dfrac{\text{median}}{\text{mean}}$

(b) $\dfrac{\text{standard deviation}}{\text{inter-quartile range}}$ *[6 marks]*

13. Evaluate $\Phi^{-1}(x)+\Phi^{-1}(1-x)$. *[3 marks]*

14. A company makes a large number of steel links for chains. They know that the force required to break any individual link is modelled by a normal distribution with mean 20 kN. The company tests chains consisting of 4 links. If any link breaks, the chain will break. A force of 18 kN is applied to all of the chains and 30% break.

(a) Estimate the probability of a single link breaking.

(b) Hence estimate the standard deviation in the breaking strength of the links. *[6 marks]*

15. Most calculators have a random number generator which generates random numbers distributed from 0 to 1. How can you use these to form random numbers that could be drawn from a normal distribution? *[4 marks]*

Summary

- Because we group continuous data, the probability of a **continuous random variable** (crv) is discussed in terms of the probability of it being in a given range. To do this we integrate a **probability density function** such that the area under the curve $f(x)$ represents the probability. The probability of the crv falling between values a and b is:

$$P(a<x<b)=\int_a^b f(x)\,dx$$

- For a function to be a probability density function, it must have the following requirements:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1, \text{ where } f(x) \geq 0$$

- For continuous random variables, the formulae for expectation and variance require integration:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x;\ \mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x;\ \mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The median, *m*, of a continuous variable satisfies $\int_{-\infty}^{m} f(x)\mathrm{d}x = \frac{1}{2}$, and the mode is the value of x at the maximum value of $f(x)$.

- One very important continuous distribution is the **normal distribution**: $X \sim N(\mu, \sigma^2)$, where μ = mean and σ^2 = variance. Calculators can provide approximate probabilities of being in any given range.

- For $X \sim N(\mu, \sigma^2)$, the **Z-score** (z) measures the number of standard deviations from the mean that a value (x) is: $z = \frac{x - \mu}{\sigma}$.

- Given a random variable $X \sim N(\mu, \sigma^2)$, Z is a new random variable that takes the values equal to the Z-scores of x, such that for every x there is a corresponding z. This is the standardised value, which always has a normal distribution $Z \sim N(0,1)$, called the **standard normal distribution**.

- The probabilities of X and Z are related:

$$\mathrm{P}(X \leq x) = \mathrm{P}\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

- If we know probabilities relating to a variable with a normal distribution we can deduce information about the data using the **inverse normal distribution**: for a given value of probability, p, the inverse normal distribution gives us the value of x such that $\mathrm{P}(X \leq x) = p$. To find x, use your GDC.

- If you need to calculate the μ and σ^2 of a normal distribution, you can use the standard normal distribution to replace values of X with their Z-scores as these follow the known distribution of $Z \sim N(0,1)$.

Introductory problem revisited

The height of many trees in a forest is measured and they have a mean of 7 m and a standard deviation of 1.5 m. Estimate the proportion of trees above 10 m tall.

If we make the reasonable assumption that heights of trees are normally distributed, this problem is asking what is the probability of being more than 2 standard deviations above the mean. This is $1 - \Phi(2) = 2.3\%$.

Mixed examination practice 24

Short questions

1. If X is a continuous random variable with pdf $f(x)=\begin{cases} k-2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

(a) Find the value of k.

(b) Find the variance of X. *[6 marks]*

2. The test scores of a group of students are normally distributed with mean 62 and variance 144.

(a) Find the percentage of students with scores above 80.

(b) What is the lowest score achieved by the top 50% of the students? *[6 marks]*

3. 200 people are asked to estimate the size of an angle. 16 give an estimate which was less than 25° and 42 give an estimate which was more than 35°. Assuming that the data follows a normal distribution, estimate the mean and the standard deviation of the results. *[6 marks]*

4. If X is a continuous random variable with pdf

$$f(x)=\begin{cases} ax+b & 1<x<5 \\ 0 & \text{otherwise} \end{cases}$$

and E(X) = 3.5, find the exact values of the constants a and b. *[5 marks]*

5. The adult female of a breed of dog has average height 0.7 m with variance 0.05 m². If the height follows a normal distribution find the probability that in six independently selected dogs of this breed exactly four are above 0.75 m tall. *[5 marks]*

6. If $Z \sim N(0,1)$, prove that for positive k:

$$\mathrm{P}(|Z|<k)=2-2\Phi(k)$$ *[5 marks]*

Long questions

1. A continuous random variable X has the probability density function

$$f(x)=\begin{cases} ax^2(5-x) & 0<x<5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant a.

(b) Evaluate the mean and the standard deviation of X.

(c) Find the probability that $X > 4$.

(d) Find the standard deviation of a normal distribution which has the same mean as X and the same probability that $X > 4$. *[12 marks]*

2. The continuous random variable X has probability density function $f(x)$ where:

$$f(x)=\begin{cases} \mathrm{e}-k\mathrm{e}^{kx} & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $k = 1$.

(b) What is the probability that the random variable X has a value that lies between $\frac{1}{4}$ and $\frac{1}{2}$? Give your answer in terms of e.

(c) Find the mean and variance of the distribution. Give your answers *exactly*, in terms of e.

The random variable X above represents the lifetime, in years, of a certain type of battery.

(d) Find the probability that a battery lasts more than six months.

A calculator is fitted with three of these batteries. Each battery fails independently of the other two. Find the probability that at the end of six months:

(e) none of the batteries has failed

(f) exactly one of the batteries has failed. *[16 marks]*

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3. A business man spends X hours on the telephone during the day. The probability density function of X is given by:

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (i) Write down an integral whose value is $E(X)$.

(ii) Hence evaluate $E(X)$.

(b) (i) Show that the median, m, of X satisfies the equation

$$m^4 - 16m^2 + 24 = 0.$$

(ii) Hence evaluate m.

(c) Evaluate the mode of X. *[11 marks]*

4. The monthly salary in Argentina is modelled by a normal distribution with an average of 1500 Pesos. 30% of people earn more than 2000 Pesos per month.

(a) Use this information to estimate the standard deviation of the monthly salary in Argentina.

(b) Find the probability that a randomly chosen individual earns more than 3000 Pesos per month.

(c) Given that someone earns more than 2000 Pesos per month find the probability that they earn more than 3000 Pesos per month.

(d) Find the probability that in three randomly chosen people at least two earn less than 2000 Pesos per month.

(e) Suggest a reason why the normal distribution may not be an appropriate model for monthly salaries. *[15 marks]*

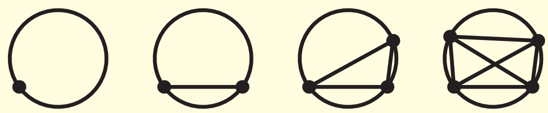
25 Mathematical induction

In this chapter you will learn:

- how to use the principle of induction to prove that patterns continue forever
- how to apply this to sequences and series
- how to apply this to differentiation
- how to apply this to number theory
- how to apply this to inequalities.

Introductory problem

If a circle has n points on its circumference, and each point is joined to each other point, what is the maximum number of regions created?



1 2 4 8

In both mathematics and science we are very interested in patterns. In science, there is no way to prove that the patterns will continue on forever, however in mathematics that is exactly what we need to do. One of the most powerful ways of doing this is a method called mathematical induction.

If something is mathematically proved does that make it true?

The mathematician Kurt Gödel proved that there are some true mathematical facts which can never be proved! This is encapsulated in Gödel's Incompleteness Theorem.

25A The principle of mathematical induction

Sequences often produce interesting patterns. Consider adding up consecutive odd numbers:

1 odd number: sum $= 1 = 1^2$

2 odd numbers: sum $= 1 + 3 = 4 = 2^2$

3 odd numbers: sum $= 1 + 3 + 5 = 9 = 3^2$

4 odd numbers: sum $= 1 + 3 + 5 + 7 = 16 = 4^2$

How can you show that this pattern continues?

You cannot keep checking forever. Suppose that you have checked (by direct calculation) that the pattern continues up to the 15th odd number; and therefore you know that the sum of the first 15 odd numbers is $1 + 3 + 5 + \ldots + 29 = 225 = 15^2$.

To check that the pattern continues, you do not have to add all the odd numbers from 1 to 31. You can use the result you already have, so

$$1 + 3 + 5 + \ldots + 29 + 31 = 225 + 31 = 256 = 16^2.$$

Mathematical induction and scientific induction are similar but there is an important difference. Mathematical induction is a logically rigorous way of proving a proposition but scientific induction cannot prove anything. However, scientific induction is a widely used and incredibly successful technique.

This example is the first documented example of mathematical induction, by the Italian mathematician Francesco Maurolico in 1575.

Building upon the previous work you have done, instead of starting all over again, is called an **inductive step**. In this example it seems fairly straightforward but in other problems this can be more difficult.

Now *suppose* that you have checked the pattern for the first k odd numbers. This means that

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

To see if the pattern still holds for the first $(k + 1)$ odd numbers, we need to add on the next term

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

Using the result for the first k odd numbers (which we *assume* we have already checked), the RHS simplifies to $(k + 1)^2$, meaning $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$ and so the pattern still holds for the first $(k+1)$ odd numbers.

We have now found out the following things about the pattern 'The sum of the first n odd numbers equals n^2':

1. the pattern holds for the first case (when $n = 1$)
2. if we *assume* the pattern up to some whole number k, then it *follows* that it will also hold for $k + 1$.

Does this prove that the pattern continues forever?
We know that it holds for $n = 1$ and because it holds for $n = 1$, it follows that it holds for $n = 2$; and because it holds for $n = 2$, it follows that it holds for $n = 3$; and so on.

We can continue this process to reach any number n, however large. Therefore the pattern holds for all positive integers.

The reasoning described in the last paragraph is called the **principle of mathematical induction**.

KEY POINT 25.1

Principle of mathematical induction

Suppose that we have a statement (or rule) about a positive integer n. If we can show that

1. the statement is true for $n = 1$
2. when we *assume* that the statement is true for $n = k$, we can *then* show that it is also true for $n = k + 1$

then the statement is true for all positive integers n.

The hardest part is undoubtedly step 2. To do this you need to make a link between one proposition and the next – the inductive step. The exact way you do this depends upon the type of problem. In the following sections we will see how to apply the principle of mathematical induction in various contexts.

25B Induction and series

Geometric and arithmetic series were covered in chapter 7.

You already know how to find an expression for the sum of a geometric or arithmetic sequence, but there are many other types of sequence. In this section we will use the principle of mathematical induction to prove a formula for the sum of the first n terms of a sequence.

KEY POINT 25.2

If $\{u_n\}$ is a sequence and $S_n = u_1 + u_2 + \ldots + u_n$ then $S_{n+1} = S_n + u_{n+1}$

When using the principle of mathematical induction, it is very important to show that you are following the correct logic. The examples below illustrate the way you must present your proof.

Worked example 25.1

Use the principle of mathematical induction to prove that, for all $n \in \mathbb{Z}^+$

$$1\times 3 + 2\times 4 + 3\times 5 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

State proposition

Proposition: $\sum_{r=1}^{r=n} n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Prove that the statement is true for $n = 1$

For $n=1$:

LHS = $1\times 3 = 3$

RHS = $\frac{1(1+1)(2\times 1+7)}{6} = \frac{1\times 2\times 9}{6} = 3$

The statement is true for $n = 1$

Assume the statement is true for $n = k$. What does that mean?

Assume it is true for $n = k$.

$1\times 3 + 2\times 4 + 3\times 5 + \ldots + k(k+2)$

$= \frac{k(k+1)(2k+7)}{6}$

continued . . .

Write down what you are trying to prove (be careful: you cannot use this – it is there to help you see how to simplify the expression)	Let $n = k + 1$ Working towards: $1\times3+2\times4+3\times5+\ldots+(k+1)(k+3)=$ $\frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}=\frac{(k+1)(k+2)(2k+9)}{6}$
Start from one side of the above equation	$LHS = 1\times3+2\times4+\ldots+k(k+2)+(k+1)(k+3)$
Use the result for $n = k$ (assumed to be true)	$=\frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)$ (using the result for $n = k$)
Simplify – take out common factors if possible and write fractions with a common denominator	$=(k+1)\left(\frac{2k^2+7k}{6}+\frac{6k+18}{6}\right)$ $=\frac{(k+1)(2k^2+13k+18)}{6}$
Factorise; look at what you are working towards to help you find factors	$=\frac{(k+1)(k+2)(2k+9)}{6}$ $=$ RHS
What have you shown?	The result is true for $n = k + 1$
Write a conclusion	The result is true for $n = 1$, and if true for $n = k$ it is also true for $n = k + 1$. Therefore the result is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Exercise 25B

1. A sequence is defined by $u_n = 2\times3^{n-1}$.
Use the principle of mathematical induction to prove that
$u_1 + u_2 + \ldots + u_n = 3^n - 1$. *[6 marks]*

2. Show that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. *[6 marks]*

3. Show that $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$. *[6 marks]*

4. Prove by induction that:

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$ *[6 marks]*

5. Prove by induction that:

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots+\frac{1}{(2n-1)\times(2n+1)}=\frac{n}{2n+1}.$$ *[6 marks]*

6. Prove that $1\times1!+2\times2!+3\times3!\ldots+n\times n!=(n+1)!-1$. *[6 marks]*

7. Use the principle of mathematical induction to show:

$$1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$$ *[6 marks]*

8. Prove that $(n+1)+(n+2)+(n+3)+\ldots+(2n)=\frac{1}{2}n(3n+1)$. *[6 marks]*

9. Prove that $\sum_{k=1}^{n}k2^k=(n-1)2^{n+1}+2$. *[6 marks]*

25C Induction and sequences

You should remember from Section 7A that sequences can be defined by a recurrence relation (inductive rule). For example, the recurrence relation

$$u_{n+1}=2u_n+3 \text{ with } u_1=7$$

produces the following sequence:

$u_1=7$
$u_2=2\times7+3=17$
$u_3=2\times17+3=37$ and so on.

A reasonable question to ask is: can we find a formula for the general term of this sequence? We can often use the number pattern to guess the formula. We must then prove that this formula is in fact correct. The principle of mathematical induction is a very effective method of proof in this context.

KEY POINT 25.3

> If a sequence is given by a recurrence formula (i.e. one where each term is found by using the previous term) we need to start by checking that the formula works for $n=1$. We then need to assume that the formula works for $n=k$ and prove that it works for $n=k+1$.

Worked example 25.2

A sequence is given by the recurrence relation $u_1 = 7,\ u_{n+1} = 2u_n + 3$ for $n \geq 1$.

Prove that the general formula for the sequence is $u_n = 5\times 2^n - 3$.

State proposition	Proposition: $u_n = 5\times 2^n - 3$
Check that the formula works for $n = 1$	For $n = 1$: RHS $= 5\times 2^1 - 3 = 10 - 3 = 7 = u_1$ So the formula works for $n = 1$.
Assume that the formula works for $n = k$ and prove that it works for $n = k + 1$	Assume the formula works for $n = k$:
Write down the formula with $n = k$	$u_k = 5\times 2^k - 3$
Write down what you are trying to prove	Let $n = k + 1$. Working towards: $u_{k+1} = 5\times 2^{k+1} - 3$
Use the recurrence relation to express u_{k+1} in terms of u_k	LHS $= u_{k+1} = 2u_k + 3$
Use the result for $n = k$	$= 2(5\times 2^k - 3) + 3$ (using the result for $n = k$)
Simplify	$= 5\times 2\times 2^k - 6 + 3 = 5\times 2^{k+1} - 3 =$ RHS
What have you shown?	The formula works for $n = k + 1$
Write a conclusion	The formula works for $n = 1$, and if it works for $n = k$ then it also works for $n = k + 1$. Therefore the formula works for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Sometimes each term in the sequence depends on more than one previous term. For example, the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n \text{ with } u_1 = 5 \text{ and } u_2 = 13$$

produces the following sequence:

$$u_1 = 5$$
$$u_2 = 13$$
$$u_3 = 5\times 13 - 6\times 5 = 35$$
$$u_4 = 5\times 35 - 6\times 13 = 97 \text{ and so on.}$$

KEY POINT 25.4

If a sequence is given by a recurrence formula where each term depends on *two* previous terms, we need to start by checking that the formula works for both $n = 1$ and $n = 2$. We then need to assume that the formula works for both $n = k$ and $n = k + 1$ and prove that it works for $n = k + 2$.

Worked example 25.3

A sequence is given by the recurrence relation $u_1 = 5$ and $u_2 = 13$, $u_{n+2} = 5u_{n+1} - 6u_n$ for $n \geq 2$. Prove that the general formula for the sequence is $u_n = 2^n + 3^n$.

State proposition

Proposition: $u_n = 2^n + 3^n$

Check that the formula works for $n = 1$ and $n = 2$

When $n = 1$:

RHS $= 2^1 + 3^1 = 5 = u_1$

So the formula works for $n = 1$.

When $n = 2$:

RHS $= 2^2 + 3^2 = 13 = u_2$

So the formula works for $n = 2$.

Assume that the formula works for $n = k$ **and** $n = k + 1$, and prove that it works for $n = k + 2$

Assume the formula works for $n = k$ and $n = k + 1$:

Write down the formula with $n = k$ and $n = k + 1$

$u_k = 2^k + 3^k$

$u_{k+1} = 2^{k+1} + 3^{k+1}$

Write down what you are trying to prove

Let $n = k + 2$

Working towards: $u_{k+2} = 2^{k+2} + 3^{k+2}$

Use the recurrence relation to express u_{k+2} in terms of u_k and u_{k+1}

LHS $= u_{k+2} = 5u_{k+1} - 6u_k$

Use the results for $n = k$ and $n = k + 1$

$= 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$

(using the results for $n = k$ and $n = k + 1$)

Expand the brackets

$= 5 \times 2^{k+1} + 5 \times 3^{k+1} - 6 \times 2^k - 6 \times 3^k$

Look at what you are working towards – group the powers of 2 and the powers of 3

$= (5 \times 2 \times 2^k - 6 \times 2^k) + (5 \times 3 \times 3^k - 6 \times 3^k)$

$= 4 \times 2^k + 9 \times 3^k = 2^2 \times 2^k + 3^2 \times 3^k = 2^{k+2} + 3^{k+2}$

$=$ RHS

continued . . .

What have you shown? — The formula works for $n = k + 2$

Write a conclusion — The formula works for $n = 1$ and $n = 2$, and if it works for $n = k$ and $n = k + 1$ then it also works for $n = k + 2$. Therefore the formula works for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Exercise 25C

1. A sequence is defined by a recurrence relation $u_{n+1} = 3u_n + 2,\ u_1 = 2$. Prove by induction that the general term of the sequence is $u_n = 3^n - 1$. *[8 marks]*

2. Given that $U_{n+1} = 5U_n + 4,\ U_1 = 4$, prove by induction that $U_n = 5^n - 1$. *[8 marks]*

3. Given that $U_{n+1} = 5U_n - 8,\ U_1 = 3$, prove by induction that $U_n = 5^{n-1} + 2$. *[8 marks]*

4. A sequence has first term 1 and subsequent terms defined by the recurrence relation $u_{n+1} = 3u_n + 1$. Prove by induction that $u_n = \dfrac{3^n - 1}{2}$. *[8 marks]*

5. A sequence is given by the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$ with $u_1 = 1$ and $u_2 = 5$. Prove that the general term of the sequence is $u_n = 3^n - 2^n$. *[9 marks]*

6. Given that $u_1 = 3, u_2 = 36, u_{n+2} = 6u_{n+1} - 9u_n$. Prove by induction that $u_n = (3n-2)3^n$. *[9 marks]*

7. Prove that if $u_0 = -1,\ u_1 = -1$ and $u_{n+2} = 5u_{n+1} - 6u_n$, then $u_n = 3^n - 2^{n+1}$. *[9 marks]*

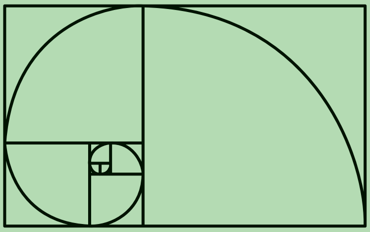
8. A sequence has first term 1 and the subsequent terms are given by the recurrence relation $u_{n+1} = \dfrac{u_n}{u_n + 1}$. Show that the nth term of the sequence is given by $u_n = \dfrac{1}{n}$. *[8 marks]*

9. The Fibonacci sequence is defined by $u_1 = u_2 = 1$, $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. Show that the nth term of the Fibonacci sequence is given by:

$$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

[12 marks]

Leonardo Fibonacci (c. 1170 – c. 1250) was an extremely influential mathematician in the middle ages, largely responsible for spreading the number system we use today. He also gave his name to the famous Fibonacci sequence. The above formula shows the link between the Fibonacci sequence and the Golden Ratio, a quantity which appears in many surprising places in mathematics.



25D Induction and differentiation

You already know how to find the second derivative of a function.

See Section 16D.

Higher order derivatives can be found by differentiating again: differentiating $\frac{d^2y}{dx^2}$ gives the third derivative $\frac{d^3y}{dx^3}$, and so on. Sometimes the derivatives produce an interesting pattern. For example, if $y = xe^x$ then (using the product rule):

$$\frac{dy}{dx} = e^x + xe^x = (x+1)e^x$$

$$\frac{d^2y}{dx^2} = 2e^x + xe^x = (x+2)e^x$$

$$\frac{d^3y}{dx^3} = 3e^x + xe^x = (x+3)e^x \text{ and so on.}$$

It appears that the nth derivative is $\frac{d^ny}{dx^n} = (x+n)e^x$.

We can prove this using the principle of mathematical induction.

KEY POINT 25.5

If we want to prove a result for an nth derivative using induction we use:

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{d}{dx}\left(\frac{d^ny}{dx^n}\right)$$

Worked example 25.4

If $y = xe^x$, prove that $\frac{d^n y}{dx^n} = (x+n)e^x$.

State the proposition	Proposition: If $y = xe^x$ then $\frac{d^n y}{dx^n} = (x+n)e^x$
Prove that the result is true for $n = 1$	$\frac{dy}{dx} = e^x + xe^x = (x+1)e^x$ The result is true for $n = 1$
Assume that the result is true for $n = k$ and prove that it is true for $n = k + 1$	Assume it is true for $n = k$: $\frac{d^k y}{dx^k} = (x+k)e^x$
Write down what you are trying to prove	Working towards: $\frac{d^{k+1} y}{dx^{k+1}} = (x+k+1)e^x$
Relate $\frac{d^{k+1}y}{dx^{k+1}}$ to $\frac{d^k y}{dx^k}$	$LHS = \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$
Use what you have assumed	$= \frac{d}{dx}\left((x+k)e^x\right)$ (using the result for $n = k$) $= e^x + (x+k)e^x$
Take out the common factor	$= e^x(x+k+1)$ $=$RHS
What have you proved?	The result is true for $n = k+1$
Write a conclusion	The result is true for $n = 1$, and if true for $n = k$ it is also true for $n = k + 1$. Therefore the result is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Exercise 25D

Prove the following by induction:

1. If $y = \dfrac{1}{1-x}$ then $\dfrac{d^n y}{dx^n} = \dfrac{n!}{(1-x)^{n+1}}$ *[8 marks]*

2. If $y = \dfrac{1}{1-3x}$ then $\dfrac{d^n y}{dx^n} = \dfrac{3^n n!}{(1-3x)^{n+1}}$ *[8 marks]*

3. $\dfrac{d^n}{dx^n}\left(xe^{2x}\right) = \left(2^n x + n2^{n-1}\right)e^{2x}$ *[8 marks]*

4. $\dfrac{d^{2n}}{dx^{2n}}\left(x\sin x\right) = (-1)^n\left(x\sin x - 2n\cos x\right)$ *[9 marks]*

5. $\dfrac{d^n}{dx^n}\left(x^2e^x\right) = \left(x^2 + 2nx + n(n-1)\right)e^x$ for $n \geq 2$ *[8 marks]*

6. Prove Leibniz's formula for the nth derivative of a product:

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k u}{dx^k}\frac{d^{n-k}v}{dx^{n-k}}$$

where $\dfrac{d^o}{dx^o}\left(f(x)\right) = f(x)$ *[10 marks]*

25E Induction and divisibility

Number theory is an important area of pure mathematics concerned with properties of natural numbers. One of the important tasks in number theory is studying divisibility. Consider the expression $f(n) = 7^n - 1$ for n = 0, 1, 2, 3…. Looking at the first few values of n:

$$f(0) = 7^0 - 1 = 0$$

$$f(1) = 7^1 - 1 = 6$$

$$f(2) = 7^2 - 1 = 48$$

$$f(3) = 7^3 - 1 = 342$$

It looks like $f(n)$ is divisible by 6 for all values of n. We can prove this using the principle of mathematical induction.

Note that in this example the first value of n is n = 0. Induction does not have to start from n = 1.

KEY POINT 25.6

If an expression is divisible by an integer d, we can write it as $A \times d$ for some integer A.

Worked example 25.5

The expression $f(n)$ is defined by $f(n) = 7^n - 1$ for all natural numbers n. Prove that $f(n)$ is divisible by 6 for all $n \in \mathbb{N}$.

State the proposition	Proposition: $f(n) = 7^n - 1$ is divisible by 6.
Prove that the statement is true for $n = 0$	$f(0) = 7^0 - 1 = 1 - 1 = 0 \quad (= 0 \times 6)$ So $f(0)$ is divisible by 6.
Assume that the statement is true for $n = k$	Assume that $f(k)$ is divisible by 6. Then $7^k - 1 = 6A$ for some $A \in \mathbb{Z}$
Think about what you are working towards	When $n = k + 1$ we are working towards: $7^{k+1} - 1 = 6B$, for some $B \in \mathbb{Z}$
Relate $f(k+1)$ to $f(k)$	$\text{LHS} = 7 \times 7^k - 1$
Using the result for $n = k$, $7^k = 6A + 1$	$= 7 \times (6A + 1) - 1$ (using the result for $n = k$)
Simplify, looking at what you are working towards (you want to take out a factor of 6)	$= 42A + 7 - 1 = 42A + 6 = 6(7A + 1) = \text{RHS}$ with $B = 7A + 1 \in \mathbb{Z}$
What have you proved?	So $f(k+1)$ is divisible by 6.
Write a conclusion	$f(0)$ is divisible by 6, and if $f(k)$ is divisible by 6 then so is $f(k+1)$. Therefore $f(n)$ is divisible by 6 for all $n \in \mathbb{N}$ by the principle of mathematical induction.

In Number theory there are other methods of proving divisibility. In particular, some very clever tests are needed to decide if extremely large numbers are prime. These are very important in code breaking and there are huge financial rewards for finding large prime numbers. Try finding out about Modular arithmetic and Fermat's Little Theorem. Some of these topics are covered in the Discrete mathematics option (Topic option 10).

Exercise 25E

1. Show that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$. *[8 marks]*

2. Show that $4^n - 1$ is divisible by 3 for all $n \geq 1$. *[8 marks]*

3. Show that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$ integers. *[8 marks]*

4. Use induction to prove that $30^n - 6^n$ is divisible by 12 for all integers $n \geq 0$. *[8 marks]*

5. Show using induction that $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. *[9 marks]*

6. Using the principle of mathematical induction, prove that $n(n^2 + 5)$ is divisible by 6 for all integers $n \geq 1$. *[9 marks]*

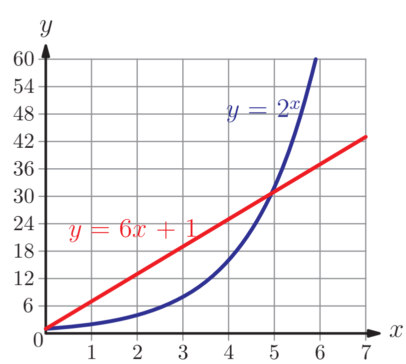
7. Use induction to show that $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$. *[9 marks]*

8. Prove, using the principle of mathematical induction, that $3^{2n+2} - 8n - 9$ is divisible by 64 for all positive integers n. *[8 marks]*

9. Show that the sum of the cubes of any three consecutive integers is divisible by 9. *[8 marks]*

25F Induction and inequalities

One of the most surprising uses of mathematical induction is in proving inequalities. For example, consider the inequality $2^n > 6n + 1$. Trying the first few integer values of n shows that this inequality is satisfied for $n \geq 5$. In fact, by looking at the graphs of $y = 2^n$ and $y = 6n + 1$ we find that the inequality is satisfied for all real numbers n greater than approximately 4.94. Note that the inequality cannot be solved algebraically. However, we can use induction to show that it holds for all integers $n \geq 5$.



Worked example 25.6

Use induction to show that the inequality $2^n > 6n+1$ holds for all integers $n \geq 5$.

State the proposition

$2^n > 6n+1$ when $n \geq 5$

Show it is true for the starting value, $n = 5$

For $n = 5$: LHS $= 2^5 = 32$ RHS $= 6 \times 5 + 1 = 31$

LHS > RHS so the inequality holds for $n = 5$

Assume that the inequality holds for some value $n = k$,

Assume that it holds for $n = k$, $k \geq 5$

$2^k > 6k+1$

Write down what you are trying to prove

Let $n = k+1$

Working towards:

$2^{k+1} > 6k+7$

Relate to the case $n = k$

LHS $= 2 \times 2^k = 2 \times (6k+1)$

(using the result for n = k)

Look at what you are working towards – separate the $6k$ term;

So, $= 12k+2$

$2^{k+1} = 6k + (6k+2)$

Use the fact that $k \geq 5$

$2^{k+1} > 6k+7$

(because $6k+2 > 7$ for $k \geq 5$)

What have you proved?

LHS > RHS, so the inequality holds for $n = k+1$

Write a conclusion

The inequality holds for $n = 5$, and if it holds for $n = k$ then it also holds for $n = k+1$. Therefore it holds for all integers $n \geq 5$ by the principle of mathematical induction.

Exercise 25F

Prove the following inequalities by induction:

1. Find the set of positive integers for which $3^n > n^3$ and prove your claim by induction. *[8 marks]*

2. $2^n > 1 + n$ for all $n > 1$ *[8 marks]*

3. $2^n > n^2$ for all $n \geq 4$ *[8 marks]*

4. $n! > 2^n$ for all $n \geq 4$ *[8 marks]*

5. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}$ for all $n>1$. *[8 marks]*

6. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>2(\sqrt{n+1}-1)$ for all $n \geq 1$. *[8 marks]*

7. Find the smallest integer N for which $3^N < N!$ Prove that $3^n < n!$ for all $n \geq$ N. *[10 marks]*

8. Show that $(1+x)^n \geq 1+nx$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. *[8 marks]*

In mathematics, no matter how many times a rule works this will never be enough to prove it. For example, it was thought that $1+1706n^2$ was never a square number. If you tried the first billion values of n, you would find that none of these are perfect squares. The first example of a square found was for 30693385322765657197397207. Just trying lots of examples does not work. However, induction does provide a logically rigorous way of showing that a pattern will continue forever.

Summary

- Mathematical induction is a powerful method used to prove a pattern continues forever.
- An **inductive step** means that you build upon previous work, rather than starting all over again.
- The principle of mathematical induction is the reasoning that if, for a statement about a positive integer n, we can show that:
  1. the statement is true for $n = 1$
  2. when we *assume* it is true for $n = k$, we can then show that it is also true for $n = k + 1$,

  then the statement is true for all positive integers n.
- Step 2 requires you to make a link between one proposition and the next – an inductive step. How to do this depends on the type of problem.

A general guide for how to make the inductive step depending upon the problem type is given in the table at the top of page 806.

Problem type	Inductive step
Series	$\sum_{r=1}^{k+1} u_r = \left(\sum_{r=1}^{k} u_r\right) + u_{k+1}$
Sequences	Use the recurrence relation
Differentiation	Differentiate again
Divisibility	Algebraic substitution
Inequality	Algebraic substitution
Anything else	Think on your feet. There will probably be a hint in the previous part of the question.

- So, the basic steps of mathematical induction are:
  - State a proposition.
  - Prove that any initial cases are true.
  - Assume that the kth case is true.
  - Link the kth case to the $(k+1)$th case.
  - Show that if the proposition is true for k, it is also true for $k+1$.
  - Write a conclusion.

Introductory problem revisited

If a circle has n points on its circumference, and each point is joined to each other point what is the maximum number of regions created?

In mathematics, we cannot always assume that a pattern will continue forever. With the opening problem, the first few terms are 1, 2, 4, 8, 16. It is very tempting to assume that the answer doubles each time, but if you try adding one more dot around the circle, you will find the answer is 31. The real formula for the number of regions with n dots is:

$$\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

Another way of writing this formula is $\binom{n}{4} + \binom{n}{2} + 1$.

Although mathematically equivalent, most people prefer this formula. Can you explain why?

Mixed examination practice 25

Short questions

1. Prove that $1\times2+2\times3+3\times4\ldots+n(n+1)=\frac{1}{3}(n+1)(n+2)$. *[6 marks]*

2. Prove that $3^{2n}+7$ is divisible by 8 when $n\in\mathbb{N}$. *[6 marks]*

3. Prove that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}\ldots+\frac{n}{(n+1)!}=\frac{(n+1)!\ -1}{(n+1)!}$. *[6 marks]*

4. Prove that $11^{n+2}+12^{2n+1}$ is divisible by 133 for $n\geq0, n\in\mathbb{N}$. *[6 marks]*

5. Prove that $\sum_{r=1}^{r=n}\frac{r}{2^r}=2-\left(\frac{1}{2}\right)^n(n+2),\ n\in\mathbb{N}$. *[6 marks]*

6. Prove using induction that:

$$\sin\theta+\sin3\theta+\ldots+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta},\ n\in\mathbb{Z}^+$$

Hence find the exact value of $\sin\frac{\pi}{7}+\sin\frac{3\pi}{7}\ldots+\sin\frac{13\pi}{7}$. *[7 marks]*

7. Prove by induction that for any positive integer n:

$$2\times4\times6\times\ldots\times(4n-2)=\frac{(2n)!}{n!}$$ *[6 marks]*

8. Prove using induction that for any positive integer n:

$$\cos x\times\cos2x\times\cos4x\times\cos8x\ \times\cdots\times\cos(2^n x)=\frac{\sin(2^n x)}{2^n\sin x}$$ *[7 marks]*

Long questions

1. (a) Use induction to prove De Moivre's theorem for all positive integers:

$$(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta$$

(b) Use induction to prove De Moivre's theorem for all negative integers.

(c) Find the modulus and the argument of $2\mathrm{i}-2$.

(d) Hence find all complex solutions of the equation $z^3=2\mathrm{i}-2$, giving your answers in the form $x+\mathrm{i}y$. *[10 marks]*

2. (a) By using the formula for $\binom{n}{r}$ show that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \le r \le n-1$.

(b) Prove by induction that $\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} = 2^n - 2$. *[8 marks]*

3. (a) Prove with induction that the sum of the first n terms of a geometric series with first term a and common ratio r is $\dfrac{a(r^n - 1)}{r-1}$.

(b) A geometric series has first term 1.2 and common ratio 0.5, how many terms of the series need to be added before the sum exceeds 2.399?

[11 marks]

4. (a) Prove by induction that for all positive integers $n^5 - n$ is divisible by 5.

(b) By factorising prove that $n^5 - n$ is also divisible by 6.

(c) Prove whether or not $n^5 - n$ is divisible by 60 for $n \ge 3$. *[17 marks]*

5. (a) Show that $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$.

(b) Prove that $\dfrac{d^n}{dx^n}(\cos x) = \cos\left(x + \dfrac{n\pi}{2}\right)$. *[10 marks]*

6. There are n lines in a plane such that no two are parallel and no three pass through the same point. Use induction to show that the number of intersection points created by these lines is $\dfrac{n(n-1)}{2}$. Prove also that the number of regions formed is $\dfrac{n(n+1)}{2} + 1$. *[9 marks]*

26 Questions crossing chapters

One of the hardest features of the International Baccalaureate® is the fact that a single question can draw upon different areas of the syllabus. This chapter brings together questions of this type, which makes it a very challenging chapter. Indeed, some of the questions are at the very top end of difficulty that could possibly feature in the examination. Do not be disheartened if you cannot finish the question. Try to extract as many marks from it as possible; this is how to get the top grades! However, hopefully you will find that although the questions look unfamiliar and daunting they are not impossible as long as you are not intimidated. In the long questions you will see that sometimes later parts are easier than earlier ones, particularly after 'show that' parts.

There is an index of what topics are covered in what questions available online (education.cambridge.org/ibdiploma). However, on your first pass through this chapter, you might find that it is a useful exercise to simply identify for yourself which topics each question links together. It is not always obvious!

Some of these questions are on the border of being within the International Baccalaureate® syllabus, but then again, so have some recent examination questions!

Mixed examination practice 26

Short questions

1. What is the probability of getting an average of 3 on two rolls of a fair die? *[6 marks]*

2. The sum of the first n terms of an arithmetic sequence is given by $S_n = 3n + 2n^2$. Find the common difference of the sequence. *[6 marks]*

3. If $f(x) = |x|$, sketch $f'(x)$ for $-2 \le x \le 2,\ x \ne 0$ *[5 marks]*

4. The continuous random variable x has probability density function:

$$f(x) = ax(b - x), 0 \le x \le b.$$

If the mode of the distribution is at $x = 4$ find the values of the constants a and b. *[6 marks]*

5. If $\boldsymbol{u} = \begin{pmatrix} x \\ x \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} x \\ 1 \\ x \end{pmatrix}$; find $\int \boldsymbol{u}.\boldsymbol{v}\ \mathrm{d}x$. *[6 marks]*

6. What is the average value of the first n terms of a geometric progression with first term a and common ratio r? *[4 marks]*

7. The function f is defined by $f(x) = (x-1)^2 + 3$.

The function g is defined by $g(x) = ax + b$, where a and b are constants.

Find the possible values of a and the corresponding values of b, if:

$$f(g(x)) = 16x^2 - 16x + 7.$$ *[6 marks]*

8. Solve the inequality $|1 + \tan 2x| < 1$ for $0 < x < \pi$. *[6 marks]*

9. Find the value of p for which the distribution B(10, p) has the largest possible variance. *[7 marks]*

10. Three data items are collected: 3, 7, x. Find the smallest possible value of the variance. *[6 marks]*

11. The discrete random variable X has the probability mass function $P(X = x) = \ln kx$ for $x = 1, 2, 3, 4$. Find the exact value of k. *[6 marks]*

12. If $\begin{pmatrix} n \\ 2 \end{pmatrix} = k$ express n in terms of k. *[6 marks]*

13. If $f(x) = \sin(x)$, give the single transformation which maps $f(x)$ to $g(x) = 1 - \cos(x)$. *[4 marks]*

14. (a) Expand $(\cos\theta + \mathrm{i}\sin\theta)^5$.

(b) Hence or otherwise express $\sin 5\theta$ in terms of $\sin\theta$. *[8 marks]*

15. The graph of $y = \ln x$ can be transformed into the graph of $y = \ln kx$ using either a horizontal stretch or a vertical translation.

(a) State the stretch factor of the horizontal stretch.

(b) Find the vertical translation vector. *[4 marks]*

16. The sequence u_n is defined by $u_n = 0.5^n$.

(a) Find the exact value of $\sum_0^{10} u_r$.

(b) Find the exact value of $\sum_0^{10} \ln(u_r)$. *[7 marks]*

17. (a) What is the argument of i?

(b) Express i in the form $re^{i\theta}$.

(c) Hence find the exact value of i^i, giving your answer in the form a^b where a and b are real numbers. *[7 marks]*

18. By taking natural logarithms of both sides or otherwise find $\frac{dy}{dx}$ in terms of x given that $y = x^{\sin x}$. *[7 marks]*

19. Starting from the fact that $a^x \times a^y = a^{x+y}$; prove by induction that $(a^x)^n = a^{nx}$ for $n \in \mathbb{N}$. *[8 marks]*

20. (a) If $y = e^{\lambda x}$; find $\frac{d^2y}{dx^2}$.

(b) If $y = e^{\lambda x}$ is a solution of the equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$; find the possible values of λ. *[6 marks]*

21. If $X \sim \text{Po}(m)$ with $m \neq 0$ and $\text{P}(X=7) = \text{P}(X=8) + \text{P}(X=9)$ find the value of m. *[6 marks]*

22. The functions f and g are defined by $f(x) = 3x+1$ and $g(x) = ax^2 - x + 5$ respectively. Find the value of a such that $f(g(x)) = 0$ has equal roots. *[7 marks]*

23. (a) If $0 < x < \frac{\pi}{2}$; use a right-angled triangle to show that $\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x}$.

(b) The solutions to the quadratic equation $ax^2 + bx + c = 0$ are $\tan\alpha$ and $\tan\beta$ where α and β are acute angles. Find a quadratic equation which has solutions $\tan\left(\frac{\pi}{2} - \alpha\right)$ and $\tan\left(\frac{\pi}{2} - \beta\right)$. *[8 marks]*

24. (a) The equation $x^4 + 10x^3 + 23x^2 - 10x - 24 = 0$ can be transformed into the equation $y^2 - 2y - 24 = 0$ using a substitution $y = ax^2 + bx$. Find the values of a and b.

(b) Hence or otherwise solve the equation $x^4 + 10x^3 + 23x^2 - 10x - 24 = 0$. *[7 marks]*

25. Solve the equation $\int_0^y x^2 + 1 \, dx = 4$. *[5 marks]*

26. For what values of x is the series $x^2 - x + (x^2 - x)^2 + (x^2 - x)^3 \ldots$ convergent? *[6 marks]*

27. $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors such that $\boldsymbol{u} \bullet \boldsymbol{v} = 0$. If $|\boldsymbol{u}| = 2$ and $|\boldsymbol{v}| = 3$, find $|\boldsymbol{u} - \boldsymbol{v}|$. *[4 marks]*

28. Prove by induction that the sum of a geometric series with first term 2 and common ratio r is $\dfrac{2(1-r^n)}{1-r}$. *[8 marks]*

29. Theo repeatedly rolls a fair die until he gets a six.

(a) Show that the probability of him getting this six on the third roll is $\dfrac{25}{316}$.

(b) p_r is the probability of getting his first six on the rth roll. Find an expression for p_r in terms of r.

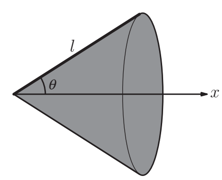
(c) Prove algebraically that $\displaystyle\sum_{r=1}^{r=\infty} p_r = 1$. *[9 marks]*

30. Evaluate exactly: $\displaystyle\int_0^{1/2} \left(\sum_{i=0}^{i=\infty} x^i \right) dx$ *[6 marks]*

31. (a) Simplify $\sin[(A+B)x] - \sin[(A-B)x]$.

(b) Hence or otherwise find $\int \sin 3x \cos 5x \, dx$. *[7 marks]*

32. A rod of length l is inclined at an angle θ to the x-axis. A cone is formed by rotating this rod around the x-axis. Find the maximum possible volume of this cone as θ varies. *[8 marks]*



33. (a) Given that $y = \arcsin x$, express x in terms of y.

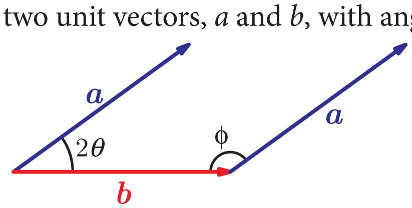
(b) Find $\dfrac{dx}{dy}$ in terms of y.

(c) Hence show that $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$. *[6 marks]*

34. (a) Show that $\dfrac{d}{dx}(\csc x) = \csc x \cot x$.

(b) Find the coordinates of the points on the curve $y = \csc x$ $(x \in [0, \pi])$ where the gradient is equal to $2\sqrt{3}$. *[6 marks]*

35. The diagram shows two unit vectors, a and b, with angle 2θ between them.



(a) Write down the size of the angle ϕ in terms of θ.

(b) Express $|\boldsymbol{a} + \boldsymbol{b}|$ in the form $k\cos^2\theta$. *[6 marks]*

36. The function f is defined for $x > 3$ by $f(x) = \ln(x^2 - 9) - \ln(x + 3) - \ln x$.

(a) Express $f(x)$ in the form $\ln(g(x))$.

(b) Find an expression for $f^{-1}(x)$.

[6 marks]

37. (a) Show that $(zw)^* = z^* w^*$.

(b) Prove by induction that $(z^n)^* = (z^*)^n$ for all positive integers n. *[10 marks]*

38. The random variable X follows Poisson distribution with mean μ. Let $p = \mathrm{P}(X = 1 \text{ or } X = 2)$.

(a) Write down an expression for p in terms of μ.

(b) Find the exact value of μ for which p is the largest possible. *[7 marks]*

39. (a) Integrate e^{kx} with respect to x.

(b) Show that for $x \in \mathbb{R}$, the imaginary part of $e^{(1+3i)x}$ is $e^x \sin(3x)$.

(c) Hence find $\int e^x \sin(3x)\,dx$. *[7 marks]*

40. (a) What transformation is required to go from the graph $y = \ln x$ to the graph $y = \ln(x^2)$?

(b) What transformation is required to go from the graph $y = \ln x$ to the graph $y = \log_{10} x$? *[6 marks]*

41. (a) Find $\frac{d}{dx}\ln y$ in terms of y and $\frac{dy}{dx}$.

(b) If $y = \frac{x^4}{(2+5x)\sqrt{x^2+1}}$ find and simplify an expression for $\ln y$.

(c) Hence find the derivative of $y = \frac{x^4}{(2+5x)\sqrt{x^2+1}}$ *[9 marks]*

42. (a) Express $e^{\frac{i\pi}{4}}$ in the form $a + ib$.

(b) Given that logarithms can be applied to complex numbers, find $\ln(1+i)$ in the form $a + ib$. *[6 marks]*

43. The probability distribution of a discrete random variable X is given by $\mathrm{P}(X = x) = \frac{4p^x}{5}$ for $x \in \mathbb{N}$. Find the value of p. *[6 marks]*

44. A continuous random variable X follows a Cauchy distribution with probability density function:

$$f(x) = \frac{1}{\pi(x^2 + 3)}.$$

Find $\mathrm{P}(-1 < x < 1)$, giving your answer in a form without π. *[6 marks]*

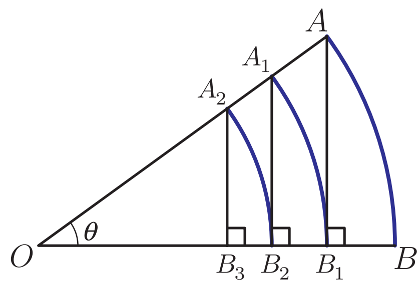
45. Prove that if $f(r) = \binom{n}{r} p^r (1-p)^{n-r}$ for $0 \le r \le n$, then $\sum_{r=0}^{r=n} f(r) = 1$. *[7 marks]*

46. (a) By considering $(1+x)^n$ or otherwise prove that $\sum_{r=0}^{r=n}\binom{n}{r}=2^n$.

(b) Evaluate $\sum_{r=0}^{r=n}(-1)^n\binom{n}{r}$. *[8 marks]*

47. The diagram shows a sector AOB of a circle of radius 1 and centre O, where $A\hat{O}B=\theta$.

The lines (AB_1), (A_1B_2), (A_2B_3) are perpendicular to OB. A_1B_1, and A_2B_2 are arcs of circles with centre O.



Calculate the sum to infinity of the arc lengths $AB+A_1B_1+A_2B_2+A_3B_3+\ldots$

[7 marks]

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48. Evaluate $\sum_{r=0}^{r=n}\binom{n}{r}\tan^{2r}\left(\frac{\pi}{3}\right)$. *[6 marks]*

49. Function $f(x)$ is such that $f(x)=\left|f(x+1)\right|-1$ for all x. By considering the transformations involved or otherwise:

(a) If $f(x)=d$, find d.

(b) If $f(x)=a\sin(bx)+c$, find a, b, and c. *[6 marks]*

50. (a) If $y=e^x+e^{-x}$, express x in terms of y.

(b) Show that the sum of all the possible values of x is zero. *[8 marks]*

51. (a) By writing $z=a+ib$ or otherwise solve $z^2=i-1$.

(b) Solve the quadratic equation $w^2+2iw=i$ where w is a complex variable. *[7 marks]*

52. (a) Find the coordinates of the image of $P(x,y)$ after a reflection in the line $y=x$ followed by a reflection in the y-axis.

(b) $f(x)$ is a 1-to-1 function. The graph of $f(x)$ is rotated 90° anticlockwise. Find the equation of the resulting graph. *[6 marks]*

53. (a) Sketch the graph $y=\left|\left|x\right|-1\right|$.

(b) Evaluate $\int_{-2}^{2}\left|\left|x\right|-1\right|dx$. *[6 marks]*

54. (a) If $z=x+iy$, find $\left|z-i\right|$ in terms of x and y.

(b) Sketch in the Argand diagram the points satisfying $\left|z-i\right|=\left|z+1\right|$.

[6 marks]

55. Find the volume of revolution when the region enclosed by the graph $y = e^x$, $y = 1$ and $x = 1$ is rotated by 2π around the line $y = 1$. *[6 marks]*

56. The probability of an event occurring is found to be $\frac{1}{7}(x^2 - 14x + 38)$, where x is known to be an integer parameter. Find all possible values of x. *[6 marks]*

57. (a) State an expression for $\sum_0^n x^n$.

(b) Hence or otherwise show that

$$1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} = \frac{1 - x^{n-1}}{(1-x)^2}.$$ *[7 marks]*

58. If ω is a solution the equation $z^n = 1$, find all possible values for $1 + \omega + \omega^2 \ldots + \omega^{n-1}$. *[5 marks]*

59. (a) Evaluate $(2+i)(3+i)$.

(b) Find the argument of $(2+i)(3+i)$.

(c) Hence or otherwise evaluate $\arctan\frac{1}{2} + \arctan\frac{1}{3}$. *[9 marks]*

60. (a) Find the real and imaginary parts of $\frac{1+e^{2i\theta}}{1-e^{2i\theta}}$ in terms of $\sin 2\theta$ and $\cos 2\theta$

(b) Hence show that $\frac{1+e^{2i\theta}}{1-e^{2i\theta}} = i\cot\theta$. *[9 marks]*

61. Use the principle of the mathematical induction to prove that:

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$ *[9 marks]*

62. (a) Find the real part of $(2+i)^n$.

(b) Prove that $(2+i)^n + (2-i)^n$ is real for all positive integer n. *[6 marks]*

63. (a) Show that $\binom{n}{1} = n$.

(b) Expand binomially $(x+h)^n$.

(c) Hence prove from first principles that $\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \in \mathbb{Z}^+$. *[6 marks]*

64. Two vectors a and b both have the same non-zero length, x. If $(a+b)\cdot(a+b) = 6x$ find the smallest possible value of x. *[6 marks]*

Long questions

1. The function f is defined by $f(x) = e^{px}(x + 1)$, where $p \in \mathbb{R}$.

(a) (i) Show that $f'(x) = e^{px}(p(x + 1) + 1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}e^{px}(p(x + 1) + n), n \in \mathbb{Z}^+.$$

(b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f. Find the **exact** value of the x-coordinate of:

(i) the minimum point

(ii) the point of inflexion.

(c) Let $p = \frac{1}{2}$. Let R be the region enclosed by the curve, the x-axis and the lines $x = -2$ and $x = 2$. Find the area of R. *[13 marks]*

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2. Daniel and Theo play a game with a biased coin. There is a probability of $\frac{1}{5}$ that the coin shows a head and a probability of $\frac{4}{5}$ that it shows a tail. They take it in turns to toss the coin. If the coin shows a head the player who tossed the coin wins the game. If the coin shows a tail, the other player has the next toss. Daniel plays first and the game continues until there is a winner.

(a) Write down the probability that Daniel wins on his first toss.

(b) Calculate the probability that Theo wins on his first toss.

(c) Calculate the probability that Daniel wins on his second toss.

(d) Show that the probability of Daniel winning is $\frac{5}{9}$.

(e) State the probability of Theo winning.

(f) They play the game with a different biased coin and find that the probability of Daniel winning is twice the probability of Theo winning. Find the probability of this coin showing a head. *[14 marks]*

3. Below is a table showing the values and gradient of $f(x)$ at various points.

x	0	1	2	3	4
$f(x)$	4	2	3	4	6
$f'(x)$	7	9	−3	4	2

(a) Is $f(x)$ a one-to-one function?

(b) Evaluate $f \circ f(3)$.

(c) The graph $y = g(x)$ is formed by translating the graph of $y = f(x)$ by a vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and then reflecting it in the x-axis. Find $g'(2)$. *[9 marks]*

4. (a) State De Moivre's theorem and prove it by induction.

write down the common ratio, r, and show that $|r| < 1$.

(b) For the infinite geometric series $1 - \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} - \frac{1}{8}e^{3i\theta} + \ldots$

(c) Find an expression for the sum of the series in the form $\dfrac{a}{b+c\mathrm{e}^{\mathrm{i}\theta}}$.

(d) Hence, or otherwise, show that:

$$1-\frac{1}{2}\cos\theta+\frac{1}{4}\cos\theta-\frac{1}{8}\cos 3\theta+\ldots-\frac{1+2\cos\theta}{5+4\cos\theta}$$ *[20 marks]*

5. A function is defined by $f(x)=2x+\frac{1}{2}\sin 2x-\tan x$ for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

(a) Find $f'(x)$.

(b) Show that the stationary points of $f(x)$ satisfy the equation $2\cos^4 x+\cos^2 x-1=0$.

(c) Hence show that the function has only one stationary point. *[11 marks]*

6. Consider the infinite geometric series $1+\cos x+\cos^2 x+\cos^3 x\ldots$ for $0<x<\pi$.

(a) Explain why the series converges.

(b) Show that the sum of the series is $\frac{1}{2}\csc^2\frac{x}{2}$.

(c) Find the exact value of $\int_{\pi/3}^{\pi/2}(1+\cos x+\cos^2 x+\cos^3 x\ldots)\mathrm{d}x$. *[14 marks]*

7. (a) Expand and simplify $\left(z-\frac{1}{z}\right)^5$.

(b) Show that $32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta+20\sin\theta$.

(c) Hence find the exact value of $\int_0^{\pi/2}\sin^5\theta\,\mathrm{d}\theta$. *[14 marks]*

8. (a) The value of the infinite series $\sum_{r=0}^{\infty}a^r$ is 1.5. Find a.

(b) Prove that $\frac{1}{1+x}=1-x+x^2-x^3\ldots$ when $|x|<k$, stating the value of k.

(c) Show that $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}\ldots$

(d) Evaluate $\ln 1.1$ to 3 decimal places. *[12 marks]*

9. (a) Sketch the graph $y=\ln x$.

(b) The tangent to this graph at the point $(p, \ln p)$ passes through the origin. Find the value of p.

(c) For what range of values of k does $\ln x = kx$ have two solutions? *[12 marks]*

10. If the polynomial $f(x)$ is divided by $(x-a)^2$, the remainder is a linear function.

(a) Explain why this statement can be written as $f(x)=(x-a)^2 g(x)+mx+c$ where $g(x)$ is a polynomial.

(b) Find an expression for $f'(x)$ in terms of $g'(x)$ and $g(x)$.

(c) Hence show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f(a)+f'(a)(x-a)$.

(d) State the condition which must be satisfied if $(x-a)^2$ is to be a factor of $f(x)$. *[12 marks]*

11. At a building site the probability, P(A), that all materials arrive on time is 0.85. The probability, P(B), that the building will be completed on time is 0.60. The probability that the materials arrive on time and that the building is completed on time is 0.55.

(a) Show that events A and B are not independent.

(b) All the materials arrive on time. Find the probability that the building will not be completed on time.

(c) There was a team of ten people working on the building, including three electricians and two plumbers. The architect called a meeting with five of the team, and randomly selected people to attend. Calculate the probability that exactly two electricians and one plumber were called to the meeting.

(d) The number of hours a week the people in the team work is normally distributed with a mean of 42 hours. 10% of the team work 48 hours or more a week. Find the probability that both plumbers work more than 40 hours in a given week. *[15 marks]*

(*© IB Organization 2002*)

12. (a) Find an expression for the sum of the series $e^{i\theta}+e^{3i\theta}+e^{5i\theta}+\ldots+e^{(2n-1)i\theta}$.

(b) Hence prove that $\cos\theta+\cos 3\theta+\cos 5\theta\ldots+\cos(2n-1)\theta=\dfrac{\sin(2n\theta)}{2\sin\theta}$.

(c) Find all solutions to the equation $\cos\theta+\cos 3\theta+\cos 5\theta=0$ when $0<\theta<\pi$. *[11 marks]*

13. From the factorial definition of binomial coefficients show that:

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$$

Prove by induction that if u and v are functions of x then:

$$\frac{d^n}{dx^n}(uv)=\sum_{i=0}^{i=n}\binom{n}{i}\frac{d^i}{dx^i}(v)\frac{d^{n-i}}{dx^{n-i}}(v)$$

[11 marks]

14. (a) Show that $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{x}{2}$.

(b) Hence use the substitution $t = \tan\frac{x}{2}$ to show that:

$$\int \frac{\sin x}{1+\cos x}\,\mathrm{d}x = \int \frac{2t}{1+t^2}\,\mathrm{d}t.$$

(c) Find the exact value of $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,\mathrm{d}x$. *[16 marks]*

15. The independent random variables X and Y have Poisson distributions such that $X \sim \mathrm{Po}(m)$ and $Y \sim \mathrm{Po}(n)$. If $Z = X + Y$:

(a) Show that $\mathrm{P}(Z = 5 \mid X = 4) = ne^{-n}$.

(b) Explain why $\mathrm{P}(Z = k) = \sum_{r=0}^{r=k} \mathrm{P}(X = r)\mathrm{P}(Y = k - r)$.

(c) Prove that $Z \sim \mathrm{Po}(m+n)$. *[15 marks]*

Examination tips

Some exam tips

- Remember to use your reading time effectively:
  - Decide which order to do the questions in. In particular, several long questions can be easier than the last few short questions, and are worth more marks!
  - Think about which questions can be done on the calculator.
  - Practise using your reading time on your practice papers.
  - Try to classify which section of the course each question is about.
- Just because you cannot do part (a) of a question does not mean you cannot do later parts. As you have seen, sometimes later parts are easier than earlier parts.
- If you cannot do an early part of a question, you can show how you would have used the answer in later parts; you will still gain marks.
- Look for links between parts of multi-part questions. They often act as hints.
- Plan your time before you go into the exam. Decide whether you work better quickly with lots of time for checking, or working slowly but not having much checking time.
- Do not get distracted if you cannot do some questions; for whatever grade you want, you do not have to get 100%!
- Practise checking your answers; this is not as easy a skill as it seems. Particularly in the calculator paper; you should be able to use your calculator to check your work.
- Scavenge for marks. A blank response is guaranteed to score zero. If you have a sensible idea write it down as some marks may be awarded for what might seem to be a minor point. However, do not waste too much time on a question where you feel uncertain about your method. Leave these questions to the end.

Answers

For questions that require a proof (normally indicated by any of the following instructions: prove, show, verify or explain) no answer has been supplied and the question number or subpart has been omitted. So, if you discover a question with a missing answer, this is because the solution is in the question and you have to derive that solution for yourself.

Chapter 1

Exercise 1A

1. (a) (i) 30 (ii) 57
   (b) (i) 13 (ii) 29
2. (a) 140 (b) 16
3. (a) 6 (b) 2 (c) 12
4. 48
5. (a) 90 (b) 48 (c) 63
6. 15
7. (a) 15 637 960
   (b) 10 104 528
8. 115 316 136
9. 8
10. (a) 720
    (b) 648
11. (a) 60
    (b) 125
12. (a) 81
    (b) 125

Exercise 1B

1. (a) (i) 120 (ii) 720
   (b) (i) 48 (ii) 360
   (c) (i) 600 (ii) 240
2. (a) (i) 40 320 (ii) 39 916 800
   (b) (i) 1080 (ii) 1 814 400
   (c) (i) 475 372 800 (ii) 357 840
3. (a) 720 (b) 40 320
   (c) $26! = 4.03\times10^{26}$ (3SF)
4. (a) 5040
   (b) 3600
5. (a) 120
   (b) 24
6. (a) $17! = 3.56\times10^{14}$ (3SF)
   (b) $16! = 2.09\times10^{13}$ (3SF)
7. 2880
8. (a) 720
   (b) 240
9. $30! = 2.65\times10^{32}$ (3SF)
10. $6\times5!\times4! = 17280$

Exercise 1C

1. (a) (i) 7 (ii) 12
   (b) (i) 56 (ii) 990
   (c) (i) 9900 (ii) 2550
   (d) (i) n (ii) $a+1$
   (e) (i) a^2-a (ii) b^2+b
   (f) (i) $\dfrac{1}{(x+8)(x+7)}$
   (ii) $\dfrac{1}{(x-4)(x-5)(x-6)}$
2. (a) (i) $\dfrac{9!}{6!}$ (ii) $\dfrac{6!}{2!}$
   (b) (i) $\dfrac{1016!}{1012!}$ (ii) $\dfrac{309!}{306!}$
   (c) (i) $\dfrac{(n+5)!}{(n+1)!}$ (ii) $\dfrac{(n+2)!}{(n-2)!}$
3. (a) (i) $9\times9!$ (ii) $131\times10!$
   (b) (i) $11\times13!$ (ii) $21\times15!$
   (c) (i) $121\times9!$ (ii) $110\times10!$
   (d) (i) $(n-1)\times(n-1)!$
   (ii) $(n^2+3n+1)\times n!$
4. $n=5$
5. $n=10$
6. $n=5$

ANSWER HINT (4, 5, 6)
Did you think about turning this into a quadratic?

Exercise 1D

1. (a) (i) 21 (ii) 792
   (b) (i) 60 (ii) 60
   (c) (i) 126 (ii) 135
   (d) (i) 136 (ii) 36
2. (a) (i) 28 (ii) 126
   (b) (i) 912 (ii) 14
   (c) (i) 1176 (ii) 980

3. (a) (i) 14 (ii) 27
(b) (i) 21 (ii) 17

4. 5005

5. (a) 35 (b) 15

6. 15 380 937

7. 36960

8. 31 500

9. $\binom{140}{12}\binom{128}{10}\binom{118}{10} = 1.62\times10^{45}$ (3SF)

10. (a) 43 680
(b) 65 520

11. (a) 35 (b) 35
(c) 31 (d) 33

12. (a) 126
(b) 120

13. 105

14. (a) 120
(b) 210

15. 24

16. $\binom{45}{15}\binom{30}{15} = 5.35\times10^{19}$ (3SF)

Exercise 1E

1. 560

2. 600

3. (a) 120
(b) 1320

4. (a) 4920
(b) 4800

5. 19557

6. 270200

7. 65 559

8. (a) 11082
(b) 48387

9. 696

Exercise 1F

1. (a) (i) 6 (ii) 5
(b) (i) 56 (ii) 110
(c) (i) 720 (ii) 1320

2. (i) 5040 (ii) 5040

3. (i) 60 (ii) 210

4. (a) (i) $n = 7$ (ii) $n = 10$
(b) (i) $n = 11$ (ii) $n = 14$

5. 7.75×10^{10} (3SF)

6. 255 024

7. 504

8. 336

9. 3 276 000

11. 186

12. 84

13. 4624

14. $n = 3$

Exercise 1G

1. $13! \times 2 = 1.25 \times 10^{10}$

2. 2 488 320

3. 30 240

4. 150×10^{14}

5. (a) 32 432 400
(b) 45 360

6. (a) 17280
(b) 5760
(c) 43200
(d) 2880

Mixed examination practice 1

Short questions

1. 210

2. 120

3. 30 240

4. 729

5. 55

6. $n = 5$

7. 8640

8. $n = 15$

9. 2947

10. 480

11. 672

12. 921 164 400

13 25 200

14. 112

Long questions

1. (a) 48
   (b) 72
   (c) 42

2. (a) 20
   (b) 22
   (c) 30

3. (a) 121 080 960
   (b) 3 991 680
   (c) 27 941 760

4. (a) We select 2 out of 4 places to put R's in.
   (b) $\binom{2n}{n}$
   (c) 20
   (d) $\binom{n+m-2}{n-1}$

5. (b) 2047
   (c) 5775

6. (a) 4495
   (b) 22
   (c) 26

Chapter 2

Exercise 2A

1. (a) (i) 6^7 (ii) 5^8
   (b) (i) a^8 (ii) x^9
   (c) (i) 7^{-3} (ii) 5^5
   (d) (i) x^2 (ii) x^5
   (e) (i) g^{-12} (ii) k^{-8}

2. (a) (i) 6^1 (ii) 5^{-2}
   (b) (i) a^{-2} (ii) x^3
   (c) (i) 5^9 (ii) 7^{15}
   (d) (i) x^6 (ii) x^{-11}
   (e) (i) 2^2 (ii) 3^{-14}
   (f) (i) g^6 (ii) k^{-8}

3. (a) (i) 2^{12} (ii) 3^{14}
   (b) (i) 5^{-4} (ii) 7^{-6}
   (c) (i) 11^2 (ii) 13^{15}
   (d) (i) 2^{17} (ii) 3^3
   (e) (i) 6^{12} (ii) 3^6

4. (a) (i) 2^{10} (ii) 3^{14}
   (b) (i) 2^9 (ii) 2^{20}
   (c) (i) 2^{13} (ii) 3^4
   (d) (i) 2^9 (ii) 3^{11}
   (e) (i) 2^{-6} (ii) 3^{-6}
   (f) (i) 2^2 (ii) 3^{10}

5. (a) (i) $8x^6$ (ii) $9x^8$
   (b) (i) $2x^6$ (ii) $3x^8$
   (c) (i) $9a^{10}$ (ii) 16
   (d) (i) $\frac{1}{2x}$ (ii) $\frac{y^2}{9}$
   (e) (i) $\frac{2}{x}$ (ii) $3y^2$
   (f) (i) $\frac{5x^2y^4}{9}$ (ii) $\frac{ab^5}{8}$
   (g) (i) $\frac{p^3}{2q^2}$ (ii) $\frac{2^7 3^{10}}{x^7}$

6. (a) (i) x^3 (ii) x^{12}
   (b) (i) $2x^5$ (ii) $\frac{1}{2x^4}$
   (c) (i) $\frac{4}{3x^3}$ (ii) $\frac{y^{12}}{x^6}$

7. (a) (i) $\frac{5}{3}$ (ii) $-\frac{3}{2}$
   (b) (i) $-\frac{1}{2}$ (ii) $-\frac{3}{4}$
   (c) (i) 4 (ii) 2
   (d) (i) 4 (ii) 0
   (e) (i) 4 (ii) 11
   (f) (i) 3 (ii) 3

8. 5×10^{-4}

9. 8cm

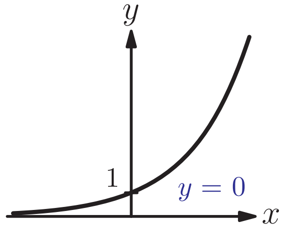
10. (a) $k=\frac{1}{3}$
    (b) $A = 16\,\text{cm}^2$

11. $2^{350} = (2^7)^{50} = (128)^{50}$
    $5^{150} = (5^3)^{50} = (125)^{50}$

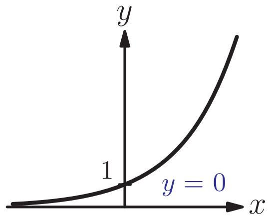
12. $b=1, a=\frac{3}{2}$

Exercise 2B

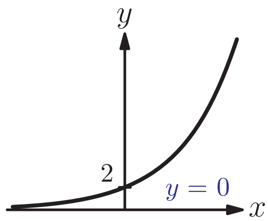
1. (a) (i)



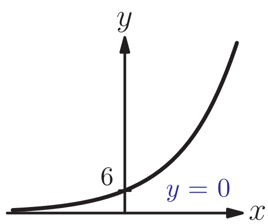
(ii)



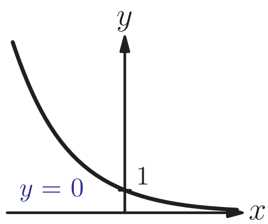
(b) (i)



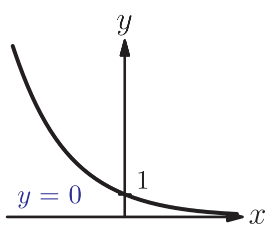
(ii)



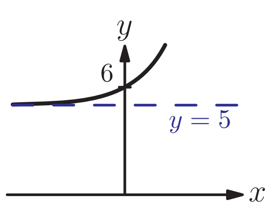
(c) (i)



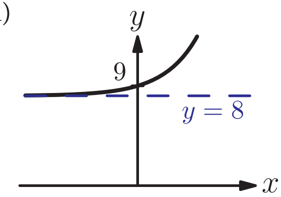
(ii)



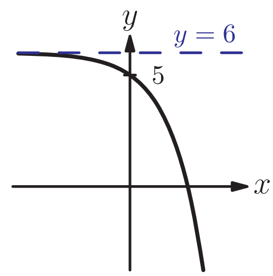
(d) (i)



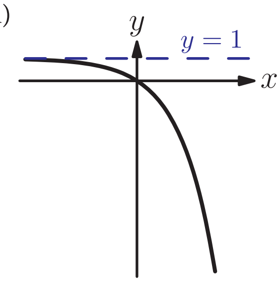
(ii)



(e) (i)



(ii)

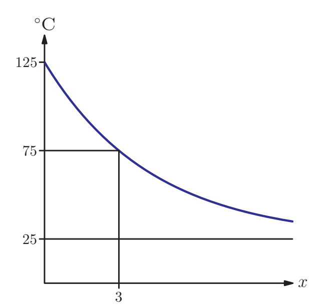


2. 13.31 m^2

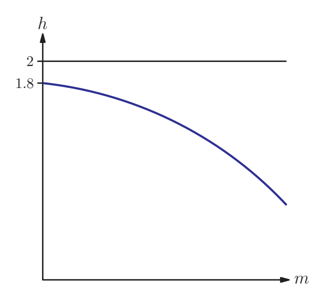
3. (a) $A = 25$, $B = 100$, $k = 3$

(b) 26°C

(c)



4. (a)

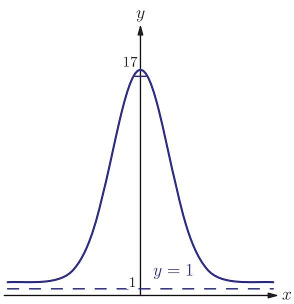


(b) 1.8 m

(c) 1.60 m

(d) The branch may not be long enough to reach the ground, or it might break before it reaches the ground.

5. (a)



(b) $x = \pm\frac{\sqrt{3}}{2}$

6. 41.2°C

7. (a) 0 m^{-1}
(b) 40 m^{-1}

Exercise 2C

1. (a) (i) 3.72 (ii) −0.283
(b) (i) 8.15 (ii) 1.36
(c) (i) 7.39 (ii) 0.049 8
(d) (i) 8.24 (ii) 0.002 74

2. $\sqrt[6]{\pi^4 + \pi^5} \approx e$

Exercise 2D

1. (a) (i) 3 (ii) 2
(b) (i) 1 (ii) 1
(c) (i) 0 (ii) 0
(d) (i) −1 (ii) −3
(e) (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$
(f) (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$
(g) (i) $\frac{2}{3}$ (ii) $\frac{3}{4}$
(h) (i) $\frac{3}{2}$ (ii) $\frac{5}{4}$
(i) (i) $\frac{3}{4}$ (ii) 2.25
(j) (i) $-\frac{1}{2}$ (ii) $-\frac{1}{2}$

2. (a) (i) 1.70 (ii) −0.602
(b) (i) −2.30 (ii) 2.30

3. (a) (i) $5\log x$ (ii) $5\log x$
(b) (i) $\log x \log y - \log y + 3\log x - 3$
(ii) $(\log x)^2 + 4\log x + 4$
(c) (i) $\frac{1}{\log a} + \frac{1}{\log b}$
(ii) $\log a + 1$

4. (a) (i) $x = 3^y$ (ii) $x = 16^y$
(b) (i) $x = a^{y+1}$ (ii) $x = a^{y^2}$
(c) (i) $x = \sqrt[3]{3y}$ (ii) $x = \sqrt{y}$

5. (a) (i) $x = 5$ (ii) $x = 2$
(b) (i) $x = 0.4$ (ii) $x = 0.25$
(c) (i) $x = 6$ (ii) $x = 100$

6. $x = 111$

7. $x = -3$

8. $x = \frac{e^2 + 1}{3}$

9. $x = 9, \frac{1}{9}$

10. $x = 81, y = 25$

11. $x = 10^{1.5} = 31.6$

12. $x = \sqrt[9]{4} = 1.17$

13. 5.50

Exercise 2E

1. (a) (i) 4 (ii) 1/2
(b) (i) 6 (ii) 3/2

2. (a) (i) $7y$ (ii) $2x + y$
(b) (i) $x + 2y - z$
(ii) $2x - y - 3z$
(c) (i) $2 - y - 5z$
(ii) $1 + y + 2z$
(d) (i) $x - 4y$
(ii) $2 + 2x + y + 2z$
(e) (i) $2 + \frac{y}{x}$ (ii) $\frac{x - z}{y} - 1$
(f) (i) $\frac{y}{x} \times 10^{x-y}$ (ii) $\frac{x + 2z}{x + y}$

3. (a) (i) $x = 2$ (ii) $x = 4$
(b) (i) $x = 9$ (ii) $x = 2$
(c) (i) $x = \frac{1}{4}$ (ii) $x = 8$
(d) (i) $x = 2^{\frac{12}{5}} = 5.28$
(ii) $x = 2^{10} = 1024$
(e) (i) $x = 8$ (ii) $x = 4$
(f) (i) $x = \frac{1}{3}$ (ii) $x = 8$

4. $\frac{1}{3}e^{\frac{3}{2}}$

5. (a) $a+2b$ (b) $2(a-b)$

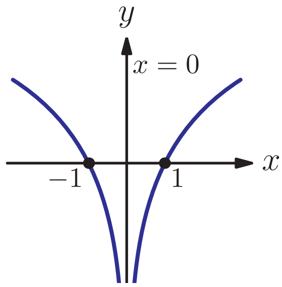
6. $x=2,\frac{1}{2}$

8. -1

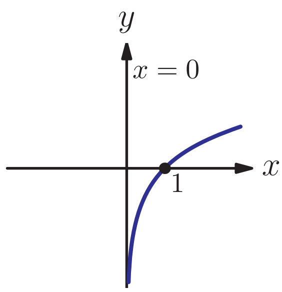
9. $a=\frac{1}{b}$

Exercise 2F

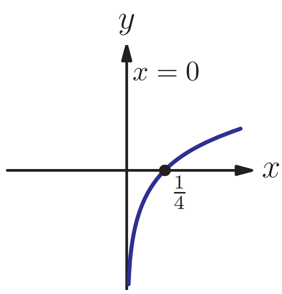
1. (a) (i)



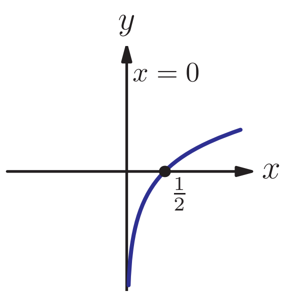
(ii)



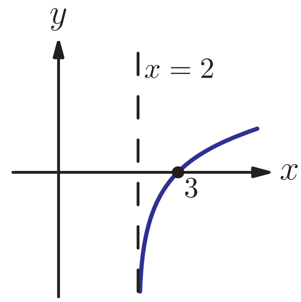
(b) (i)



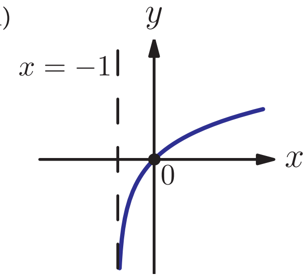
(ii)



(c) (i)



(ii)



Exercise 2G

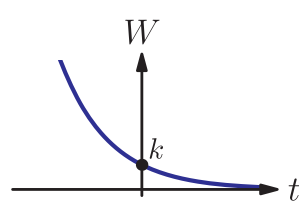
1. (a) (i) 2.45 (ii) 116
   (b) (i) −0.609 (ii) 4.62
   (c) (i) −1.71 (ii) 0.527
   (d) (i) 1.11 (ii) −2.98

2. (a) 100
   (b) 48 299
   (c) 2.24 h

3. (a) 64
   (b) 4.96 h

4. (a) 4.96 units
   (b) 138.8 mins

5. (a)



   (b) 2.3 mins

6. $\dfrac{\ln\left(\frac{5}{4}\right)}{\ln\left(\frac{1}{36}\right)}$

7. $x=10+\log_7 3$

8. 11.3 min

9. (b) 0.742

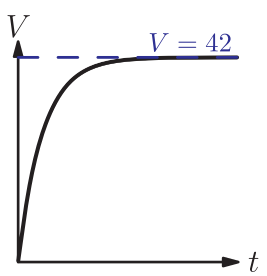
Mixed examination practice 2

Short questions

1. $x = \pm 24$
2. (a) $2a + \frac{b}{2} - c$
   (b) $\frac{a-1}{2}$
   (c) $\frac{b-c}{2}$
3. $x = e^{\frac{4}{3}} = 3.79, y = e^{\frac{10}{3}} = 28.0$
4. $x = 1 \pm \sqrt{1 - e^y}$
5. $x = \frac{\ln 3}{\ln 2}$
6. $a = \ b^{-2}$
7. $x = 5^{\frac{5}{3}}$ or $5^{\frac{-5}{3}}$
8. $x = e^2$ or e^{-2}

Long questions

1. (a)



   (b) 0 ms^{-1}
   (c) 42 ms^{-1}
   (d) 3.71s

2. (a) $k = 37000, a = \left(\frac{22}{37}\right)^{0.1} = 0.949$
   (b) 2750
   (c) 2039
   (d) $k = 7778, a = \left(\frac{10000}{7778}\right)^{0.1} = 1.025$
   (e) 2.5%

3. (a) $y = 3x^2$
   (b) $y = e^6 x^4$
   (c) $y = 2e^{3x-3}$
   (d) 2

Chapter 3

Exercise 3A

1. (a) Order 3, lead coefficient 3
   (b) Order 5, lead coefficient −1
   (c) No
   (d) No
   (e) No
   (f) No
   (g) Order 7, lead coefficient 2
   (h) Order 0, lead coefficient 1

2. (a) (i) $6x^3 + 8x^2 - 29x + 14$
   (ii) $3x^3 + 16x^2 + 23x + 6$
   (b) (i) $2x^4 - 15x^3 + 4x^2 + 4x - 1$
   (ii) $2x^4 - 7x^3 - 30x^2 + 6x + 15$
   (c) (i) $b^4 + b^3 - 3b^2 + 14b - 4$
   (ii) $r^4 - 11r^3 + 33r^2 - 62r + 14$
   (d) (i) $-x^6 + 2x^5 + 5x^4 - 10x^3 - x^2 + 5$
   (ii) $-x^6 + 2x^4 + x^3 - x^2 - x$

3. (a) (i) $x^2 + 5x - 1$
   (ii) $x^2 + x - 6$
   (b) (i) $3x^2 + 2x - 2$
   (ii) $5x^2 - 2$
   (c) (i) $x^3 - 2x^2 + 3x + 7$
   (ii) $x^3 - x^2 + x + 7$
   (d) (i) $x^2 + 5$
   (ii) $x - 2$

4. (a) (i) $x^3 + x^2 + 3$
   (ii) $x^3 + x^2 + 2$
   (b) (i) $2x^2 + 3$
   (ii) $x - 3$

5. (a) (i) $a = 4, b = -6$
   (ii) $a = 3, b = 1$
   (b) (i) $a = b = 2$
   (ii) $a = 0, b = -3$
   (c) (i) $a = 2, b = -2$
   (ii) $a = 2, b = 5$
   (d) (i) $a = -4, b = -6$
   (ii) $a = 10, b = 3$
   (e) (i) $a = \pm 2, b = 2$
   (ii) $a = \pm 2, b = \mp 5$

7. (a) Yes (b) No

Exercise 3B

1. (a) (i) 3 (ii) –2
   (b) (i) –7 (ii) 5
   (c) (i) –2 (ii) –4
   (d) (i) –8 (ii) 3

2. (a) No (b) No
   (c) Yes (d) Yes
   (e) No (f) Yes
   (g) No (h) Yes
   (i) No (j) No

3. (a) (i) $(x+1)(x-1)(x+2)$
   (ii) $(x+1)(x-2)(x+2)$
   (b) (i) $(x-2)^2(x-3)$
   (ii) $(x+2)^3$
   (c) (i) $(x-1)(x^2-2x+10)$
   (ii) $(x-3)(x^2+x+5)$
   (d) (i) $(x-1)(2x-1)(3x-1)$
   (ii) $(x+2)(4x+3)(3x-5)$

4. $a=1,\ b=-18$

5. $a=-44,\ b=48$

6. $k=0,4$

7. $k=-\dfrac{1}{2}$

8. (a) $a=2,\ b=59$ (b) $(x+8)$

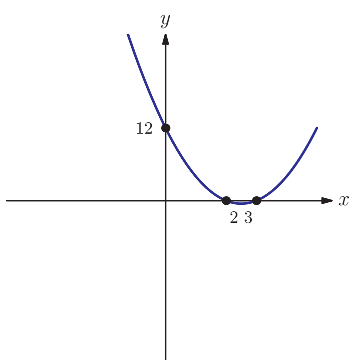
9. (a) $a=-12,\ b=22$ (b) 0

10. 14

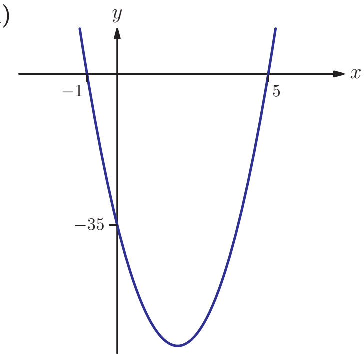
11. $a=37,\ b=-30$

Exercise 3C

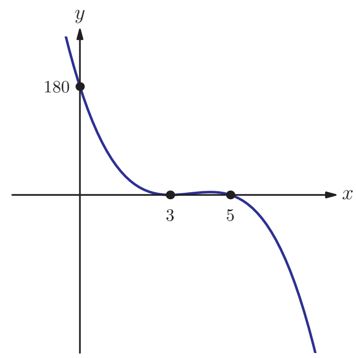
1. (a) (i)



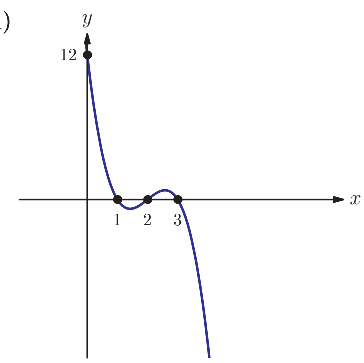
(ii)



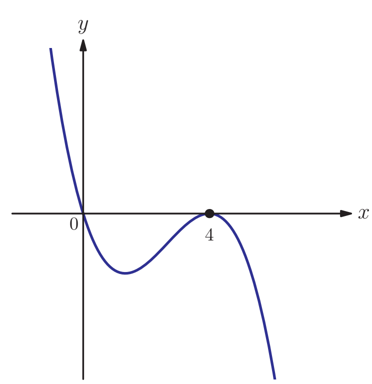
(b) (i)



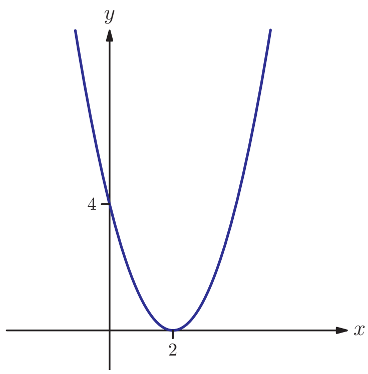
(ii)



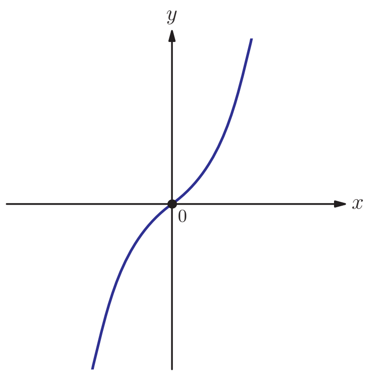
(c) (i)



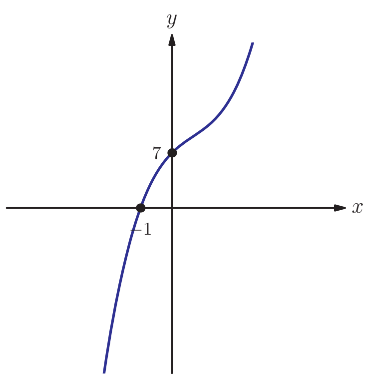
(ii)



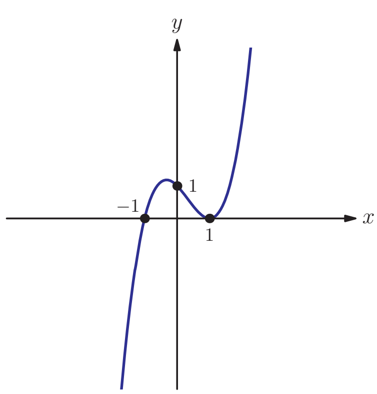
(d) (i)



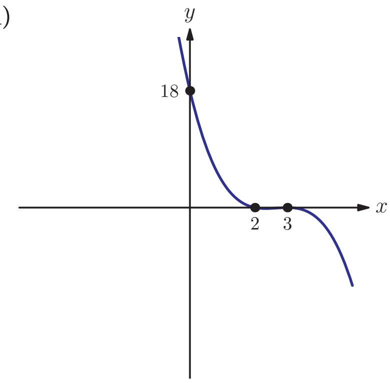
(ii)



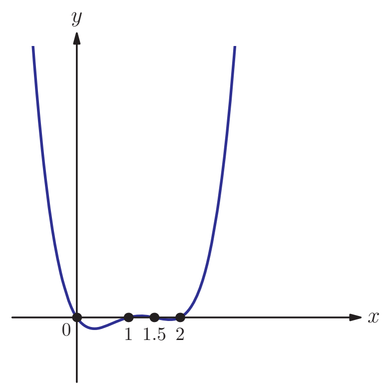
(e) (i)



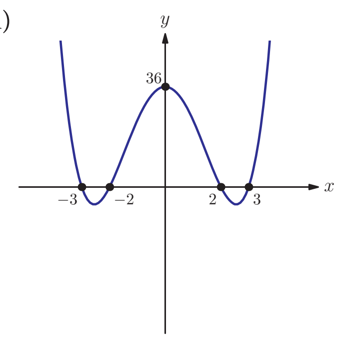
(ii)



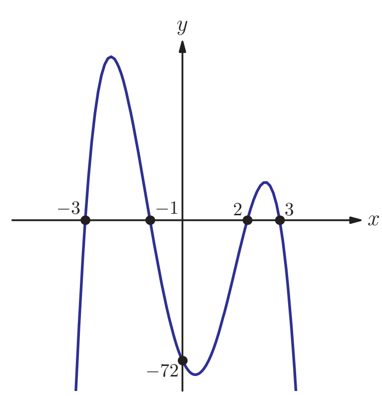
2. (a) (i)



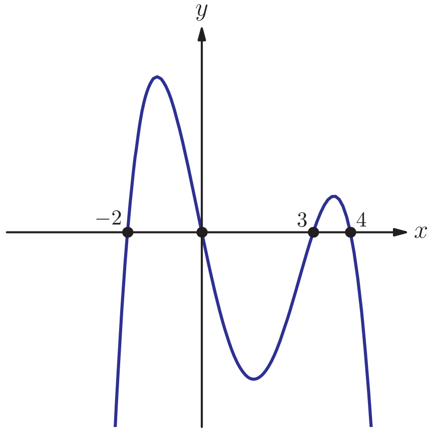
(ii)



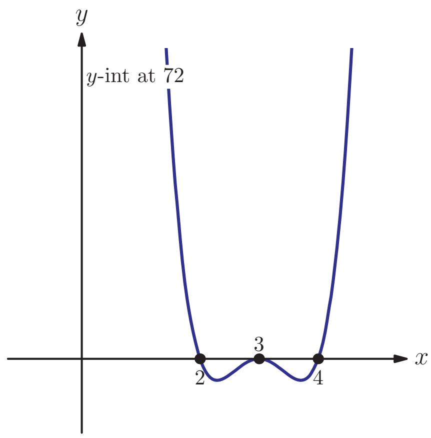
(b) (i)



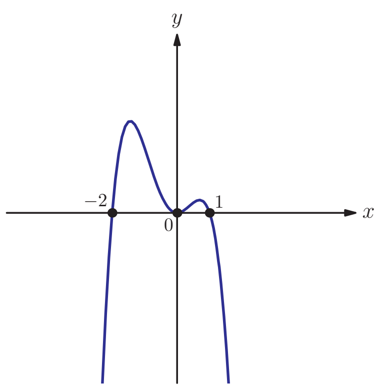
(ii)



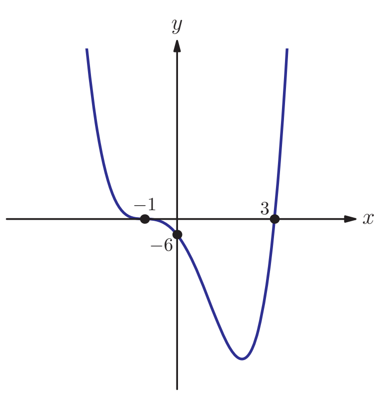
(c) (i)



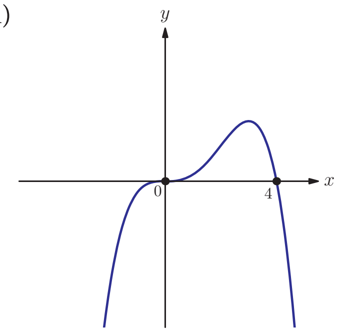
(ii)



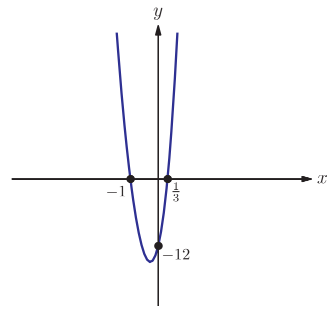
(d) (i)



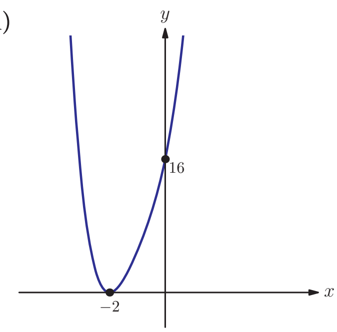
(ii)



(e) (i)



(ii)



3. (i) A1, B3, C2 (ii) A3, B1, C2

4. (a) (i) $a=2, b=-8, c=6$
(ii) $a=5, b=5, c=-10$

(b) (i) $a=-\dfrac{1}{4}, b=\dfrac{3}{4}, c=1$
(ii) $a=-1, b=0, c=1$

5. (a) (i) $y=2(x-1)(x-4)(x+2)$
(ii) $y=6x(x+2)(x-3)$

(b) (i) $y=-5x(x-1)(x+2)$
(ii) $y=-(x-1)(x+2)(x+4)$

(c) (i) $y=-(x+1)(x-2)^2$
(ii) $y=(x+1)^2(x-2)$

(d) (i) $y=x(x+2)(x+3)(2x-1)$
(ii) $y=-2(x-3)(x-4)(x+1)(x-2)$

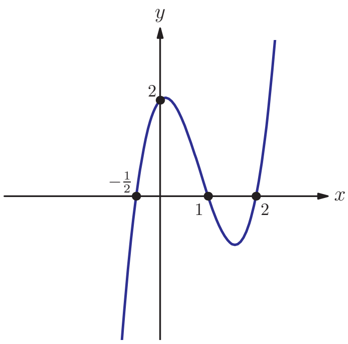
(e) (i) $y=-3x^2(x-1)(x-3)$
(ii) $y=5x(x-2)^2(2x+1)$

(f) (i) $y = 2(x-3)^2(x+1)^2$

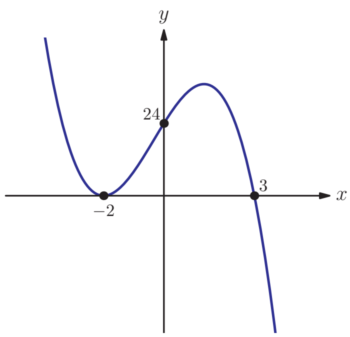
(ii) $y = -(x-3)^2(x-1)^2$

6. (b) $(x-2)(x-1)(2x+1)$

(c)



7.

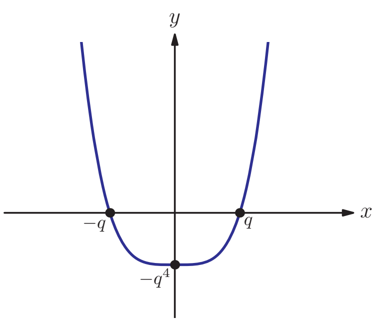


8. (a) $p=2, q=-8, r=-6, s=36$

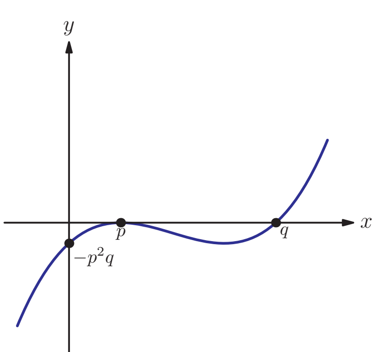
(b) $p=-1, q=3, r=0, s=0$

9. (a) $(x-q)(x+q)(x^2+q^2)$

(b)



10. (a)



(b) 1

Exercise 3D

1. (a) (i) 36 (ii) 68
(b) (i) −47 (ii) −119
(c) (i) 0 (ii) 0
(d) (i) 49 (ii) 49

2. (a) Two (b) None
(c) One (d) Two

3. (a) (i) $x = \frac{3 \pm \sqrt{5}}{2}$

(ii) $x = \frac{1 \pm \sqrt{5}}{2}$

(b) (i) $x = -1, \frac{2}{3}$

(ii) $x = \frac{3 \pm \sqrt{7}}{2}$

(c) (i) $x = \frac{4}{3}, -1$

(ii) $x = \frac{1}{2}, -1$

(d) (i) $x = 2 \pm \sqrt{7}$

(ii) $x = 1, -\frac{3}{2}$

4. (a) (i) $k < \frac{1}{4}$

(ii) $k < \frac{25}{12}$

(b) (i) $k = \frac{3}{5}$

(ii) $k = -\frac{1}{24}$

(c) (i) $k \geq -\frac{5}{4}$ (ii) $k \leq \frac{1}{16}$

(d) (i) $k > \frac{3}{8}$ (ii) $k > -\frac{25}{12}$

(e) (i) $k = \frac{17}{4}$ (ii) $k = \frac{55}{32}$

(f) (i) $k = 1$ (ii) $k = \frac{1}{32}$

(g) (i) $k < 0$ (ii) $k < 0$

5. $m = \pm\sqrt{2}$

6. $k = \frac{11}{2} \pm \sqrt{30}$

7. $c \geq \frac{17}{16}$

8. $0 < k < 6$

9. $-9 < k < -1$

10. $m \le -8$ or $m \ge 0$

11. $m < -\dfrac{9}{16}$

12. $k = \pm 9$

Mixed examination practice 3

Short questions

1. $k+2$

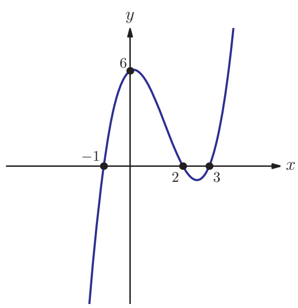
2. a, c negative, b positive, $b^2 - 4ac = 0$

3. $a=1,\ b=2,\ c=-12,\ d=-18,\ e=27$

4. $a=1,\ b=0$

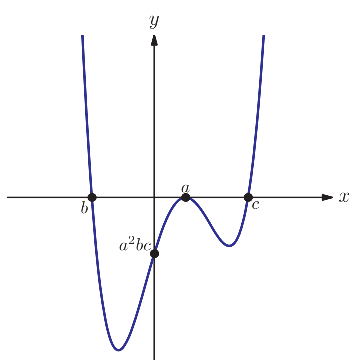
5. (b) $f(x) = (x-2)(x+1)(x-3)$

   (c)



6. $(a,b) = \pm\left(\dfrac{5}{3}, -\dfrac{4}{3}\right), \pm\left(-\dfrac{1}{3}, \dfrac{8}{3}\right)$

7.



8. $3 \pm 2\sqrt{2}$

9. $-4\sqrt{3} < k < 4\sqrt{3}$

10. $k \le -\sqrt{5} - \dfrac{1}{2}$ or $k \ge \sqrt{5} - \dfrac{1}{2}$

11. $a = -10, b = -18$

12. (a) $k-1, 1$

    (b) $-3, 5$

14. $-2\sqrt{2} \le k \le 2\sqrt{2}$

Long questions

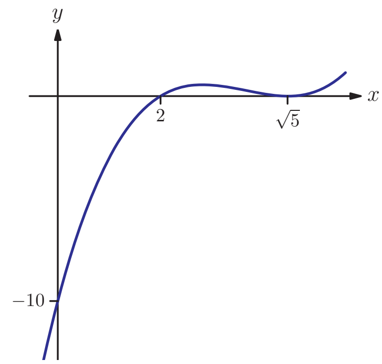
1. (a) $(0, -a)$

   (b) $x = -\dfrac{b}{2}$

   (d) $b > 7$ or $b < -5$

2. (b) $p = \pm 2\sqrt{5}$

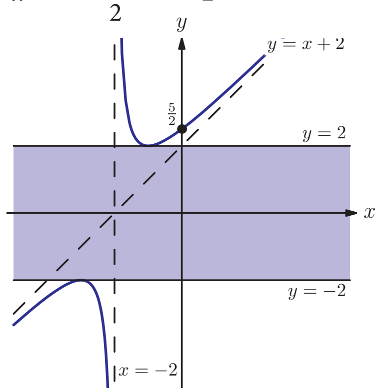
   (c)



3. (b) $x = -2, y = x + 2$

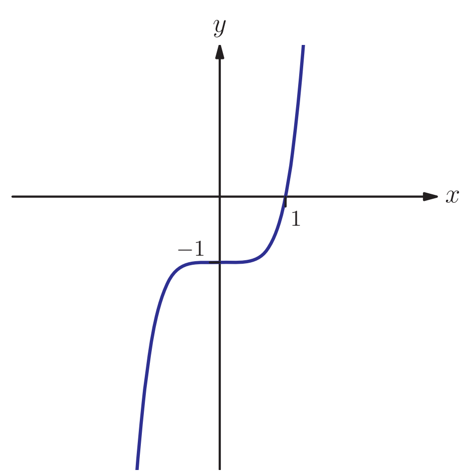
   (c) $x = \dfrac{y \pm \sqrt{y^2 - 4}}{2} - 2$

   (e)



4. (a) 5

   (c)



Chapter 4

Exercise 4A

1. (a) (i) $x=3$ (ii) $x=-1$

(b) (i) $x=\frac{1}{2},-\frac{3}{5}$

(ii) $x=3,-3$

(c) (i) $x=27,1$

(ii) $x=16,\frac{1}{81}$

(d) (i) $x=7,\pm\sqrt{3}$

(ii) $x=\frac{6}{5},\ 1,\ 4$

(e) (i) $x=2$

(ii) $x=1-\log_2 7$

2. (a) (i) $x=-3,1,4$

(ii) $x=-1,-3,5$

(b) (i) $x=2,\frac{3\pm\sqrt{5}}{2}$

(ii) $x=1,\frac{5\pm\sqrt{17}}{2}$

3. (a) (i) $x=1,2,3$

(ii) $x=-2,1,3$

(b) (i) $x=1,-1$

(ii) $x=-3,2,4$

4. $x=\frac{2}{3}\pm 2.$

5. $x=0,4,-4$

Exercise 4B

1. (a) (i) $a=\pm\sqrt{3},\pm\sqrt{7}$

(ii) $x=\pm 2,\pm\sqrt{3}$

(b) (i) $x=-\sqrt[3]{5},\sqrt[3]{1.5}$

(ii) $a=1,-2$

(c) (i) $x=\pm\sqrt{2+\sqrt{6}}$

(ii) $x=\pm\sqrt{6}$

(d) (i) $x=4,16$

(ii) $x=16,36$

(e) (i) $x=\ln 4$

(ii) $x=\ln 4,\ln 5$

(f) (i) $x=1,\ \log_5 10$

(ii) $x=2,\log_2 3$

(g) (i) $x=1,\sqrt{2}$ (ii) $x=3,9$

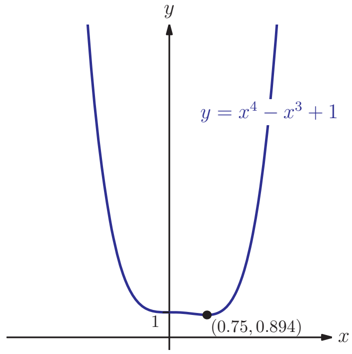
2. $x=0,2$

3. $x=0,\ \log_a 5$

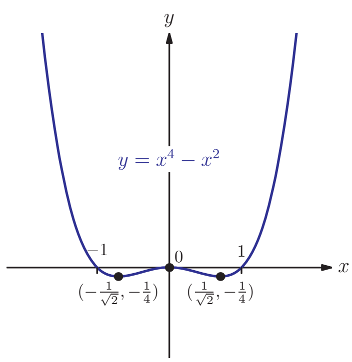
4. $x=2,32$

Exercise 4C

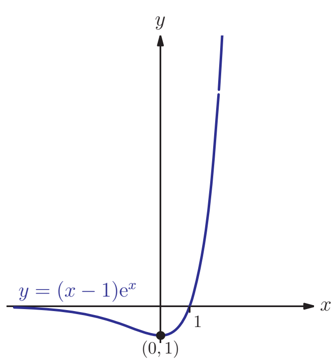
1. (a) (i)



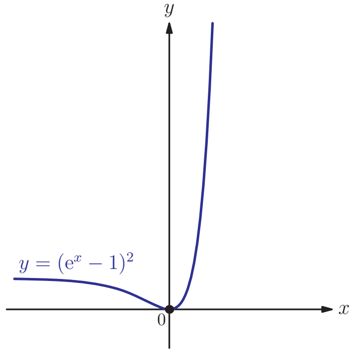
(ii)



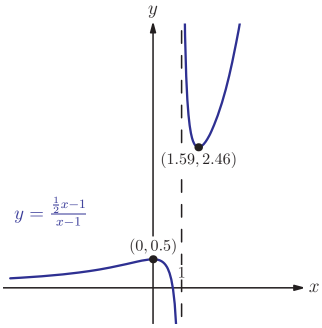
(b) (i)



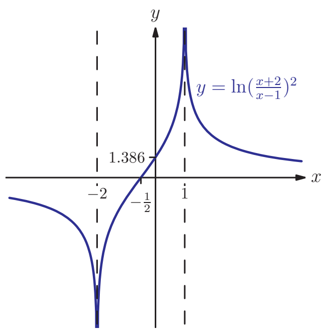
(ii)



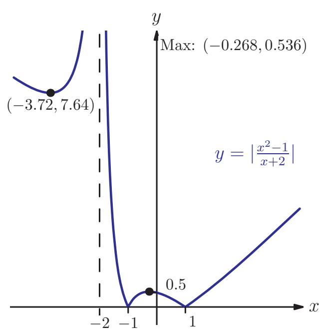
(c) (i)



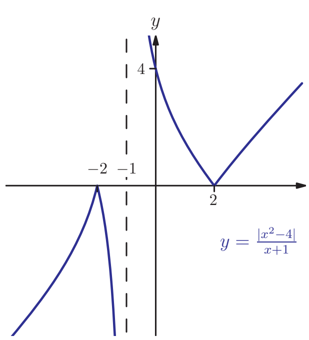
(ii)



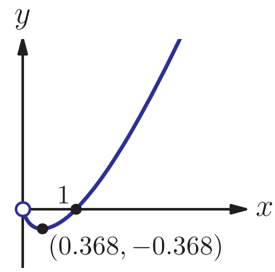
(d) (i)



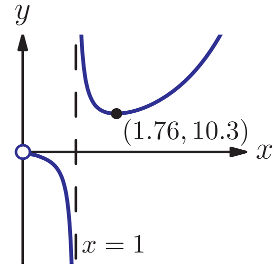
(ii)



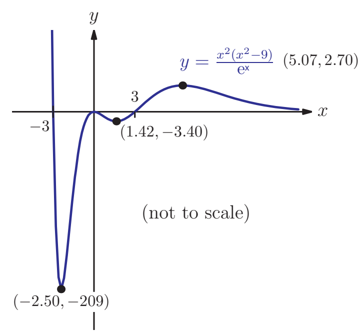
2.



3.



4.



Exercise 4D

1. (a) (i) $x=-1.88, 0.347, 1.53$
(ii) $x=-4.49$
(b) (i) $x=0$
(ii) -1.74
(c) (i) No solution
(ii) $x=3.59$

2. $x=1.53$

3. (a) 3 (b) 1 (c) 1

Exercise 4E

1. (a) (i) $(-2,-3),(1,0)$
(ii) $(3,0)$
(b) (i) $(-3,-9),(4,5)$
(ii) No intersection

2. (a) (i) $(3,1), \left(\frac{-11}{5},\frac{-8}{5}\right)$
(ii) $(-3,3), (5,-1)$

(b) (i) $(1,3), (3,1)$
(ii) $(-3,-5), (-5,-3)$
(c) (i) $(-1,6),(2,3)$
(ii) $(1,-3), (-1,-5)$

4. $-1\pm 2\sqrt{6}$

5. $\pm 6\sqrt{2}$

7. $x=3, y=1$ or $y=3, x=1$

8. $x=\sqrt{3}$, $y=9\sqrt{3}$

Exercise 4F

1. (a) (i) $x=3, y=-1, z=1$
(ii) $x=-2, y=2, z=1$
(b) (i) $x=3, y=-3, z=0$
(ii) $x=-2, y=1, z=0$
(c) (i) $x=3, y=-1, z=2$
(ii) $x=-1, y=0, z=3$

2. (a) Unique solution
(b) No solutions
(c) Infinitely many solutions
(d) No solutions
(e) Unique solution

3. (a) (i) $x=2-t, y=1+t, z=t$
(ii) $x=2t+1, y=t-1, z=t$
(b) $x=2t+1, y=t, z=0$
(c) $x=5-t+2s, y=s, z=t$

4. -2

5. -9

6. $x=1,\ y=z=\dfrac{2-a}{3}$

7. (a) $a=1$ or 2
(b) $x=1+\dfrac{t}{5}, y=\dfrac{3t}{5}, z=t$

8. (a) -2
(b) 9
(c) $x=t+5, y=t-1, z=t$

Exercise 4G

1. (a) (i) $x<3.08$ (ii) $x>1.58$
(b) (i) $x\le 4.11$ (ii) $x\ge 7.54$
(c) (i) $x<1.91$ (ii) $x<0.743$
(d) (i) $x\ge 1.36$ (ii) $x<1.08$
(e) (i) $x>-1.77$ (ii) $x\le 4.49$
(f) (i) $-2<x<0$ or $x>2$
(ii) $x<-\sqrt{6}$ or $0<x<\sqrt{6}$

2. (a) $0<x<1.30$ or $x>12.7$
(b) $1<x<1.30$ or $x>12.7$

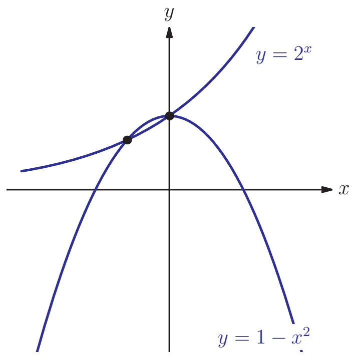
Exercise 4H

2. $4p^3+27q^2=0$

Mixed examination practice 4

Short questions

1. (a)



(b) 2

2. 0.541

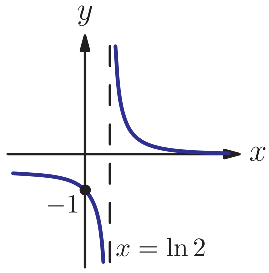
3. $-1.12<x<-0.379$ or $x>-2.02$

4. $x=2, y=-2, z=-3$

5. $x=e^3$

6. (a) $x=\pm 2, \pm 3$
(b) $-3\le x\le -2$ or $2\le x\le 3$

7. (a)



(b) $x=\ln 2$

8. (a) -7
(b) $x=5-t, y=t, z=9$

9. ± 5

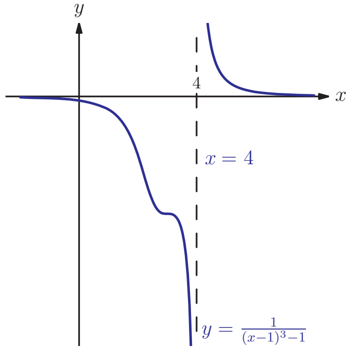
10. $0<x<1$ or $1.43<x<8.61$

11. $x=1$

Long questions

1. (a) (i) $k = 27$
   (ii) $x = 4$

   (b) (i)



   (ii) $x = 4, y = 0$

   (c) $(1.44, -1.38)$

2. (a) $x = \pm\sqrt{2}$

   (b) (i) $k = -1$ (ii) $x = 1 \pm \sqrt{2}$

   (c) $x < 1 - \sqrt{2}$ or $x > 1 + \sqrt{2}$

3. (a) 0
   (b) 5
   (c) $x = 4 - t, y = -1, z = t$

Chapter 5

Exercise 5A

1. (a) Function: many-to-one
   (b) Relation
   (c) Relation
   (d) Function: one-to-one
   (e) Function: one-to-one
   (f) Function: one-to-one

2. (a) (i) Yes (ii) Yes
   (b) (i) Yes (ii) Yes
   (c) (i) No (ii) No

Exercise 5B

1. (a) (i) 24 (ii) 140
   (b) (i) 14 (ii) 4
   (c) (i) $3z^2 - z$ (ii) $3a^2 - a$
   (d) (i) $3x^2 + 5x + 2$
   (ii) $3x^2 - 13x + 14$
   (e) (i) $-x$ (ii) $57x^2 - 11x$
   (f) (i) $\dfrac{3-x}{x^2}$ (ii) $3x - \sqrt{x}$

2. (a) (i) 3 (ii) 7
   (b) (i) 0 (ii) 1
   (c) (i) $1 + \log_{10} y$ (ii) $1 + \log_{10} z$
   (d) (i) $2 + \log_{10} x$ (ii) $3 + \log_{10} y$
   (e) (i) $4 + 5\log_{10} x$ (ii) 1

3. (a) (i) 4 (ii) -8
   (b) (i) $3x + 1 - \sqrt{y}$ (ii) $6x + 4 - 2\sqrt{x}$
   (c) (i) $24x + 2 - 6\sqrt{x}$ (ii) $6x^2 + x - \sqrt{x^2 + 1}$

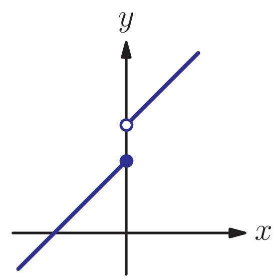
Exercise 5C

1. (a) Domain: $\mathbb{R}$ Range: $]0, \infty[$
   (b) Domain: $\mathbb{R}$ Range: $]0, \infty[$
   (c) Domain: $]0, \infty[$ Range: $\mathbb{R}$
   (d) Domain: $]0, \infty[$ Range: $\mathbb{R}$

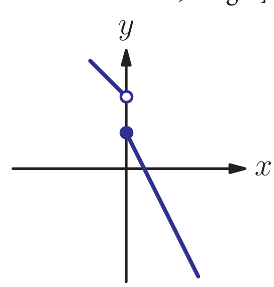
2. (a) (i) $x \neq -2$ (ii) $x \neq 7$
   (b) (i) $x \neq 2$ or -4 (ii) $x \neq \pm 3$
   (c) (i) $y \geq 1$ (ii) $x \geq -3$
   (d) (i) $a > 1$ (ii) $x < \dfrac{2}{5}$
   (e) (i) $x \neq 0$ or -1 (ii) $x \geq -1$
   (f) (i) $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$ (ii) $x \leq -3$ or $x \geq 1$
   (g) (i) $x \geq 0$ (ii) $x \geq -\dfrac{3}{2}$

3. (a) (i) $y \leq 7$ (ii) $y \geq 3$
   (b) (i) $y \geq 12$
   (ii) $y \geq 5, y \in \mathbb{Z}$
   (c) (i) $y \geq 0$ (ii) $y \geq 0$
   (d) (i) $y \leq -1$ or $y > 0$
   (ii) $y > 0$

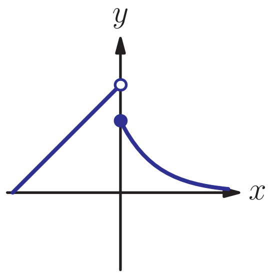
4. (a) (i) Not continuous, range $]-\infty, 2] \cup]3, \infty[$



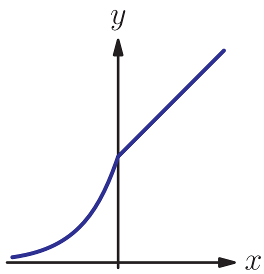
   (ii) Not continuous, range $]-\infty, 1] \cup]2, \infty[$



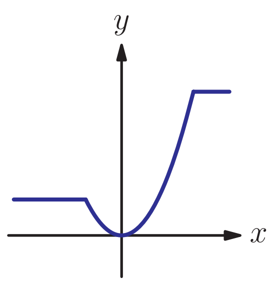
(b) (i) Not continuous, range $]-\infty, 3[$



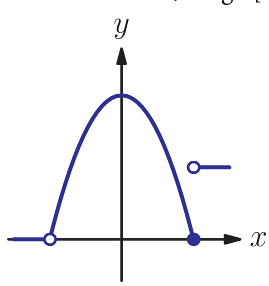
(ii) Continuous, range $]0, \infty[$



(c) (i) Continuous, range $[0, 4]$



(ii) Not continuous, range $[0, 2]$



5. $x \geq 5$

6. $x \geq 1, x \neq 2, x \neq 3$

7. a = 15

8. $x < -2$ or $x > -1$

9. $x \leq \frac{1}{2}$ or $x > 12$

10. (a) (i) $a \leq x < b$ (ii) $\varnothing$

(b) $f(a) = \begin{cases} \ln(b-a) & \text{for } a < b \\ \text{undefined} & \text{for } a \geq b \end{cases}$

Exercise 5D

1. (a) (i) 5 (ii) 26
(b) (i) $9x+8$ (ii) $9x^2+12x+5$
(c) (i) $9\sqrt{a}+17$
(ii) $y^4-4y^3+8y^2-8y+5$
(d) (i) $9x^2+17$
(ii) $27z^2+36z+17$

2. (a) (i) x^2 (ii) x^3
(b) (i) $3x-5$ (ii) x^2+5x+6
(c) (i) $x+4$ (ii) $x^{\frac{2}{3}}$
(d) (i) $\ln(\ln x)$
(ii) $\ln\left(\frac{x+1}{3}\right)$

3. $x = 0, -2$

4. $x = -\frac{1}{3}$

5. (a) $y \neq 2$ (b) $x = 1.5$
(c) Domain: $x < -1$ or $x \geq 1.5$
Range: $y > 0$ and $y \neq \sqrt{2}$

6. (a) $\sqrt[3]{2x+3}$ (b) $2\sqrt[3]{x}+3$

7. (a) $a = \frac{-4}{3}$, $b = \frac{-2}{3}$ (b) $y \geq 0$

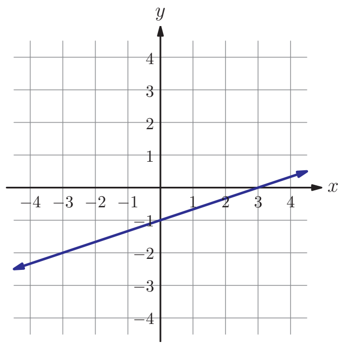
8. (a) x^2 is not always > 3
(b) $x \in]-\infty, -\sqrt{3}[\cup]\sqrt{3}, \infty[$

9. $\frac{x-2}{6}$

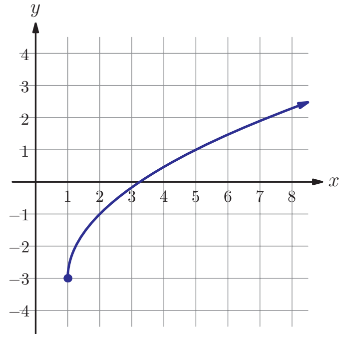
Exercise 5E

1. (a) (i) $\frac{x-1}{3}$ (ii) $\frac{x+3}{7}$
(b) (i) $\frac{2x}{3x-2}, x \neq \frac{2}{3}$ (ii) $\frac{x}{1-2x}, x \neq \frac{1}{2}$
(c) (i) $\frac{xb-a}{x-1}, x \neq 1$ (ii) $\frac{x-1}{bx-a}, x \neq \frac{a}{b}$
(d) (i) $1-x$ (ii) $\frac{x-2}{3}$
(e) (i) $\frac{x^2+2}{3}, x \geq 0$ (ii) $\frac{2-x^2}{5}, x \geq 0$
(f) (i) $\frac{1-e^x}{5}$ (ii) $\frac{e^x-2}{2}$
(g) (i) $2\ln\left(\frac{x}{7}\right)$, $x > 0$ (ii) $\frac{1}{10}\ln\left(\frac{x}{9}\right)$, $x > 0$
(h) (i) $5-\sqrt{y+19}$ (ii) $\sqrt{y+10}-3$

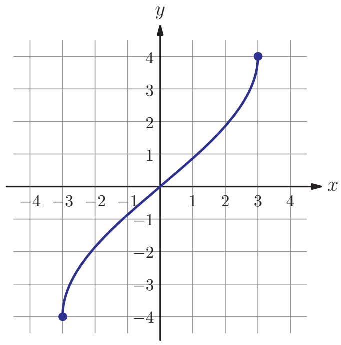
2. (a)



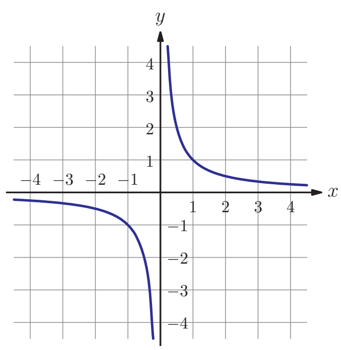
(b)



(c)



(d)



3. (a) (i) x, yes (ii) x, yes

(b) (i) $4x$, no (ii) $\frac{x}{16}$, no

(c) (i) $20+x$, no (ii) $2+x$, no

(d) (i) x, yes (ii) x, yes

(e) (i) x, yes (ii) x, yes

(f) (i) $\frac{x^4}{3}$, no (ii) x^9, no

(g) (i) $\sqrt[4]{x}$, no (ii) $\sqrt[9]{x}$, no

(h) (i) $\frac{x-1}{x}$, no (ii) $\frac{1+x}{2+x}$, no

(i) (i) x, yes (ii) x, yes

4. $x \geq 0$

5. (a) -1 (b) 1

6. -23

7. $f^{-1}(x)=\frac{1}{2}\ln\left(\frac{x}{3}\right)$

8. $(f \circ g)^{-1}(x)=\sqrt[3]{\frac{x-3}{2}}$

9. (a) $\ln 3$

10. $x=-1$

11. $f^{-1}(x)=-\sqrt{\frac{4x+4}{1-x}},\ x \neq 1$

12. (a) $k=0,\ f^{-1}(x)=-\sqrt{x},\ x \geq 0$

(b) $k=-1,\ f^{-1}(x)=\sqrt{x-2}-1,\ x \geq 2$

(c) $k=0,\ f^{-1}(x)=-x,\ x \geq 0$

13. (a) $f^{-1}(x)=\frac{e^x}{3}+1$

(b) $g \circ f(x)=3x-3,\ x>1$

14. (a) $k=2$

(b) (i) $]-\infty,0] \cup]2,\infty[$

(ii) $f^{-1}(x)=\begin{cases} 1+\sqrt{-x}, & x \leq 0 \\ 1-\sqrt{x-2}, & x>2 \end{cases}$

15. (b) $k=-3$

Exercise 5F

1. (a) (i) $\left(-\frac{1}{3},0\right),\left(0,\frac{1}{3}\right)$ (ii) $\left(-\frac{5}{2},0\right),(0,5)$

(b) (i) $\left(\frac{3}{2},0\right),\left(0,-\frac{3}{7}\right)$ (ii) $\left(\frac{5}{3},0\right),\left(0,-\frac{5}{2}\right)$

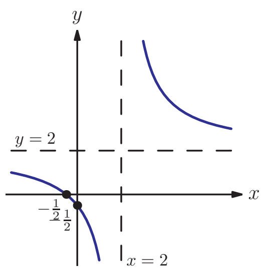
2. (a) (i) $x=1, y=4$ (ii) $x=7, y=2$

(b) (i) $x=\frac{1}{2}, y=\frac{3}{2}$ (ii) $x=\frac{5}{3}, y=\frac{4}{3}$

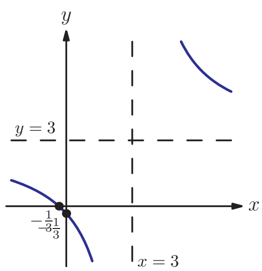
(c) (i) $x=\frac{-5}{2}, y=\frac{-1}{2}$ (ii) $x=\frac{2}{3}, y=\frac{-2}{3}$

(d) (i) $x=2, y=0$ (ii) $x=\frac{-1}{2}, y=0$

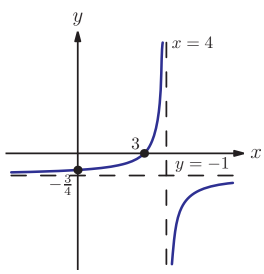
3. (a) (i)



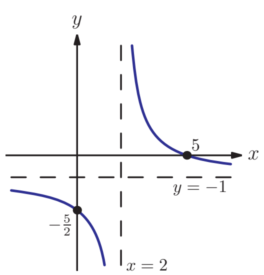
(ii)



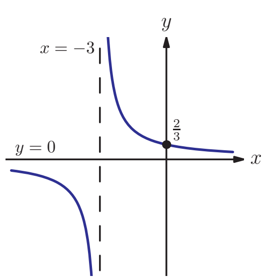
(b) (i)



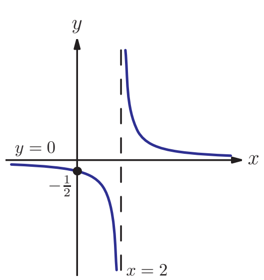
(ii)



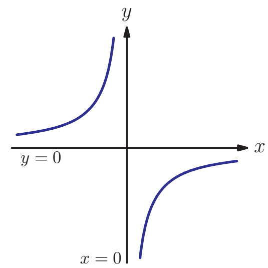
(c) (i)



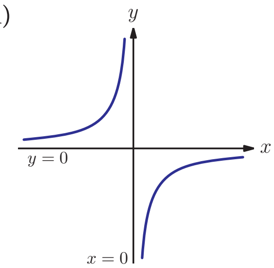
(ii)



(d) (i)



(ii)



4. (a) (i) Domain $x \neq 0$, Range $y \neq 0$, $f^{-1}(x) = \dfrac{3}{x}$

(ii) Domain $x \neq 0$, Range $y \neq 0$, $f^{-1}(x) = \dfrac{7}{x}$

(b) (i) Domain $x \neq 3$, Range $y \neq 0$, $f(x) = \dfrac{2+3x}{x}$

(ii) Domain $x \neq 0$, Range $y \neq 0$, $f^{-1}(x) = \dfrac{5^x - x}{x}$

(c) (i) Domain $x \neq \dfrac{1}{3}$, Range $y \neq \dfrac{2}{3}$,

$f^{-1}(x) = \dfrac{x+1}{3x-2}$, $x \neq \dfrac{2}{3}$

(ii) Domain $x \neq \dfrac{-1}{2}$, Range

$y \neq 2$, $f^{-1}(x) = \dfrac{-x-5}{2x-4}$, $x \neq 2$

(d) (i) $x \neq -1$, Range

$y \neq 2$, $f^{-1}(x) = \dfrac{5-2x}{x+2}$, $x \neq -2$

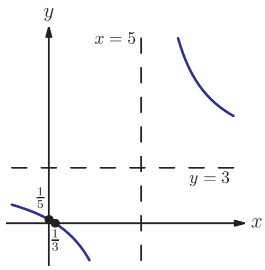
(ii) Domain $x \neq \dfrac{3}{4}$, Range

$y \neq \dfrac{3}{4}$, $f^{-1}(x) = \dfrac{3x-1}{4x-3}$, $x \neq \dfrac{3}{4}$

5. $x = \dfrac{4}{5}$, $y = \dfrac{-3}{5}$

6. (a) Domain $x \neq -3$, Range $y \neq 0$

(b) $f^{-1}(x) = \dfrac{1-3x}{x}$

7. 

8. (a) $y \neq \frac{a}{2}$

(b) $f^{-1}(x) = \frac{8x+3}{2x-a}, x \neq \frac{a}{2}$

(c) 8

Mixed examination practice 5

Short questions

1. (a) $3^x - 3$

(b) $\sqrt[3]{\ln\left(\frac{x}{3}\right)+1}$

2. (a) $y = \log_2 x$ (b) (1,0)

3. (a) $x = 5, y = -4$

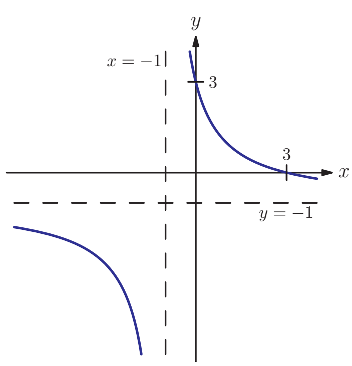
(b) $f^{-1}(x) = \frac{5x+3}{x+4}$

4. (a) $(x-3)^2 + 1$ (b) $\sqrt{x-1}+3$

(c) $x \geq 1$

5. (a) $(x-3)^2 - 7$ (b) $y \geq -7$ (c) $\sqrt{x+7}+3$

6. (a) $y \in \mathbb{R}, y \neq -1$

(b) 

(c) $f^{-1}(x) = \frac{3-x}{x+1}, x \neq -1, y \neq -1$

7. (a) (i) $]0, 3] \cup]5, \infty[$

(ii) $f^{-1}(x) = \begin{cases} \ln\left(\frac{3}{x}\right), & 0 < x \leq 3 \\ 5 - x, & x > 5 \end{cases}$

Domain: $]0, 3] \cup]5, \infty[$

(b) $p = 5$

8. (a) $a = -2, b = 1$ (b) $y \geq 0$

Long questions

1. (a) 10

(b) $4 - x^2$

(c) Reflection in the line $y = x$

(d) (i) $\sqrt{x-1}$ (ii) $y > 3$

(iii) $x > 10$

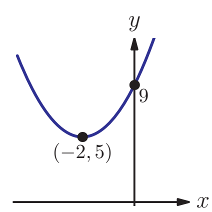
(e) $x = -4, 1$

2. (a) (i) 15 (ii) $y \in \mathbb{R}$

(iii) $\frac{3x+5}{x-1}$ (iv) $4x+3$

(b) $f(x)$ can be 1, which is not in the domain of g.

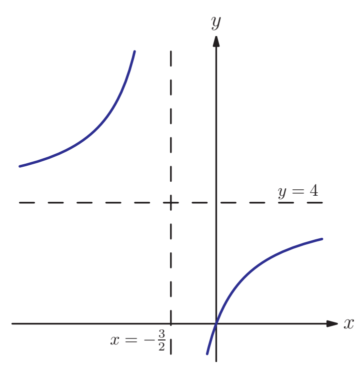
(c) (i) $\frac{x+3}{x-1}$ (ii) $x \neq 1$ (ii) $y \neq 1$

3. (a) $(x+2)^2 + 5$ (b) 

(c) Range of $f(x)$ is $y \geq 5$, Range of $g(x)$ is $y > 0$

(d) $y > 9$

4. (a) $y = \frac{8x}{2x+3}$

(b) 

(c) (i) $\frac{16x+8k}{4x+2k+3}$ (ii) $x = -\frac{2k+3}{4}, y = 4$

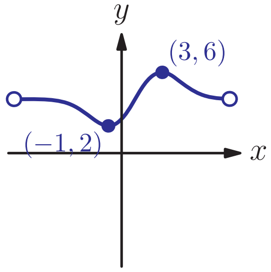
(iii) $f(x) = f^{-1}(x) = \frac{16x-76}{4x-16}$

5. (b) $f\left(\frac{1}{x}\right) + 2f(x) = \frac{2}{x} + 1$ (c) $\frac{1}{3}\left(\frac{4}{x} - 2x + 1\right)$

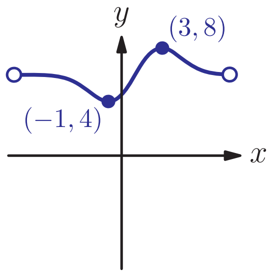
Chapter 6

Exercise 6A

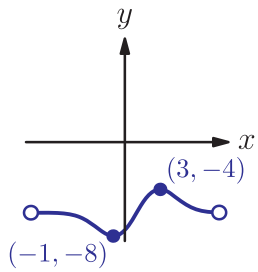
1. (a) (i)



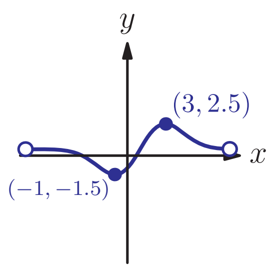
(ii)



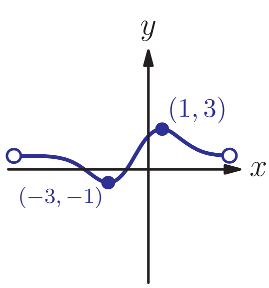
(b) (i)



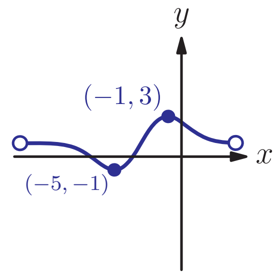
(ii)



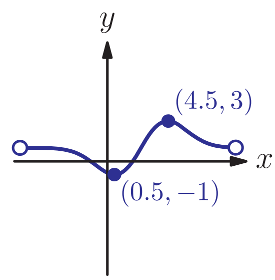
(c) (i)



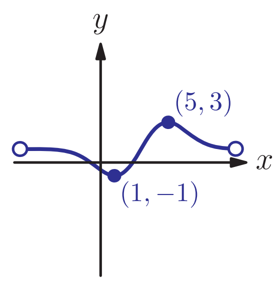
(ii)



(d) (i)



(ii)



2. (a) (i) $y = 3x^2 + 3$ (ii) $y = 9x^3 - 7$

(b) (i) $y = 7x^3 - 3x + 4$ (ii) $y = 8x^2 - 7x + 6$

(c) (i) $y = 4(x-5)^2$ (ii) $y = 7(x+3)^2$

(d) (i) $y = 3(x+4)^3 - 5(x+4)^2 + 4$

(ii) $y = (x-3)^3 + 6(x-3) + 2$

3. (a) (i) Vertically down 5 units
(ii) Vertically down 4 units

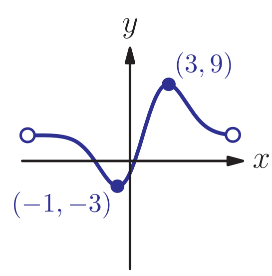
(b) (i) Left 1 unit (ii) Left 5 units

(c) (i) Right 4 units (ii) Right 5 units

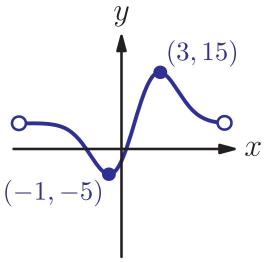
(d) (i) Left 3 units (ii) Right 2 units

Exercise 6B

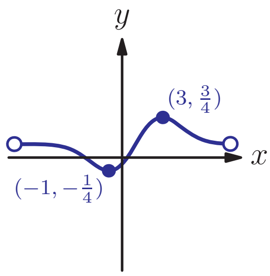
1. (a) (i)



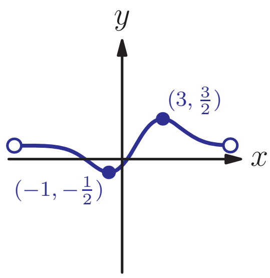
(ii)



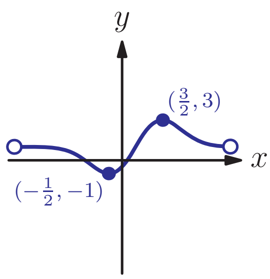
(b) (i)



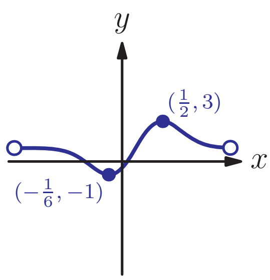
(ii)



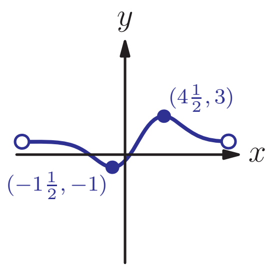
(c) (i)



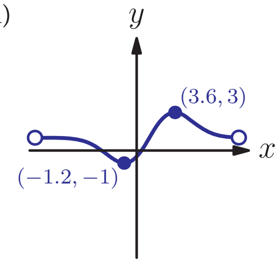
(ii)



(d) (i)



(ii)



2. (a) (i) $y = 21x^2$

(ii) $y = 18x^3$

(b) (i) $y = \frac{1}{3}(7x^3 - 3x + 6)$

(ii) $y = \frac{4}{5}(8x^2 - 7x + 1)$

(c) (i) $y = x^2$

(ii) $y = 7\left(\frac{x}{5}\right)^2$

(d) (i) $y = 3(2x)^3 - 5(2x)^2 + 4$

(ii) $y = \left(\frac{3x}{2}\right)^3 + 6\left(\frac{3x}{2}\right) + 2$

3. (a) (i) Vertical stretch, scale factor 4

(ii) Vertical stretch, scale factor 6

(b) (i) Horizontal stretch, scale factor $\frac{1}{3}$

(ii) Horizontal stretch, scale factor $\frac{1}{4}$

(c) (i) Horizontal stretch, scale factor 2

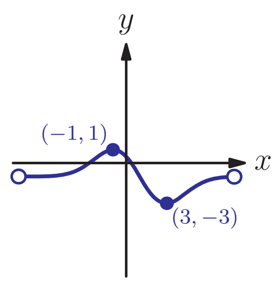
(ii) Horizontal stretch, scale factor 5

(d) (i) Horizontal stretch, scale factor $\frac{1}{3}$

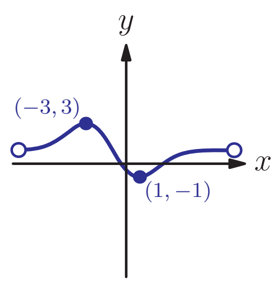
(ii) Horizontal stretch, scale factor 2

Exercise 6C

1. (a)



(b)



2. (a) (i) $y = -3x^2$

(ii) $y = -9x^3$

(b) (i) $y = -7x^3 + 3x - 6$

(ii) $y = -8x^2 + 7x - 1$

(c) (i) $y = 4x^2$

(ii) $y = -7x^3$

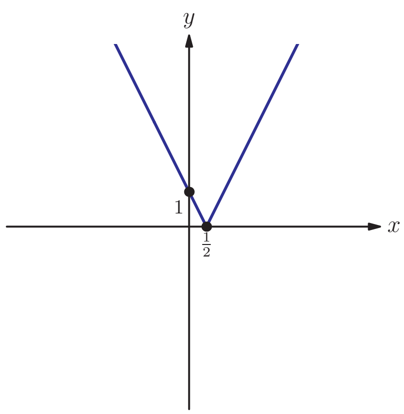
(d) (i) $y = -3x^3 - 5x^2 + 4$

(ii) $y = -x^3 - 6x + 2$

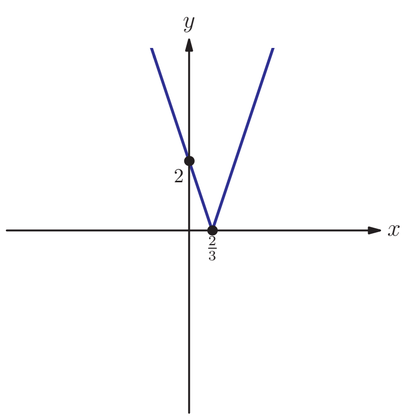
3. (a) (i) Reflection in the x-axis
(ii) Reflection in the x-axis
(b) (i) Reflection in the y-axis
(ii) Reflection in the y-axis
(c) (i) Reflection in the x-axis
(ii) Reflection in the x-axis
(d) (i) Reflection in the y-axis
(ii) Reflection in the y-axis

Exercise 6D

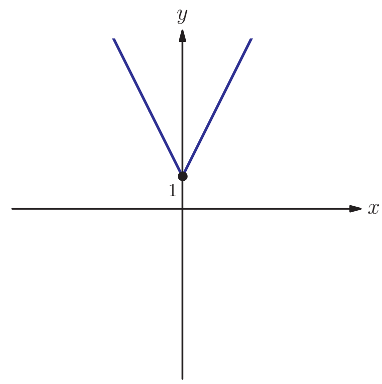
1. (a) (i)



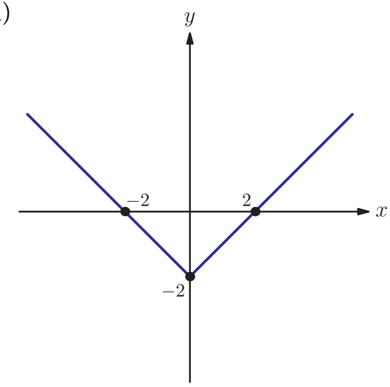
(ii)



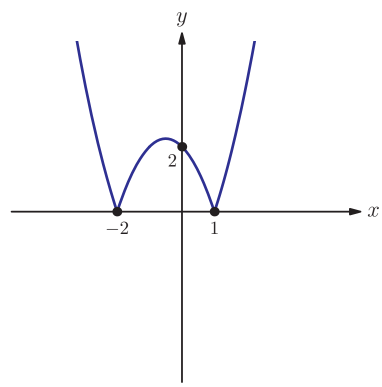
(b) (i)



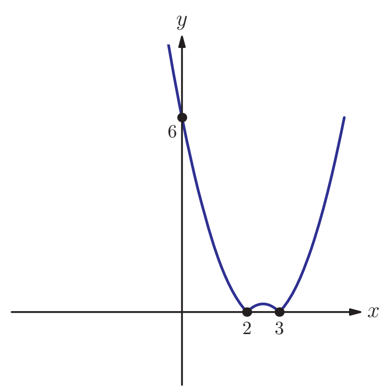
(ii)



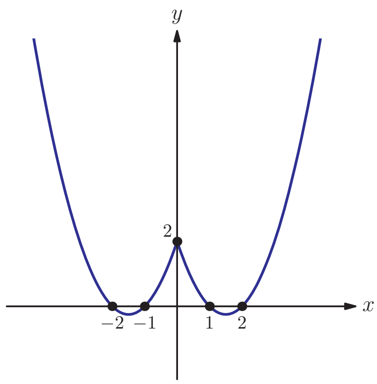
(c) (i)



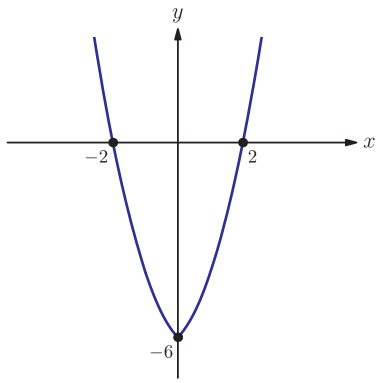
(ii)



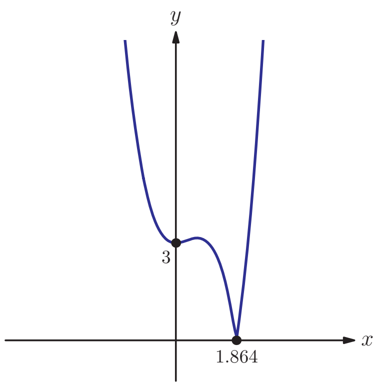
(d) (i)



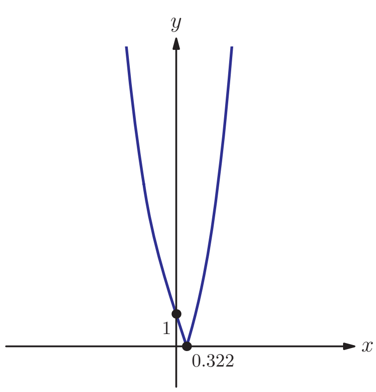
(ii)



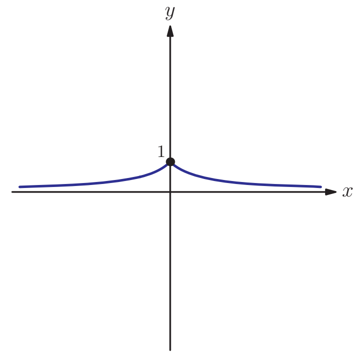
2. (a) (i)



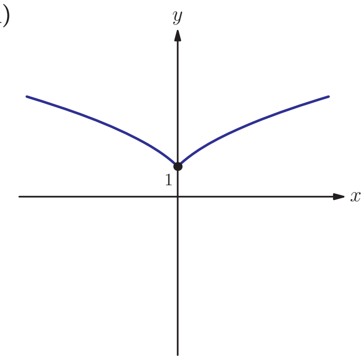
(ii)



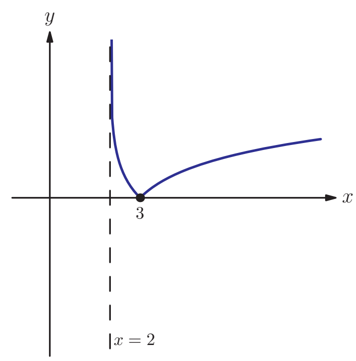
(b) (i)



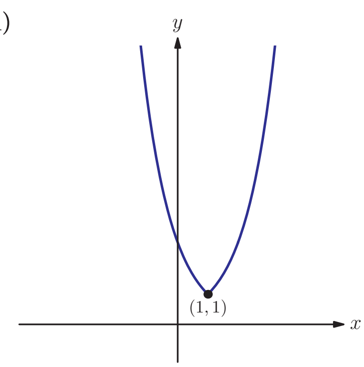
(ii)



(c) (i)



(ii)



3. (a) (i) $x = \pm 4$ (ii) $x = \pm 18$

(b) (i) $x = 0, 4$ (ii) $x = -1, \frac{1}{3}$

(c) (i) $x = 0, -8$ (ii) $x = -\frac{2}{3}, 8$

(d) (i) $x = \frac{2}{3}, 4$ (ii) $x = \frac{1}{4}, \frac{9}{2}$

4. (a) (i) $x = -2, 3$ (ii) $x = -\frac{1}{2}$

(b) (i) $x = \frac{1}{2}$ (ii) $x = -1, -2$

(c) (i) $x = 4, -\frac{4}{3}$ (ii) $x = 1, -\frac{1}{3}$

(d) (i) No solutions (ii) No solutions

5. (a) (i) $x \in]-\infty, -5[\cup]5, \infty[$

(ii) $x \in]-\infty, -2[\cup]2, \infty[$

(b) (i) $-3 < x < 3$ (ii) $-10 < x < 10$

(c) (i) $x \in \left]-\infty, \frac{-5}{2}\right[\cup \left]\frac{3}{2}, \infty\right[$

(ii) $-\frac{1}{3} < x < \frac{5}{3}$

(d) (i) $\frac{4}{3} < x < 6$ (ii) $x \in]-\infty, 1[\cup]5, \infty[$

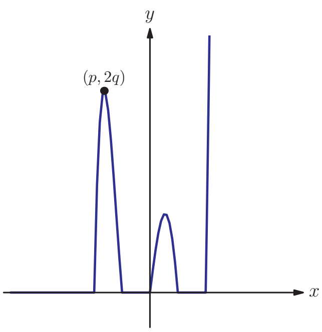
(e) (i) $-\frac{4}{3} < x < 4$ (ii) $-1 < x < 1$

6. $x \in \mathbb{R}$ **7.** $0.472, 8.47$

8. $2 \le x \le 5$ **9.** $x = 0, 4, -4$.

10. $\frac{q^2}{2}$

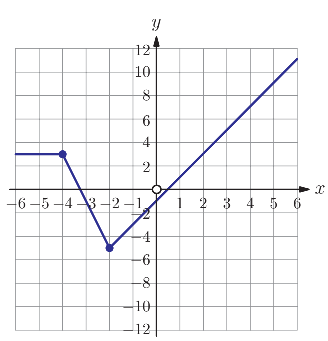
11.



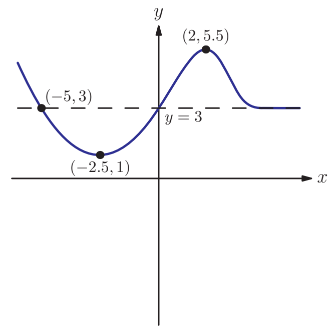
Exercise 6E

1. (a) $y = p(f(x) + c)$ (b) $y = f\left(\frac{x}{q} + d\right)$

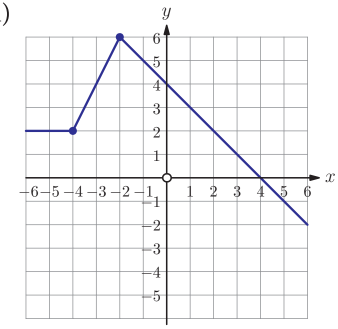
2. (a) (i)



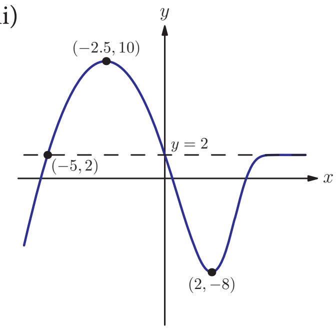
(ii)



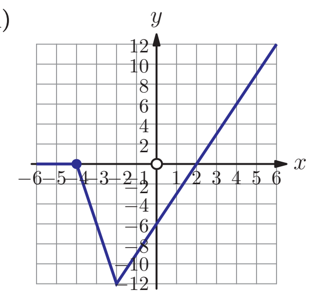
(b) (i)



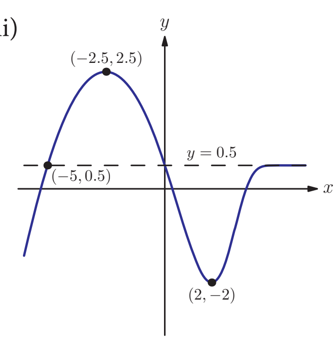
(ii)



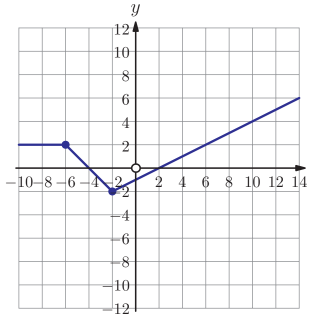
(c) (i)



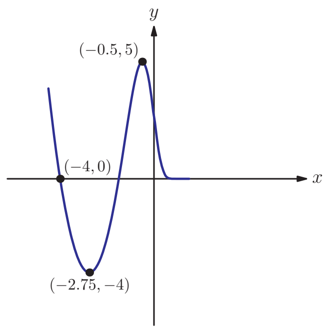
(ii)



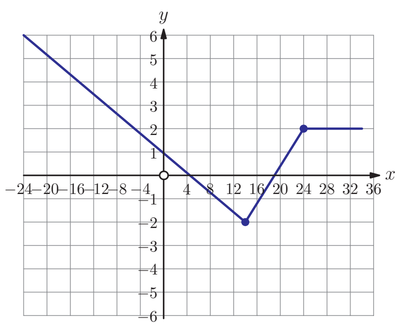
(d) (i)



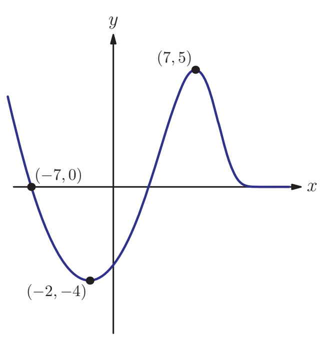
(ii)



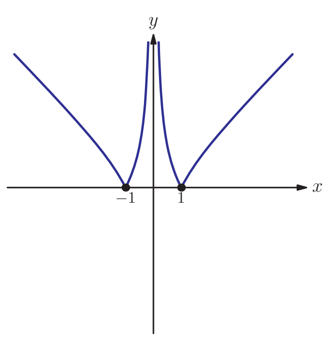
(e) (i)



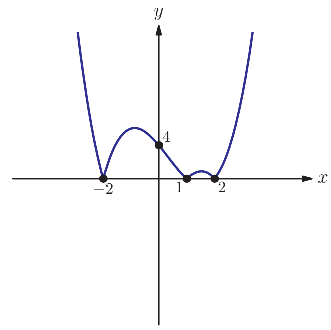
(ii)



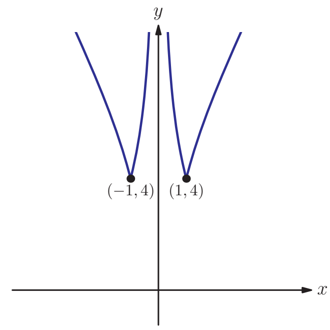
3. (a) (i)



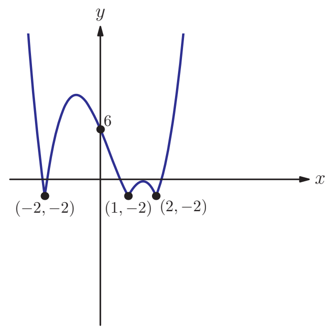
(ii)



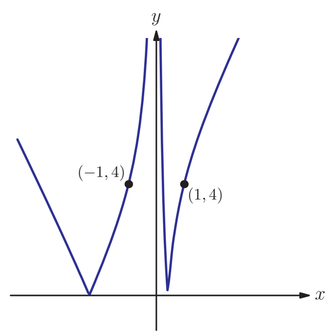
(b) (i)



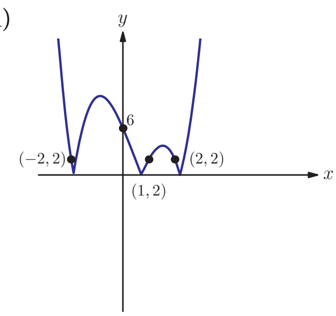
(ii)



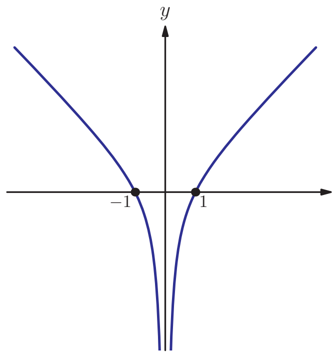
(c) (i)



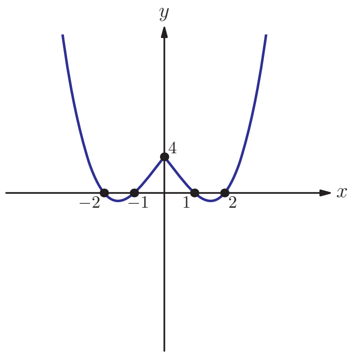
(ii)



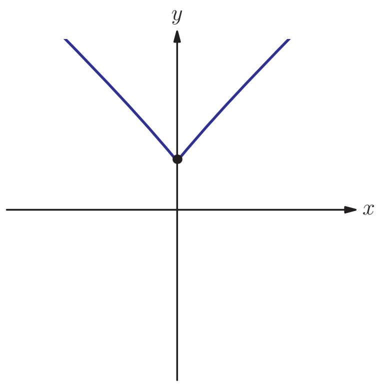
4. (a) (i)



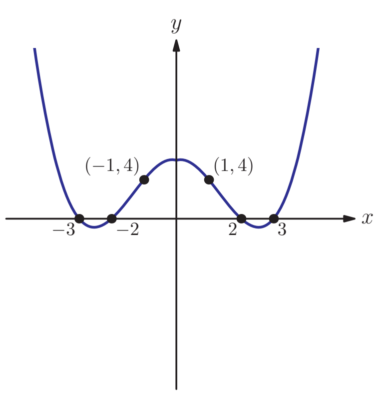
(ii)



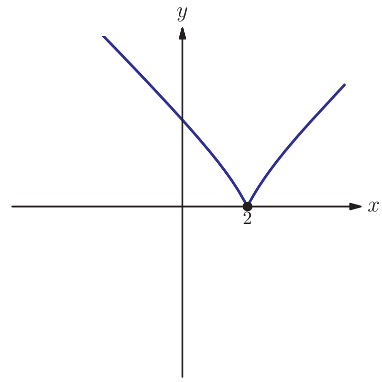
(b) (i)



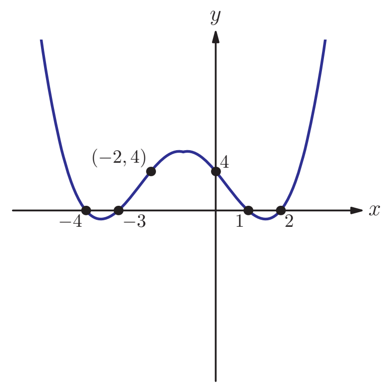
(ii)



(c) (i)



(ii)



5. (a) (i) $k(x)=2f(x)-6$ Stretch sf 2 relative to $y=0$ and translation $\begin{pmatrix}0\\-6\end{pmatrix}$

(ii) $k(x)=5f(x)+4$ Stretch sf 5 relative to $y=0$ and translation $\begin{pmatrix}0\\4\end{pmatrix}$

(b) (i) $h(x)=5-3f(x)$ Stretch sf 3 relative to $y=0$, reflection in $y=0$ and translation $\begin{pmatrix}0\\5\end{pmatrix}$

(ii) $h(x)=4-8f(x)$ Stretch sf 8 relative to $y=0$, reflection in $y=0$ and translation $\begin{pmatrix}0\\4\end{pmatrix}$

6. (a) (i) $g(x)=6x^2-6$

(ii) $g(x)=x^2+1$

(b) (i) $g(x)=x^2+4$

(ii) $g(x)=7x^2-4$

(c) (i) $g(x)=4-2x^2$

(ii) $g(x)=6-2x^2$

(d) (i) $g(x)=5-x^2$

(ii) $g(x)=-3-3x^2$

7. (a) (i) $g(x)=f(x+1)$ translation $\begin{pmatrix}-1\\0\end{pmatrix}$.

(ii) $g(x)=f(x-3)$ Or reflection in $x=0$ then translation $\begin{pmatrix}3\\0\end{pmatrix}$.

(b) (i) $h(x)=f(2x)$ Horizontal stretch sf $\frac{1}{2}$

(ii) $h(x)=f\left(\frac{x}{3}\right)$:Stretch sf 3 relative to $x=0$

(c) (i) $k(x)=f(2x+2)$:Translation $\begin{pmatrix}-2\\0\end{pmatrix}$ then stretch sf $\frac{1}{2}$ relative to $x=0$

(ii) $k(x)=f(3x-1)$: Translation $\begin{pmatrix}1\\0\end{pmatrix}$ then stretch sf $\frac{1}{3}$ relative to $x=0$

8. (a) (i) $g(x)=32x^2-16x-2$

(ii) $g(x)=8x^2+16x+4$

(b) (i) $g(x)=8x^2+64x+124$

(ii) $g(x)=\frac{9x^2}{2}-9x+\frac{1}{2}$

(c) (i) $g(x)=2x^2-12x+14$

(ii) $g(x)=2x^2+12x+14$

9. (a) (i) $g(x)=2f(x+1)-2$: Vertical stretch sf 2 then translation by $\begin{pmatrix}-1\\-2\end{pmatrix}$

(ii) $g(x)=3f(x-4)-40$: Vertical stretch sf 3 then translation by $\begin{pmatrix}4\\-40\end{pmatrix}$

(b) (i) $g(x)=f(x-3)-4$: translation by $\begin{pmatrix}3\\-4\end{pmatrix}$

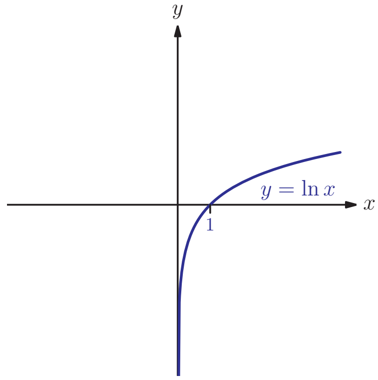
(ii) $g(x)=-4f(x-1)+8$: Vertical stretch sf 4 and reflection in x-axis then translation by $\begin{pmatrix}8\\1\end{pmatrix}$

10. (i) $a=5,\ b=7$

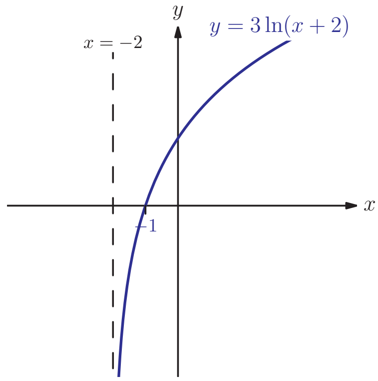
(ii) $a=2,\ b=3,\ c=-5$

11. $h(x)=4^{x+1}+16x-4$

12. (a)



(b)



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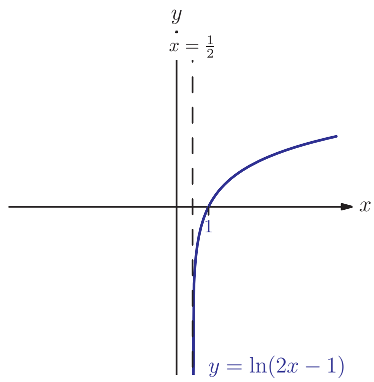
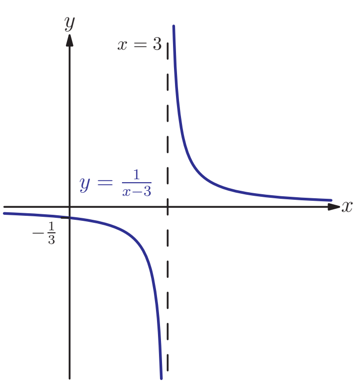
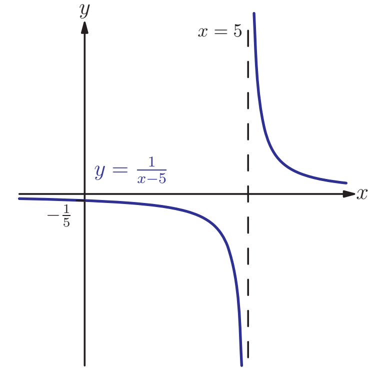


Exercise 6F

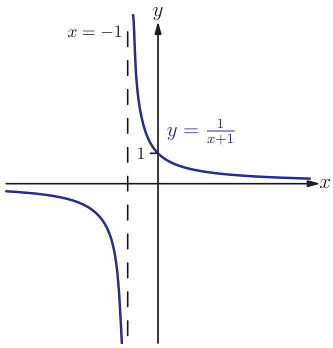
1. (a) (i)



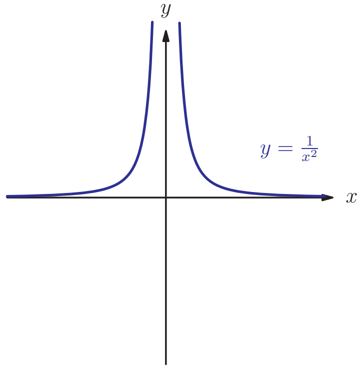
(ii)



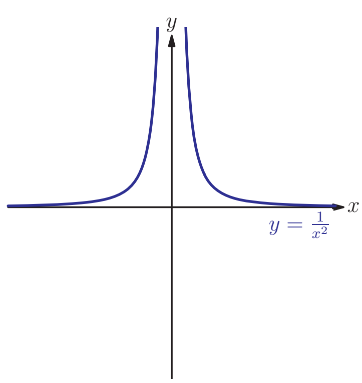
(b) (i)



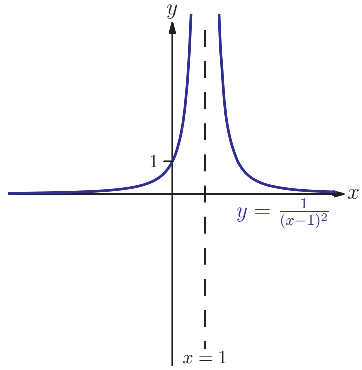
(ii)



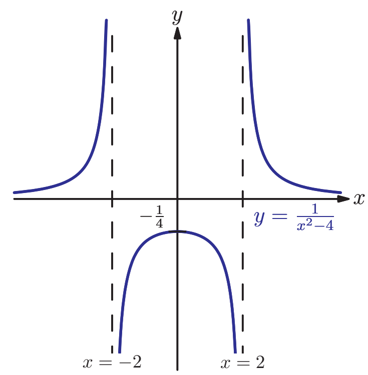
(c) (i)



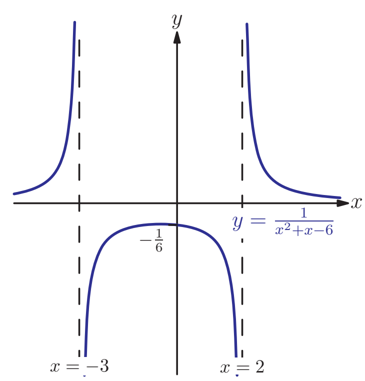
(ii)



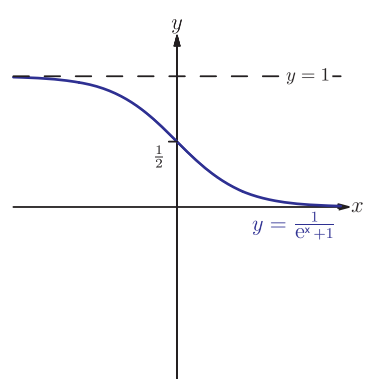
(d) (i)



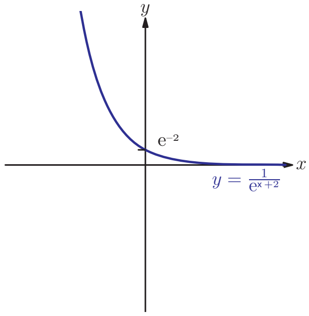
(ii)



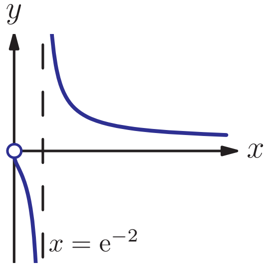
(e) (i)



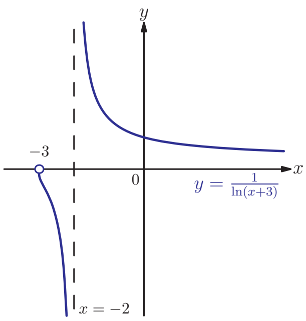
(ii)



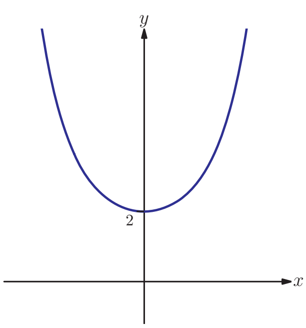
(f) (i)



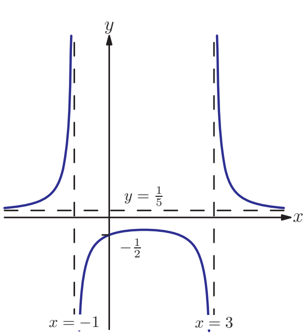
(ii)



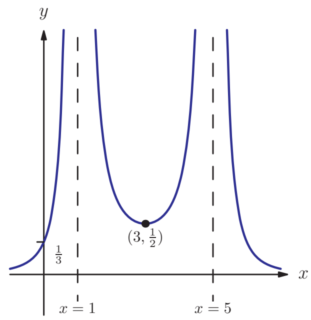
2. (a)



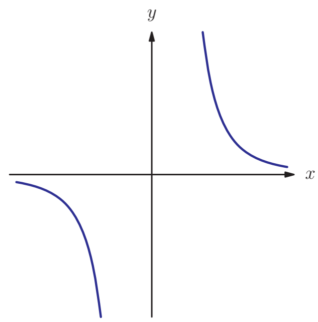
(b)



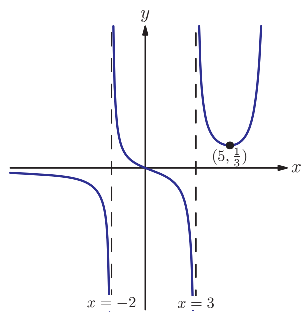
(c)



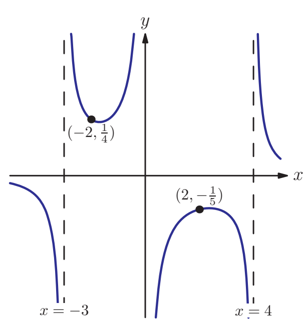
(d)



(e)

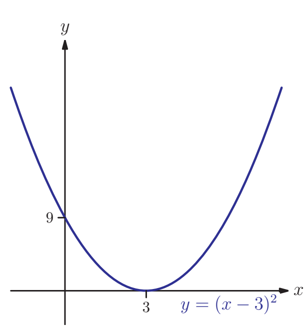


(f)

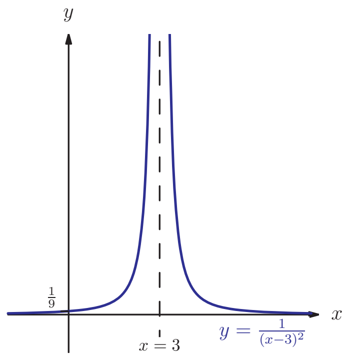


3. (a) $p = 3, q = 0$

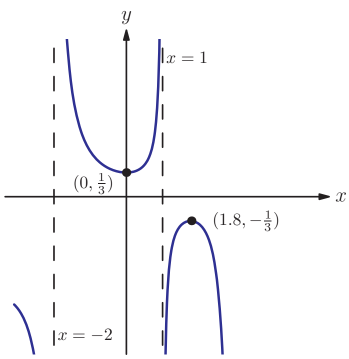
(b) (i)



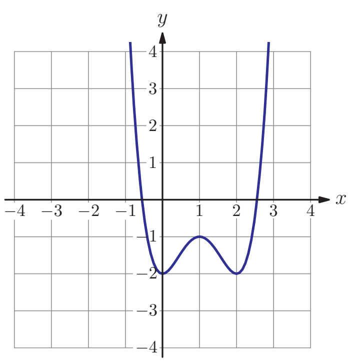
(ii)



4.

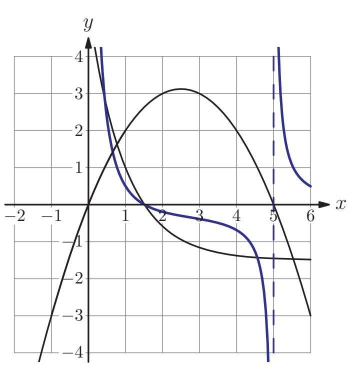


5.



6. (a) $x = 0, x = \pm\sqrt{\ln 4}$

7.

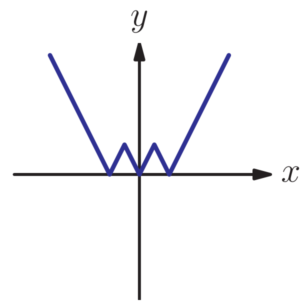


Exercise 6G

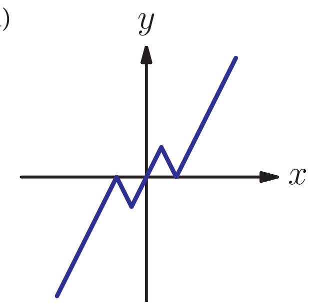
1 (a) (i) even (ii) even
(b) (i) odd (ii) odd
(c) (i) neither (ii) neither
(d) (i) odd (ii) even
(e) (i) neither (ii) neither

2. (a) Two-fold rotational symmetry about the origin

(b) (i)



(ii)



9. (a) $x = -3$

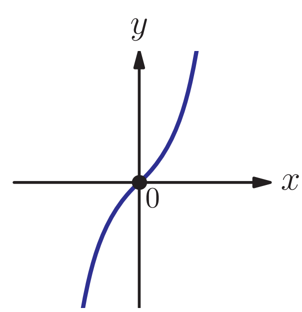
(b) $a = 3$

10. Reflective symmetry through $x = 5$

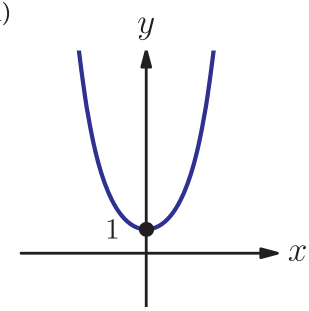
11. $f \circ f(4) = 4$: f is self-inverse if reflective in $y = x$

12. (b) $\frac{1}{2}\left(f(x) + f(-x)\right)$

(d) (i)



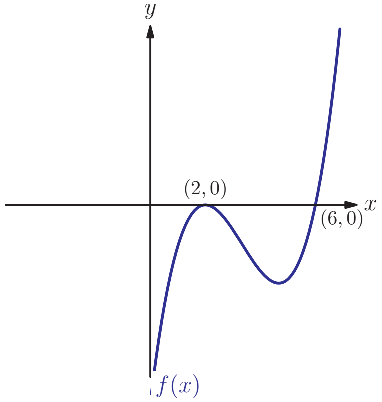
(ii)



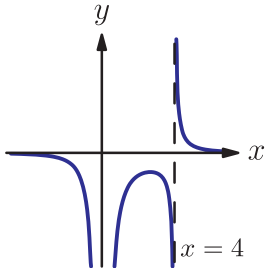
Mixed examination practice 6

Short questions

1. (a)



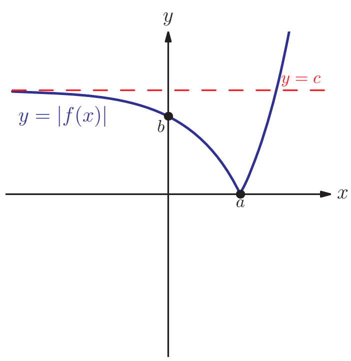
(b)



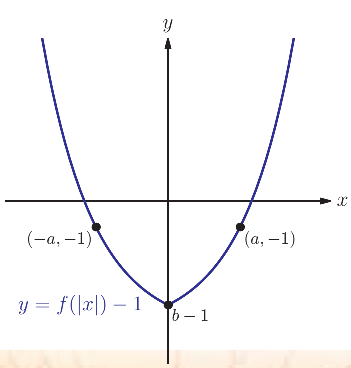
2. $y = 2x^2 - 12x^2 + 24x - 18$

3. $\frac{1}{3} < x < 1$

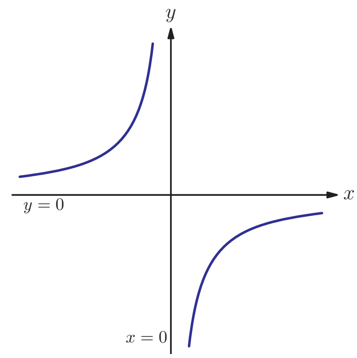
4. (a)



(b)



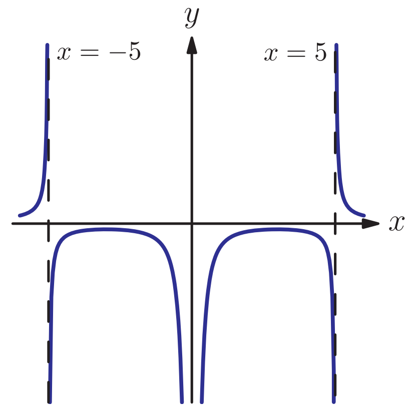
5. (a)



(b) Vertical stretch with scale factor 3 and reflection in the x-axis (or y-axis)

(c) $f^{-1}(x) = -\frac{3}{x}$

6. (a)

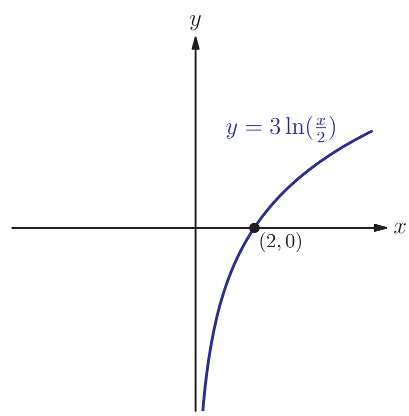


(b) $\left(-3, -\frac{1}{5}\right), \left(3, -\frac{1}{5}\right)$

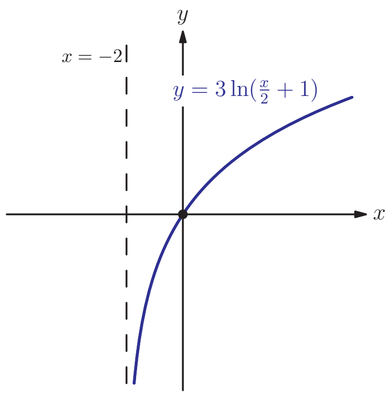
7. Translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and vertical stretch with scale factor (sf)3.

8. (a) Horizontal stretch with sf 2; vertical stretch with sf 3

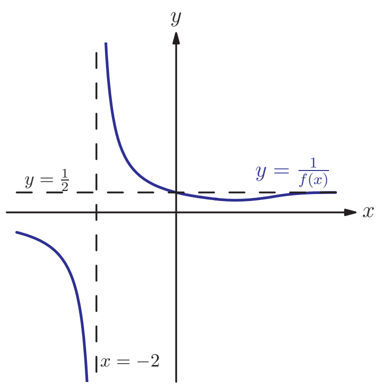
(b)



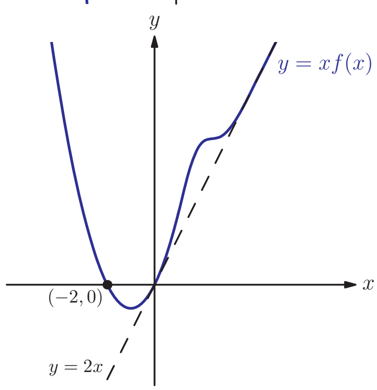
(c)



9. (a)



(b)



10. $x \geq 0$

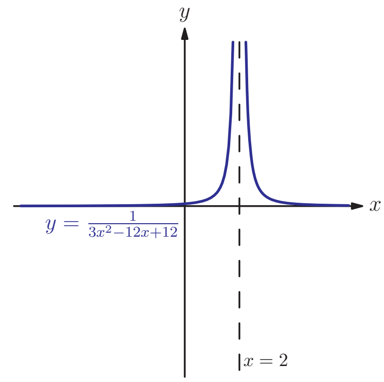
Long questions

1. (a) Translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and vertical stretch with sf 3.

(b) Translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and translation by $\begin{pmatrix} 0 \\ 10 \end{pmatrix}$

(c) Translation by $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ and vertical stretch with scale factor 3.

(d)



2. (a) $y=3$

(b) $p=3, q=1$

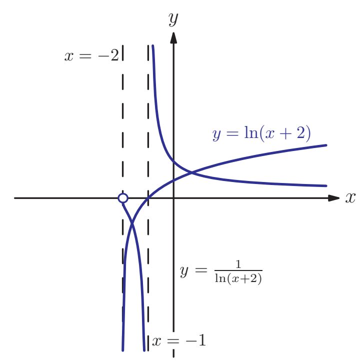
(c) Translation with vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(d) $f^{-1}(x)=\dfrac{2x-5}{x-3}, x \neq 3$

(e) Reflection in the line $y=x$

3. (a) Translation by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

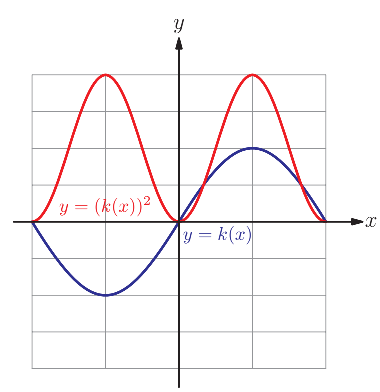
(b)



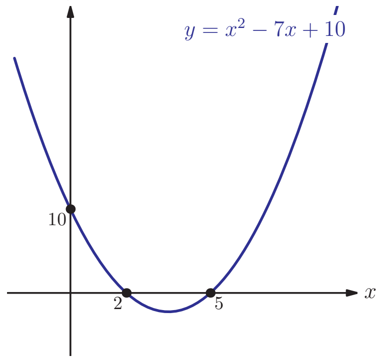
(c) (i) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(ii) $a=-1, b=6, c=-10,$
$d=-1$

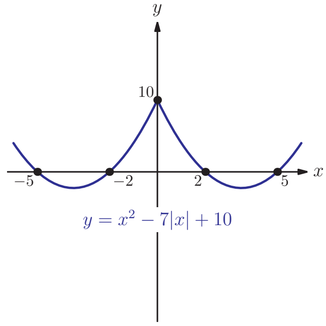
(d)



4. (a)



(c)

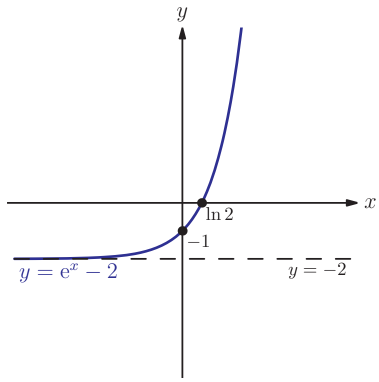


(d) $x = \pm\frac{10}{7}$ (e) $x = \pm 3, \pm 4$

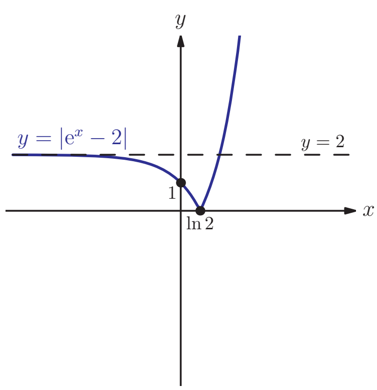
5. (a) -18 (b) 6

(c) $p = 3, q = 17$ (d) $x \in \mathbb{R}$

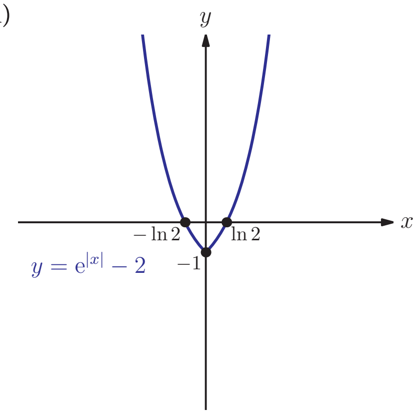
6. (a)



(b) (i)



(ii)



(c) $x = \ln\left(2 - \sqrt{3}\right), x \geq \ln 2$

Chapter 7

Exercise 7A

1. (a) (i) 3.1, 8.1, 13.1, 18.1, 23.1

(ii) 10, 6.2, 2.4, −1.4, −5.2

(b) (i) 0, 1, 4, 13, 40

(ii) 1, −1, −19, −181, −1639

(c) (i) 2, 3, 6, 18, 108

(ii) $2, 1, \frac{1}{2}, \frac{1}{2}, 1$

(d) (i) 3, 4, 8, 9, 13

(ii) −3, 3, −5, 7, −9

(e) (i) 0, 4, 8, 12, 16

(ii) 13, 11, 9, 7, 5

2. (a) (i) 5, 8, 11, 14, 17

(ii) −4.5, −3, −1.5, 0, 1.5

(b) (i) 0, 7, 26, 63, 124

(ii) 5, 20, 45, 80, 125

(c) (i) 3, 9, 27, 81, 243

(ii) $4, 2, 1, \frac{1}{2}, \frac{1}{4}$

(d) (i) 1, 4, 27, 256, 3125

(ii) 1, 0, −1, 0, 1

3. (a) (i) $u_{n+1} = u_n + 3,\ u_1 = 7$

(ii) $u_{n+1} = u_n - 0.8,\ u_1 = 1$

(b) (i) $u_{n+1} = 2u_n,\ u_1 = 3$

(ii) $u_{n+1} = 1.5u_n,\ u_1 = 12$

(c) (i) $u_{n+1} = u_n + n + 1,\ u_1 = 1$

(ii) $u_{n+2} = 2(u_{n+1} + u_n)$,
$u_1 = 1, u_2 = 2$

4. (a) (i) $u_n = 2n$

(ii) $u_n = 2n - 1$

(b) (i) $u_n = 2^n$ (ii) $u_n = 5^n$

(c) (i) $u_n = n^2$ (ii) $u_n = n^3$

(d) (i) $u_n = \frac{n}{n+1}$

(ii) $u_n = \frac{2n-1}{2^n}$

5. (a) $u_2 = 4,\ u_3 = 8,\ u_4 = 16$

(b) (i) $u_n = 2^{n-1}$

Exercise 7B

1. (a) (i) 27 (ii) 39

(b) (i) 120 (ii) $\frac{665}{48}$

(c) (i) $14b$ (ii) $19p$

2. (a) (i) $\sum_{2}^{43} r$ (ii) $\sum_{3}^{30} 2r$

(b) (i) $\sum_{2}^{7} \frac{1}{2^r}$ (ii) $\sum_{0}^{5} \frac{2}{3^r}$

(c) (i) $\sum_{r=2}^{10} 7ra$ (ii) $\sum_{r=0}^{19} r^b$

Exercise 7C

1. (a) (i) $u_n = 9 + 3(n-1)$

(ii) $u_n = 57 + 0.2(n-1)$

(b) (i) $u_n = 12 - (n-1)$

(ii) $u_n = 18 - \frac{1}{2}(n-1)$

(c) (i) $u_n = 1 + 3(n-1)$

(ii) $u_n = 9 + 10(n-1)$

(d) (i) $u_n = 4 - 4(n-1)$

(ii) $u_n = 27 - 7(n-1)$

(e) (i) $u_n = -17 + 11(n-1)$

(ii) $u_n = -32 + 10(n-1)$

2. (a) (i) 33 (ii) 29
(b) (i) 100 (ii) 226

3. (a) $a_n = 5 + 8(n-1)$ (b) 50

4. 121

5. 25

6. 17

7. $a = 2,\ b = -3$

8. (b) 456 pages

Exercise 7D

1. (a) (i) 3060 (ii) 1495
(b) (i) 9009 (ii) 23 798
(c) (i) −204 (ii) 1470
(d) (i) 667.5 (ii) 14.25

2. (a) (i) 13 (ii) 32 (iii) 53

(b) $\frac{x}{2}$

3. $a = 15,\ d = -8$

4. (a) $S_n = \frac{n}{2}(3n+1)$ (b) 30

5. 559

6. $a = 14,\ d = -8$

7. 55

8. $u_n = 6n - 5$

9. $\theta = 20°$

10. 10 300

11. 23 926

Exercise 7E

1. (a) (i) $u_n = 6 \times 2^{n-1}$

(ii) $u_n = 12 \times \left(\frac{3}{2}\right)^{n-1}$

(b) (i) $u_n = 20 \times \left(\frac{1}{4}\right)^{n-1}$

(ii) $u_n = \left(\frac{1}{2}\right)^{n-1}$

(c) (i) $u_n = (-2)^{n-1}$

(ii) $u_n = 5 \times (-1)^{n-1}$

(d) (i) $u_n = ax^{n-1}$

(ii) $u_n = 3\times(2x)^{n-1}$

2. (a) (i) 13 (ii) 7
(b) (i) 10 (ii) 10
(c) (i) 10 (ii) 8

3. (a) (i) 15 (ii) 31
(b) (i) 33 (ii) 17

4. 39 366

5. 10

6. 16

7. 2.5 or −1.82

8. ±384

9. $a = 7$ or -3.52

10. $a = -2,\ b = 4$

11. $m = 7$

Exercise 7F

1. (a) (i) 17 089 842 (ii) 2303.4375
(b) (i) 514.75 (ii) 9.487 171
(c) (i) 39 368 (ii) 9840
(d) (i) 191.953 125 or 63.984 375
(ii) 24 414 062.5 or 16 276 041.67

2. (a) (i) $r = 3$ (ii) $r = 0.2$
(b) (i) $r = -6$ (ii) $r = -0.947$

3. (a) 5 (b) $S_n = \dfrac{375(5^n - 1)}{4}$

4. $a = 5,\ r = \frac{3}{2}$

5. (a) $1 + x + x^2 + x^3$

(b) $(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$

Exercise 7G

1. (a) (i) $\dfrac{27}{2}$ (ii) $\dfrac{196}{3}$

(b) (i) $\dfrac{1}{3}$ (ii) $\dfrac{26}{33}$

(c) (i) Divergent
(ii) Divergent

(d) (i) $\dfrac{25}{3}$ (ii) $\dfrac{18}{5}$

(e) (i) Divergent (ii) $\dfrac{7}{3}$

2. (a) (i) $|x| < 1$ (ii) $|x| < 1$

(b) (i) $|x| < \dfrac{1}{3}$ (ii) $|x| < \dfrac{1}{10}$

(c) (i) $|x| < \dfrac{1}{5}$ (ii) $|x| < \dfrac{1}{3}$

(d) (i) $|x| < 4$ (ii) $|x| < 12$

(e) (i) $|x| < 3$ (ii) $|x| < \dfrac{4}{5}$

(f) (i) $|x| > 2$ (ii) $|x| > \dfrac{1}{2}$

(g) (i) $1 < x < 2$ (ii) $0 < x < 4$

(h) (i) $\dfrac{1}{2} < x < 1$ (ii) $x < -\dfrac{1}{2}$

(i) (i) $|x| < 1$ (ii) $|x| < \dfrac{1}{\sqrt[3]{4}}$

3. $-\dfrac{54}{3}$

4. (a) $27 = \left(\dfrac{1-\left(\dfrac{-1}{3}\right)^n}{2}\right)$

(b) $S_\infty = \dfrac{27}{2}$

5. (a) $\dfrac{2}{3}$ (b) 9

6. $\dfrac{1}{8}$

7. (a) $|x| < \dfrac{3}{2}$ (b) 5

8. 9

9. (a) $1 < x < \dfrac{5}{3}$ (b) 7

10. (a) $x < 0$ (b) $x = -3$

11. (a) 3 (b) ∞

Exercise 7H

1. (a) £34.78

(b) £1194.05

2. (a) \$60 500

(b) 22 years

3. (a) 5000×1.063^n

(b) \$6786.35

(c) (i) $5000 \times 1.063^n > 10000$

(ii) 12 years

4. (a) 10 (b) 23.7%

5. (a) \$265.33

(b) 235 months

6. (a) 12 days

(b) Day 102

7. (a) 0.8192 m

(b) 15.85 m

8. (b) $25000(1.04^n - 1)$

(c) Year 29

Mixed examination practice 7

Short questions

1. 97.2
2. (a) 1, 5, 9

(b) $4n - 3$

3. 13th
4. 2
5. $d = 0, -\frac{1}{4}$
6. 4.5
7. 19 264
8. $\ln\left(\frac{a^{69}}{b^{138}}\right)$

Long questions

1. (a) $10000 + 800n$

(b) 10000×1.05^n

(c) $n < 19$ years

2. (a) $2n - 1$ (b) 6

(c) 64

3. (a) n (b) $\frac{n(n+1)}{2}$

(c) $\frac{n(n-1)}{2} + 1$

(e) 32

4. (b) $150000 \times 1.06^n - \frac{500000(1.06^n - 1)}{3}$

(c) 40 years

Chapter 8

Exercise 8A

1. (a) 4
(b) 35
(c) 7
(d) 56

2. (a) $792x^5y^7$ (b) $11440a^7b^9$

(c) $10c^3d^2$ (d) $36a^2b^7$

(e) $15x^2y^4$

Exercise 8B

1. (a) (i) 216 (ii) 20

(b) (i) $560x^3y^4$ (ii) $-280x^3y^4$

(c) (i) -5 (ii) 78 030

2. (a) (i) 56 (ii) 80

(b) (i) -672 (ii) -32

3. (a) (i) $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

(ii) $729 + 1458x + 1215x^2 + 540x^3 + 135x^4 + 18x^5 + x^6$

(b) (i) $243x^5 + 405x^4y + 270x^3y^2$

(ii) $16c^4 - 32c^3d + 24c^2d^2$

(c) (i) $8x^6 - 36x^5 + 54x^4 - 27x^3$

(ii) $8x^{-3} + 60x^{-2}y + 150x^{-1}y^2 + 125y^3$

(d) (i) $16z^8 + 96z^5 + 216z^2 + 216z^{-1} + 81z^{-4}$

(ii) $27x^3y^3 + 135x^3y + 225x^3y^{-1} + 125x^3y^{-3}$

4. (a) n (b) $\frac{1}{2}n^2 - \frac{1}{2}n$

(c) $\frac{1}{6}n^3 - 6n^2 + \frac{1}{3}n$

5. (a) $80x\,y^4$ (b) $-80x^2y^3$
6. 720
7. $-945x^5$
8. 79 200 000
9. 14
10. 12
11. 9
12. 7

Exercise 8C

1. (a) (i) -4 (ii) 126
   (b) (i) -5 (ii) -28
2. (i) 15 (ii) 40
3. (i) 5733 (ii) -272
4. (a) (i) $3x^7 - 17x^8 + 16x^7$
   (ii) $-x^7 + 16x^6 - 105x^5$
   (b) (i) $1 + x - 4x^2$
   (ii) $128 + 64x - 96x^2$

ANSWER HINT (5,6,7)

In questions 5,6 and 7 there are algebraic tricks that make the expansions much easier.

5. $y^6 + 18y^7 + 135y^8 + 540y^9$
6. $1 - 10x^2 + 45x^4$
7. $1 - 20x + 190x^2 - 1140x^3$
8. $m = 3, n = 15$ and $m = -5, n = -17$
9. $n = 5, k = 2$ and $n = 17, k = -1$

Exercise 8D

1. (a) $1 + 35x + 525x^2 + 4375x^3$; 1.407
   (b) $64 + 576x + 2160x^2$; 64.5782
2. (a) $81 - 540x + 1350x^2$
   (b) 80.4614
3. (a) $128 + 2240x + 16800x^2$
   (b) 130.257
4. (a) $128 + 1344x + 6048x^2$
   (b) (i) 322.28 (ii) 142.0448
   (c) part (ii) Smaller value of x means higher order terms much smaller and therefore less important.

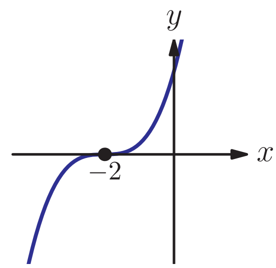
Mixed examination practice 8

Short questions

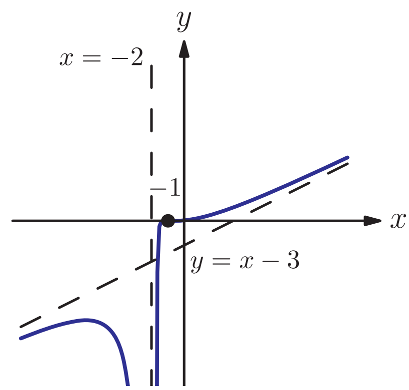
1. -101376
2. $232 - 164\sqrt{2}$
3. (a) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
   (b) 32.808 040 1001
4. $243 + 162x - 2484x^2$
5. $x^8 - 8x^5 + 24x^2 - 32x^{-1} + 16x^{-4}$
6. 3 or -3
7. $m = -8, n = -34$ or $m = 5, n = 31$

Long questions

1. (a)



   (b) $x^3 + 6x^2 + 12x + 8$
   (c) 8.012 006 001
   (d) $x = -4$
2. (a) $x = -2$, $\left(0, \frac{1}{16}\right)$, $(-1,0)$
   (b) $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$,
   $g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16$
   (c) (i) $k = 3, a = 10$ (ii) $y = x - 3$
   (d)

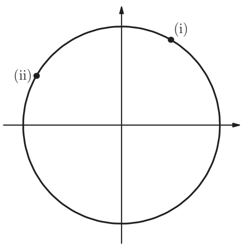


3. (a) $7 + 5\sqrt{2}$
   (b) $\binom{n}{k}\left(\sqrt{2}\right)^k$ (d) 24
4. (c) $\dfrac{r+2}{n-r-1}$

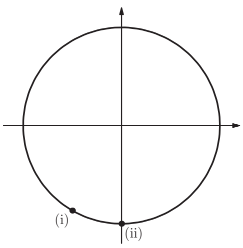
Chapter 9

Exercise 9A

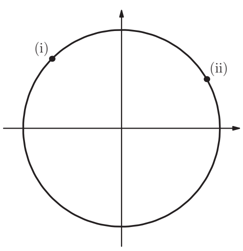
1. (a)



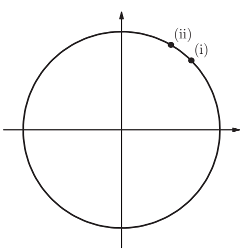
(b)



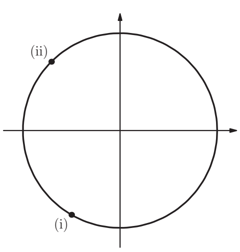
(c)



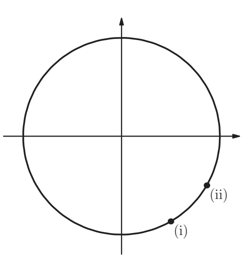
2. (a)



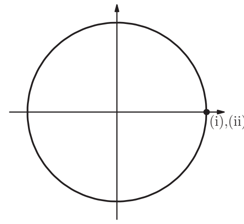
(b)



(c)



(d)



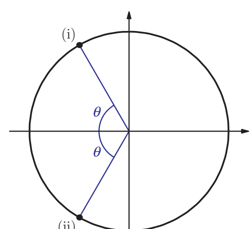
3. (a) (i) $\frac{3\pi}{4}$ (ii) $\frac{\pi}{4}$
(b) (i) $\frac{\pi}{2}$ (ii) $\frac{3\pi}{2}$
(c) (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{6}$
(d) (i) $\frac{5\pi}{18}$ (ii) $\frac{4\pi}{9}$

4. (a) (i) 5.585 (ii) 0.349
(b) (i) 4.712 (ii) 1.571
(c) (i) 1.134 (ii) 2.531
(d) (i) 1.745 (ii) 1.449

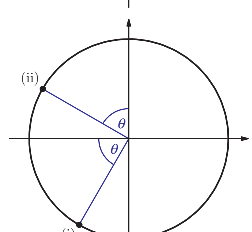
5. (a) (i) 60° (ii) 45°
(b) (i) 150° (ii) 120°
(c) (i) 270° (ii) 300°
(d) (i) 69.9° (ii) 265°

6. (a) (i) 0.733 (ii) 2.932
(b) (i) $\frac{9\pi}{8}$ (ii) $\frac{3\pi}{8}$
(c) (i) 75° (ii) 99°
(d) (i) 92.8° (ii) 156.4°

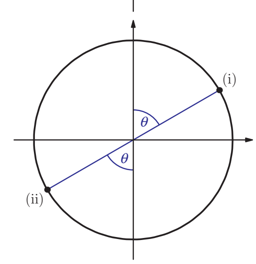
7. (a)



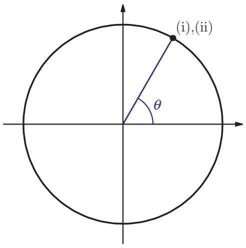
(b)



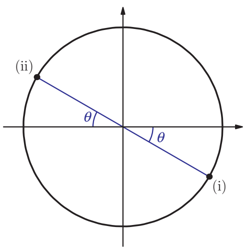
(c)



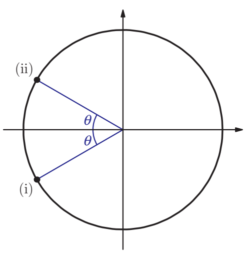
(d)



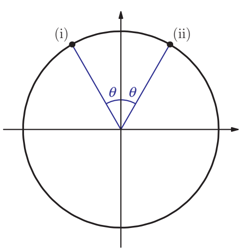
8. (a)



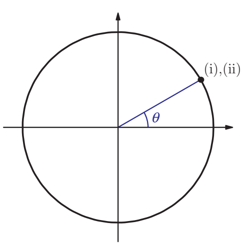
(b)



(c)



(d)



Exercise 9B

1. (a) $\sin x \approx 0.85 \quad \cos x \approx 0.5$
(b) $\sin x \approx -1 \quad \cos x \approx 0$
(c) $\sin x \approx 0.35 \quad \cos x \approx -0.95$

2. (a) (i) 1 (ii) 0
(b) (i) 1 (ii) −1
(c) (i) −1 (ii) 0

3. (a) (i) 0 (ii) −1
(b) (i) −1 (ii) 1
(c) (i) 0 (ii) 0

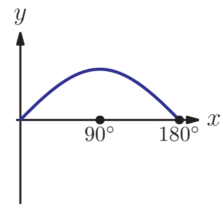
4. (a) −0.809 (b) 0.809
(c) 0.809 (d) −0.809

5. (a) −0.866 (b) −0.866
(c) −0.866 (d) 0.866

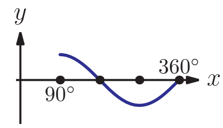
6. (a) 0.766 (b) 0.766
(c) −0.766 (d) −0.766

7. (a) 0.766 (b) 0.766
(c) −0.766 (d) −0.766

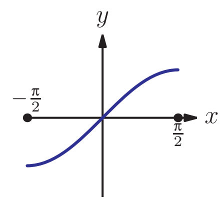
8. (a) (i)



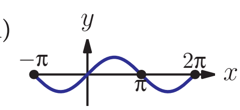
(ii)



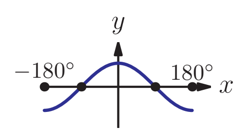
(b) (i)



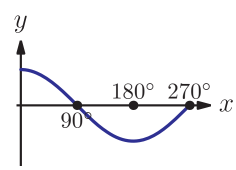
(ii)



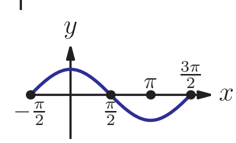
9. (a) (i)



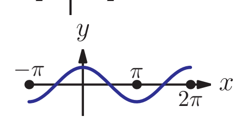
(ii)



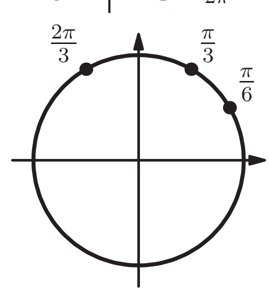
(b) (i)



(ii)



10. (a)



(b) (i) 0.5 (ii) −0.5

11. (a) (i) 0.315 (ii) 0.629
(b) (i) 0.752 (ii) −0.711

12. (a) (i) 0.669 (ii) −0.978
(b) (i) −0.766 (ii) −0.682

13. $-2\cos x$

14. $\sin x$

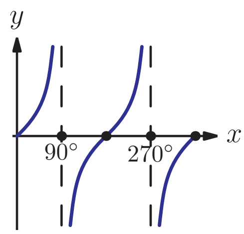
Exercise 9C

1. (a) $\tan x \approx \dfrac{0.5}{0.85} = 0.59$

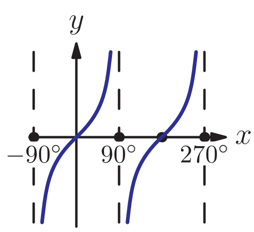
(b) $\tan x \approx \dfrac{-0.5}{-0.85} = 0.59$

(c) $\tan x \approx \dfrac{-0.5}{0.85} = -0.59$

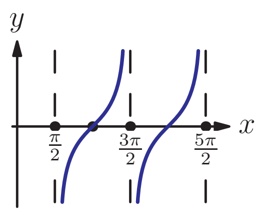
2. (a) (i)



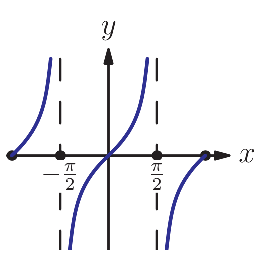
(ii)



(b) (i)



(ii)



3. (a) (i) 2.57 (ii) 80.70
(b) (i) −0.76 (ii) −1.62

4. (a) (i) 0.625 (ii) −0.213
(b) (i) 0 (ii) 1.28

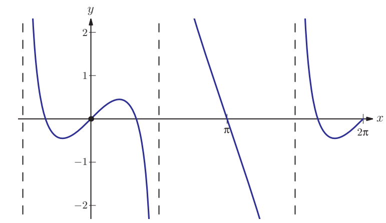
5. (a) $-\tan x$ (b) $-\dfrac{1}{\tan x}$

(c) $\tan x$ (d) $\tan x$

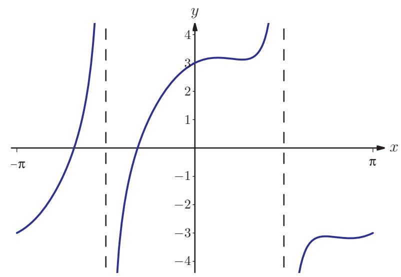
6. (a) $-\tan\theta°$ (b) $-\tan\theta°$

(c) $\dfrac{1}{\tan\theta°}$ (d) $\tan\theta°$

7. (a)



(b)



8. (a) 0,180,360 (b) 57.9,122

9. (a) $(0.805, 1.12), (5.48, -1.12)$
(b) $(-1.11, 2.24), (2.03, -2.24)$

10. (a) 1.87,5.07 (b) 0,1.57,6.28

Exercise 9D

1. (a) $-\dfrac{\sqrt{2}}{2}$ (b) 0

(c) $-\dfrac{\sqrt{2}}{2}$ (d) −1

2. (a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$

(c) $-\dfrac{1}{2}$ (d) $-\sqrt{3}$

3. (a) $\dfrac{\sqrt{2}}{2}$ (b) $\dfrac{\sqrt{2}}{2}$

(c) $-\dfrac{\sqrt{2}}{2}$ (d) 1

4. (a) $-\dfrac{1}{2}$ (b) $-\dfrac{\sqrt{3}}{2}$

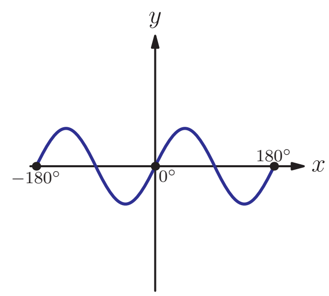
(c) $\dfrac{\sqrt{3}}{3}$ (d) $\dfrac{-\sqrt{3}}{3}$

5. (a) $\dfrac{3}{4}$ (b) $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

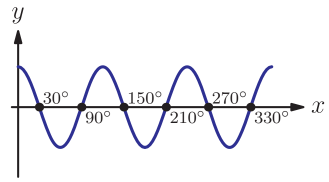
(c) $\dfrac{1-\sqrt{3}}{2}$

Exercise 9E

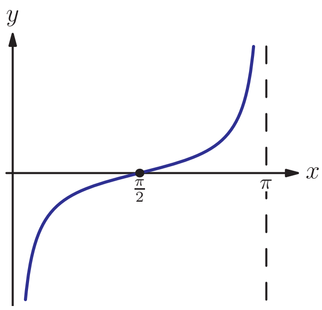
1. (a) (i)



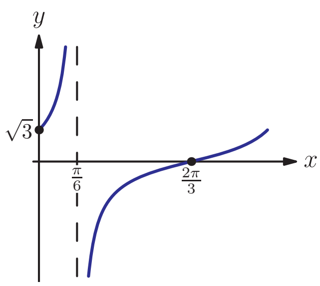
(ii)



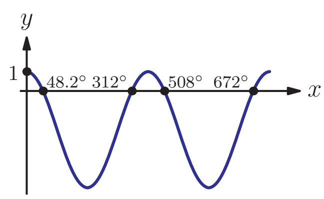
(b) (i)



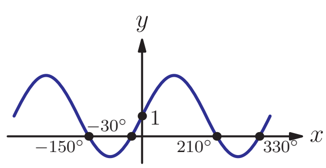
(ii)



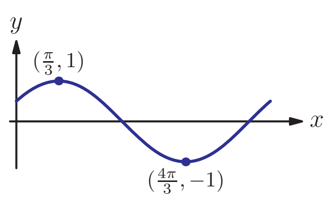
(c) (i)



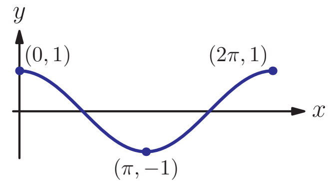
(ii)



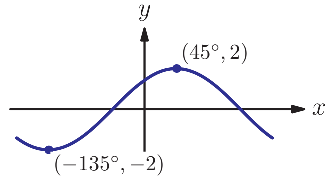
2. (a) (i)



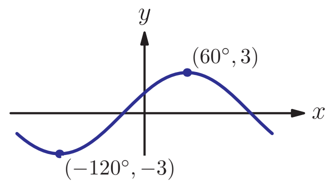
(ii)



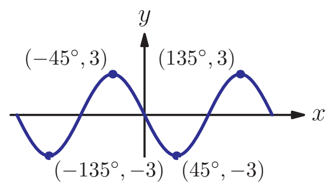
(b) (i)



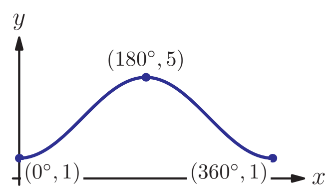
(ii)



(c) (i)



(ii)



3. (a) Amp: 3 Period: $\frac{\pi}{2}$

(b) Amp: ∞ Period: $\frac{\pi}{3}$

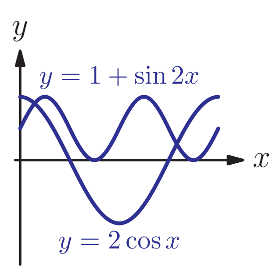
(c) Amp: 1 Period: $\frac{2\pi}{3}$

(d) Amp: 2 Period: 2

4. $p = 5, q = 2$

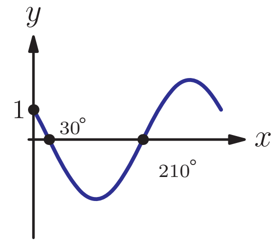
5. $a = 2, b = 20°$

6. (a)



(b) 2 (c) 8

7. (a)



(b) $(120°,-2),(300°,2)$

(c) $(120°,-3),(300°,1)$

Exercise 9F

1. $a=1.5$ $b=\dfrac{\pi}{6}$ $m=4.5$
2. (a) 9 m, 23 m (b) 1:42 am to 10:18 am
3. (a) $a=5, k=\dfrac{\pi}{5}$ (b) 6.02s, 8.98s
4. (a) 110, 130 cm (b) $\dfrac{\pi}{200}$s (c) $\dfrac{\pi}{400}$s

Exercise 9G

1. (a) (i) 0.927 (ii) 0.201
   (b) (i) −1.25 (ii) −0.927
2. (a) (i) $\dfrac{\pi}{6}$ (ii) $\dfrac{\pi}{6}$
   (b) (i) $-\dfrac{\pi}{3}$ (ii) $\dfrac{3\pi}{4}$
   (c) (i) $-\dfrac{\pi}{2}$ (ii) $\dfrac{\pi}{4}$
3. (a) (i) 44.4° (ii) 17.5°
   (b) (i) 128.3° (ii) 138.6°
   (c) (i) 81.1° (ii) −82.0°
4. (a) (i) 0.6 (ii) −0.3
   (b) (i) -2 (ii) −1
5. (a) (i) $\dfrac{\sqrt{3}}{2}$ (ii) $\dfrac{\sqrt{2}}{2}$
   (b) (i) $\dfrac{1}{2}$ (ii) $\sqrt{3}$
6. (a) (i) $\dfrac{\pi}{3}$ (ii) $\dfrac{5\pi}{6}$
   (b) (i) $\dfrac{\pi}{3}$ (ii) π
7. (a) 0.866 (b) −0.433
   (c) 0.141
8. (b) $\arcsin x=\dfrac{\pi}{2}-\arccos x$ (c) $x=1$
9. (a) $y=\arccos(0.6-\sin x)$

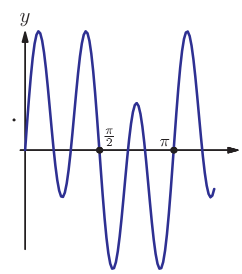
   $y=\arcsin(\cos x-0.2)$

   (b) $x=0,\ y=0.927$

Mixed examination practice 9

Short questions

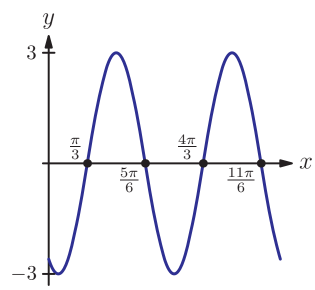
1. (a) 1.4 m (b) 2.09 m
2. period = π



3. (a) 78.5 s (b) 377 m (c) 4.8 ms^{-1}
4. (a) π

   (b) $\left(\dfrac{\pi}{3},0\right),\left(\dfrac{5\pi}{6},0\right),\left(\dfrac{4\pi}{3},0\right),\left(\dfrac{11\pi}{6},0\right)$

   (c)



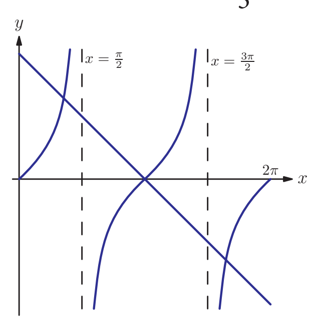
5. $a=5, b=\dfrac{\pi}{4}$

Long questions

1. (a) (i) $\dfrac{2\pi}{3},\dfrac{3}{2}$ (ii) $k=\dfrac{\pi}{6},\ c=\dfrac{1}{2}$

   (b) $0,-\dfrac{2\pi}{3},\ -2\pi,\ -\dfrac{8\pi}{3},\ -4\pi$

   (c) (i) 8 (ii) $\dfrac{10\pi}{3}-\alpha,\ \alpha+2\pi$
2. (a)



   (b) (i) $\pi,\ 2\pi-x_0$ (ii) Infinitely many

   (c) (i) s, c (iii) $\sqrt{3},\ \dfrac{1}{\sqrt{3}}$ (iv) $\dfrac{\pi}{6},\dfrac{\pi}{3}$

   (d) 1.2 (e) (ii) $x_1=2\pi-x_0$
3. (a) −1; π

   (b) (i) Translation $\begin{pmatrix}-\dfrac{\pi}{6}\\0\end{pmatrix}$ and verticle stretch with scale factor 2

(ii) $-2; \frac{5\pi}{6}$

(c) (i) No; $\cos(A) \geq -1$ so $2\cos(A) + 3 \geq 1$, never 0
(ii) $[1, 5]$

Chapter 10

Exercise 10A

1. (a) (i) $30°, 150°$ (ii) $45°, 135°$
(b) (i) $60°, 300°$ (ii) $30°, 330°$
(c) (i) $240°, 300°$ (ii) $210°, 330°$
(d) (i) $45°, 225°$ (ii) $60°, 240°$

2. (a) (i) $\frac{\pi}{6}, \frac{11\pi}{6}$ (ii) $\frac{\pi}{4}, \frac{7\pi}{4}$
(b) (i) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (ii) $\frac{5\pi}{6}, \frac{7\pi}{6}$
(c) (i) $\frac{\pi}{4}, \frac{3\pi}{4}$ (ii) $\frac{\pi}{3}, \frac{2\pi}{3}$
(d) (i) $\frac{\pi}{6}, \frac{7\pi}{6}$ (ii) $\frac{3\pi}{4}, \frac{7\pi}{4}$

3. (a) (i) $26.7°, 153.3°$ (ii) $44.4°, 135.6°$
(b) (i) $138.6°, -138.6°$ (ii) $101.5°, -101.5°$
(c) (i) $18.4°, 198.4°, 378.4°, 558.4°$
(ii) $53.1°, 233.1°, 413.1°, 593.1°$
(d) (i) $-138.2°, -41.8°, 221.8°, 318.2°$
(ii) $-165.5°, -14.5°, 194.5°, 345.5°$

4. (a) (i) $0.644, 5.64, 6.93, 11.9$
(ii) $0.841, 5.44, 7.12, 11.7$
(b) (i) $-2.21, -0.927, 4.07, 5.36$
(ii) $-2.78, -0.36, 3.50, 5.93$
(c) (i) $-0.588, 2.55$ (ii) $-1.25, 1.89$
(d) (i) $0, 6.28, 12.6$
(ii) $1.57, 4.71, 7.85, 11.0$

5. (a) (i) $30°, 150°$ (ii) $45°, 135°$
(b) (i) $\pm 180°$ (ii) $-90°$
(c) (i) $60°, 240°$ (ii) $45°, 225°$
(d) (i) $\pm 225°, \pm 135°$ (ii) $\pm 210°, \pm 150°$

6. (a) (i) $\pm\frac{5\pi}{3}, \pm\frac{\pi}{3}$ (ii) $\pm\frac{11\pi}{6}, \pm\frac{\pi}{6}$
(b) (i) $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, (ii) $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
(c) (i) $-\frac{\pi}{6}, \frac{5\pi}{6}$ (ii) $-\frac{\pi}{4}, \frac{3\pi}{4}$
(d) (i) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ (ii) $\pi, 2\pi$
(e) $-\frac{5\pi}{4}, -\frac{7\pi}{4}$

7. (a) (i) $5.74°, 174°$ (ii) $-14.5°, 195°$
(b) (i) $1.11, 5.17$ (ii) $1.00, 5.28$
(c) (i) $1.03, -2.11$ (ii) $1.14, 4.29$

8. $-\frac{\pi}{6}, -\frac{5\pi}{6}$

Exercise 10B

1. (a) (i) $\pm 2.19, \pm 0.955$, (ii) $\pm 2.26, \pm 0.886$
(b) (i) $48.2°, 132°, 228°, 312°$
(ii) $52.2°, 128°, 232°, 308°$

2. (a) (i) $0°, 180°, 360°$ (ii) $\frac{\pi}{2}, \frac{3\pi}{2}$
(b) (i) $0, \pm\pi, 0.848, 2.29$ (ii) $\pm\frac{\pi}{2}, \pm 1.91$
(c) (i) $0.944, 1.30, 4.09, 4.44$
(ii) $\frac{3\pi}{4}, \frac{7\pi}{4}, 0.464, 3.61$
(d) (i) $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ (ii) $\frac{\pi}{2}, \frac{3\pi}{2}, 0.983, 4.12$
(e) (i) $60°$ (ii) No solutions

3. (a) (i) $35.3°, 145°, 215°, 325°$
(ii) $22.1°, 97.9°, 142°, 218°, 262°, 338°$
(b) (i) $0.266, 1.45, 2.36$,
(ii) $0.706, 3.01, 3.85, 6.15$
(c) (i) $-71.6°, 108°$ (ii) $-132°, 48.4°$

4. (a) (i) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
(ii) $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$
(b) (i) $67.5°, 112.5°, 247.5°, 292.5°$
(ii) $\pm 20°, \pm 100°, \pm 140°$
(c) (i) $\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}$ (ii) $\frac{\pi}{12}, \frac{7\pi}{12}$

5. (a) (i) $270°, 330°$ (ii) $0, \frac{2\pi}{3}$
(b) (i) $\frac{\pi}{6}, -\frac{\pi}{2}$ (ii) $75°, 345°$
(c) (i) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (ii) π

6. $1.01, 2.13$

7. (a) $-\frac{1}{2}$
(b) $210°, 330°$

8. $0, \pm\pi$

9. $\pm\sqrt{\frac{\pi}{6}}, \pm\sqrt{\frac{5\pi}{6}}$

Exercise 10C

1. (a) $\cos x = \frac{\sqrt{8}}{3}$ $\tan x = \frac{1}{\sqrt{8}}$
(b) $\cos x = \frac{3}{5}$ $\tan x = \frac{4}{3}$

2. (a) $\sin x = -\frac{\sqrt{8}}{3}$ $\tan x = \sqrt{8}$
(b) $\cos x = -\frac{\sqrt{7}}{4}$, $\tan x = \frac{3}{\sqrt{7}}$

3. (a) (i) $-\frac{2\sqrt{6}}{5}$ (ii) $-\frac{\sqrt{3}}{2}$
(b) (i) $-\frac{4}{3}$ (ii) 0

4. (a) $\pm\frac{3}{\sqrt{13}}$ (b) $\pm\frac{1}{\sqrt{5}}$

5. (a) 3 (b) 1
(c) −2 (d) −2
(e) 1 (f) $\frac{3}{2}$

6. (a) (i) $4 - \sin^2 x$ (ii) $2\cos^2 x - 1$

7. (a) $\frac{1}{1+t^2}$ (b) $\frac{t^2}{1+t^2}$
(c) $\frac{1-t^2}{1+t^2}$ (d) $\frac{2+3t^2}{t^2}$

9. $5 - \frac{2}{\cos^2 x}$

10. $\frac{1}{1 - 2\sin^2 x + \sin^4 x}$

Exercise 10D

1. (a) (i) 33.7° (ii) 59.0°
(b) (i) 0.322 (ii) 1.89
(c) (i) 2.11, 5.25 (ii) 2.21, 5.36
(d) (i) −113°, 66.8° (ii) −101°, 78.7°

2. (a) (i) $\frac{\pi}{12}, \frac{5\pi}{12}$ (ii) $\frac{\pi}{6}, \frac{2\pi}{3}$
(b) (i) $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$ (ii) $\frac{\pi}{4}$

3. (a) $0°, 135°, 180°, 315°, 360°$
(b) $-2.55, 0, 0.588, \pm\pi$
(c) 26.6° (d) 2.50, 5.64

4. (a) (i) 45°, 135°, 225°, 315°
(ii) 54.7°, 125°, 235°, 305°
(b) (i) 45°, 135°, 225°, 315°
(ii) 0, 180°, 360°

5. −180°, −30°, 0, 30°, 180°

6. ±41.8°, ±138°

7. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{6}$

8. $-0.253, -2.89, -\frac{\pi}{2}$

9. $\frac{1}{3}$

10. (a) $\frac{2}{3}, -\frac{1}{2}$
(b) 48.2°, 120°, 240°, 312°

11. (b) $0.464, -2.68, \frac{\pi}{4}, -\frac{3\pi}{4}$

Mixed examination practice 10

Short questions

1. −31.8°, 148°

3. ±2.41, ±0.730

5. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$

6. 48.2°, 311.8°, 120°, 240°

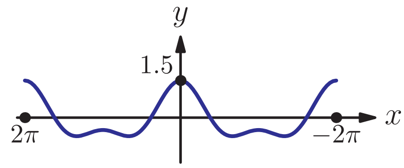
7. $-\frac{7\pi}{24}, -\frac{\pi}{24}, \frac{17\pi}{24}, \frac{23\pi}{24}$

8. (a) $\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$
(b) $\pm\frac{\pi}{6}, \pm\frac{\pi}{3}$

Long questions

1. (a) 3π m
(b) 5.05 m
(c) 1.50 m

2. (a)



(b) 2π

(c) $0, \pm\pi, \pm 2\pi$

(d) 1.2

(e) (ii) $2\pi - x_0$

3. (a) $k = \pm 4$

(c) (i) 1

(ii) $\pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$

(iii) $k = 5$

(iv) 7

Chapter 11

Exercise 11B

1. (a) (i) 6.04 (ii) 14.4

(b) (i) 10.6 cm (ii) 23.3 cm

2. (a) (i) 49.7° (ii) 59.2°

(b) (i) 74.6° or 105° (ii) 62.0° or 118°

(c) (i) 50.9° (ii) 54.4°

3. 21.0°, 29.0°, 8.09 cm

4. 10.4 cm, 49.9°, 95.1°; 269 cm, 130°, 15°

5. 9.94 cm

Exercise 11C

1. (a) (i) 5.37 (ii) 3.44

(b) (i) 8.00 (ii) 20.5

2. (a) (i) 60.6° (ii) 120°

(b) (i) 81.5° (ii) 100°

3. (i) 106° (ii) 36.2°

4. 6.12 km

5. 7.95

6. 4.4

7. $2\sqrt{2} + \sqrt{41}$

Exercise 11D

1. (a) (i) 10.7 cm^2 (ii) 24.3 cm^2

(b) (i) 27.6 cm^2 (ii) 26.2 cm^2

2. (a) 81.7 (b) 60.9

3. 17.7 cm, 29.7 cm^2

4. $4\sqrt{3}$ cm^2

Exercise 11E

1. (a) $\sqrt{134}$ cm (b) $4\sqrt{6}$ cm

2. A = 59.7°, B = 47.5°, C = 72.3°, Area = 85.6 cm^2

3. 62.5°

4. $4\sqrt{11} = 13.3$ cm

5. (a) 12.0 cm (b) 17.0 cm

6. (a) 18.8 cm (b) 23.1 m

7. (a) $RA = \frac{h}{\tan\alpha}$ $RB = \frac{h}{\tan\beta}$

(b) 13 m

Exercise 11F

1. (a) 7.8 cm (b) 1.8 cm

2. (a) 82.2 cm (b) 6.84 cm

3. 25 cm

4. (a) 0.938 (b) 53.7°

5. 2.53

6. 7.5 cm

7. 6.69 cm

8. 15.7 cm

9. 31.6 cm

10. $\left(\frac{25\pi}{6} + 10\right)$ cm

11. 5 cm

12. $\frac{6\pi}{5}$

Exercise 11G

1. (a) 16.25 cm^2 (b) 0.072 cm^2

2. (a) 463 cm^2 (b) 4.79 cm^2

3. 0.8

4. 167°

5. 9.49 cm

6. 11.3 cm

7. 5.14 cm^2

8. 48.4 cm^2

9. 2 cm or 1.5 cm

10. 2.54

Exercise 11H

1. (a) (i) 0.935 cm (ii) 3.39 cm
   (b) (i) 21.7 cm (ii) 15.8 cm
2. (a) (i) 1.89 cm (ii) 6.99 cm
   (b) (i) 52.5 cm (ii) 37.1 cm
3. (a) (i) 0.0595 cm^2 (ii) 1.21 cm^2
   (b) (i) 149 cm^2 (ii) 70.1 cm^2
4. (a) $12.5(\theta - \sin\theta)$ (b) 2.08
5. (b) 70.1° 3.67 cm^2

Mixed examination practice 11

Short questions

1. (a) $\frac{\pi}{3}$ (b) 28.9 cm^2
   (c) 23.8 cm
2. 80 cm^2
3. 58.7°, 121°
4. (a) 8.09 m (b) 6.58 m
5. (a) 10.2 cm (b) 18.8 cm
6. $2\sqrt{43}$
7. 4 cm or 13 cm
8. 12.3 cm^2
9. (a) 1.14 cm^2 (b) 2.00 cm^2
10. (a) $\pi - 2\theta$
    (b) $54 - 2\pi$ cm^2
11. 7.23 cm^2
12. (a) $\cos B = \frac{23}{32}$
    (b) $\sin B = \frac{3\sqrt{55}}{32}$
    (c) $\frac{15\sqrt{55}}{4}$ cm^2
13. (b) 7

Long questions

1. (a) $MB^2 = \left(\frac{x}{2}\right)^2 - 5x\cos\theta + 25$
   (c) 41.4°
2. (b) $\sqrt{2}r$
   (c) $\frac{\pi r^2}{4}$
   (d) $\left(\frac{\pi}{2} - 1\right)r^2$
3. (b) $\frac{1}{2}r^2(2\pi - \theta)$
   (d) 2.50
4. (a) $\frac{4.42}{3x^2}$
   (b) $\frac{3x^2 - 2x - 3}{2x^2}$
   (c) (ii) $x = 1.24,\ \theta = 1.86$
   $x = 2.94,\ \theta = 0.171$
5. (b) $-1 \le \cos\theta < \frac{-\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2} < \cos\theta \le 1$
   (c) $0 < \theta < \frac{\pi}{6}$ or $\frac{5\pi}{6} < \theta < \pi$
6. (a) (ii) $\sqrt{x^2 + 100}$
   (c) 38.7°
   (d) 5.63
   (e) (ii) $\frac{40}{3}$
7. (a) (i) 5 (ii) 144
   (b) (i) $z = 10 - x$
   (ii) $z^2 = x^2 + 36 - 12x\cos Z$
   (e) (i) 12
   (ii) Isosceles
8. (a) $\frac{\pi}{2}$, right angle between a tangent and a radius
   (b) ABO_2P is a rectangle, because there are right angles at A and B, and AB is parallel to PO_2.
   (c) 24.5 cm
   (d) 1.369
   (e) 85.6 cm

Chapter 12

Exercise 12A

1. (a) (i) $-\frac{7}{8}$ (ii) $\frac{1}{9}$

(b) (i) $\frac{2\sqrt{2}}{3}$ (ii) $\frac{4}{5}$

(c) (i) $\frac{4\sqrt{2}}{9}$ (ii) $\frac{24}{25}$

2. (a) $\frac{2-\sqrt{2}}{4}$ (b) $\frac{2-\sqrt{3}}{4}$ (c) $\frac{\sqrt{3}+2}{4}$

3. $\sqrt{2}-1$

4. (a) $\cos(6A)$

(b) $2\sin 10x$ (c) $3\cos b$

(d) $\frac{5}{2}\sin\left(\frac{2x}{3}\right)$

5. (a) $0, \pi, 2\pi$

(b) $90°$

(c) $-\frac{\pi}{2}, \frac{\pi}{2}, 0.305, 2.84$

(d) $0°, 180°, 360°$

7. $0.955, -0.955, 2.19, -2.19$

9. (a) $\pm\frac{\sqrt{3}}{2}$

(b) $-\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

10. (a) $8\cos^4\theta - 8\cos^2\theta + 1$

(b) $8\sin^4\theta - 8\sin^2\theta + 1$

11. (b) $\frac{1-\cos x}{1+\cos x}$

12. $\frac{2a-b}{4a}$

Exercise 12B

1. (a) $\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

(b) $\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$

(c) $-\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$

(d) $-\sin x$

2. (a) $\frac{\sqrt{6}-\sqrt{2}}{4}$

(b) $\frac{\sqrt{2}+\sqrt{6}}{4}$

(c) $-2-\sqrt{3}$

3. (a) $\frac{56}{65}$ (b) $\frac{8+3\sqrt{5}}{15}$

4. (b) $\sqrt{2}\cos x$

5. (a) $\frac{\tan\theta - 1}{\tan\theta + 1}$

(b) $-\frac{1}{2}, -\frac{1}{3}$ (c) $2.68, 2.82$

6. (a) $\sin\left(x+\frac{\pi}{4}\right)$, 1, $x = \frac{\pi}{4}$

(b) $2\cos(x-25°)$, 2, $x = 25°$

8. (a) $4\cos^3 A - 3\cos A$

(b) $\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

9. (b) $x = \frac{\pi}{3}$

10. (b) $\frac{2}{9}$

11. (b) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12. (a) $\frac{17}{4}$

(b) $\frac{3}{4}$

Exercise 12C

1. (a) $2\sqrt{13}\sin(x+0.983)$

(b) $\sqrt{10}\sin(x+0.322)$

2. (a) $2\sqrt{2}\sin(x-45°)$

(b) $2\sin(\theta-60°)$

3. (a) $2\sqrt{2}\cos\left(x+\frac{\pi}{6}\right)$

(b) $5\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

4. (a) $\sqrt{85}\cos(x-0.709)$

(b) $13\cos(x-0.395)$

5. (a) $13\sin(x+1.18)$

(b) Vertical stretch with scale factor 13; Translation 1.18 units to the left

6. (a) $\sqrt{58}\sin(x-1.16)$

(b) $y \in \left[-\sqrt{58}, \sqrt{58}\right]$

7. (a) $\sqrt{41}\cos(x+0.896)$

(b) 0.675

8. (a) $2\cos\left(x-\frac{\pi}{3}\right)$

(b) minimum : $\left(\frac{4\pi}{3},-2\right)$, maximum: $\left(\frac{\pi}{3},2\right)$

9. $-\pi, -\frac{3\pi}{4}, 0, \frac{\pi}{4}, \pi$

Exercise 12D

1. (a) (i) 2.760 (ii) 1.480
(b) (i) −2.670 (ii) 1.212
(c) (i) 1.051 (ii) 0.5774

2. (a) (i) $\frac{2\sqrt{3}}{3}$ (ii) $\sqrt{2}$

(b) (i) $-\sqrt{2}$ (ii) $-\frac{2\sqrt{3}}{3}$

(c) (i) −1 (ii) $\frac{\sqrt{3}}{3}$

(d) (i) −1 (ii) 0

3. $\csc A = \frac{5}{4}$, $\sec B = \frac{3\sqrt{5}}{5}$

4. (a) (i) 1.05, 5.24

(ii) 1.23, 5.05

(b) (i) 0.730, 2.41
(ii) 0.379, 2.76

(c) (i) 0.197, 3.34
(ii) 1.11, 4.25

(d) (i) 0.615, 2.53, 3.76, 5.67
(ii) 0.126, 1.44, 3.27, 4.59

5. (a) (i) $-\frac{\pi}{6}, -\frac{5\pi}{6}$ (ii) $-\frac{\pi}{2}$

(b) (i) $\frac{\pi}{6}, -\frac{5\pi}{6}$

(ii) $\frac{\pi}{4}, -\frac{3\pi}{4}$

(c) (i) 0

(ii) $\frac{5\pi}{6}, -\frac{5\pi}{6}$

(d) (i) $\frac{\pi}{2}, -\frac{\pi}{2}$ (ii) $-\frac{\pi}{4}, \frac{3\pi}{4}$

6. (a) (i) $\frac{5}{3}$ (ii) $\frac{\sqrt{29}}{5}$

(b) (i) $2\sqrt{6}$ (ii) $2\sqrt{2}$

(c) (i) $-\frac{1}{\sqrt{10}}$ (ii) $-\frac{2}{\sqrt{5}}$

(d) (i) $\pm\frac{3}{\sqrt{7}}$ (ii) $\pm\frac{2}{\sqrt{3}}$

8. 1.08

9. (a) $(0.715, 2.39)$, $(-0.715, -2.39)$

(b) $]-\infty, -2.39] \cup [2.39, \infty[$

12. (b) 1, 2

(c) $\frac{\pi}{4}, \frac{5\pi}{4}, 1.11, 4.25$

15. $\arccos\left(\frac{1}{x}\right)$

Mixed examination practice 12

Short questions

1. $-\frac{3}{2}$

2. (a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

(b) $-2\pi, -\pi, 0, \pi, 2\pi$

3. (b) $\frac{2\pi}{3}, \frac{4\pi}{3}$

4. 20.9°, 69.1°

5. (a) $2\sqrt{5}\sin\left(2x+\frac{\pi}{6}\right)$

(b) (i) $1+\frac{2}{5}\sqrt{5}$ (ii) $x=\frac{2\pi}{3}$

6. (a) $\sin(\arcsin x) = x$

(c) $\frac{1}{\sqrt{2}}$

Long questions

1. (a) $AB = 2r\sin\theta$, $BC = 2r\cos\theta$
(b) $2r^2\sin\theta\cos\theta$
(c) $r^2\sin\theta\cos\theta$

(d) $\frac{1}{2}$

2. (b) −1

(c) $1+\sqrt{2}$

3. (a) $a = 1.2$, $p = \frac{2\pi}{3}$

(b) amplitude = 0.9, period = 3

(c) $1.5\sin\left(\frac{2\pi}{3}x + 0.927\right)$

(d) amplitude = 1.5, period = 3

(e) 1.06

(f) 0.058, 0.557

4. (a) $r = 2, \alpha = \frac{\pi}{6}$

(b) $[-2, 2]$

(c) $\frac{\pi}{2}, \frac{7\pi}{6}$

5. (a) $(t+1)(t^2 - 4t + 1)$

(c) 1

(d) $\tan 15° = 2 - \sqrt{3}$,

$\tan 75° = 2 + \sqrt{3}$

Chapter 13

Exercise 13A

1. (a) (i) $\boldsymbol{b}$ (ii) $\boldsymbol{a}+\boldsymbol{b}$

(b) (i) $-\boldsymbol{a}$ (ii) $-\frac{1}{2}\boldsymbol{a}$

(c) (i) $\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}$

(ii) $\frac{1}{2}\boldsymbol{b}-\frac{1}{2}\boldsymbol{a}$

2. (a) (i) $\boldsymbol{a}+\frac{4}{3}\boldsymbol{b}$ (ii) $\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}$

(b) (i) $-\frac{3}{2}\boldsymbol{a}+\boldsymbol{b}$

(ii) $-\frac{1}{2}\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}$

(c) (i) $\frac{3}{2}\boldsymbol{a}-\boldsymbol{b}$

(ii) $-\frac{4}{3}\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}$

3. (a) (i) $\begin{pmatrix}4\\0\\0\end{pmatrix}$ (ii) $\begin{pmatrix}0\\-5\\0\end{pmatrix}$

(b) (i) $\begin{pmatrix}3\\0\\1\end{pmatrix}$ (ii) $\begin{pmatrix}0\\2\\-1\end{pmatrix}$

4. (a) $\boldsymbol{b}-\boldsymbol{a}$

(b) $\frac{1}{2}\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}$

(c) $4\boldsymbol{a}-3\boldsymbol{b}$

5. (a) $\overrightarrow{AB} = \begin{pmatrix}1\\2\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}0.5\\1\end{pmatrix}$

(b) $(10, -2)$

6. (a) $\begin{pmatrix}1\\-3\\7\end{pmatrix}$ (b) $\begin{pmatrix}3.5\\-0.5\\1.5\end{pmatrix}$

7. $\begin{pmatrix}3\\-4\end{pmatrix}$

8. $\begin{pmatrix}1.6\\0.8\\1.8\end{pmatrix}$

9. (a) $\frac{3}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}-2\boldsymbol{k}$

(b) $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

10. $\begin{pmatrix}0\\-1\\6\end{pmatrix}$

Exercise 13B

1. (a) (i) $\begin{pmatrix}21\\3\\36\end{pmatrix}$ (ii) $\begin{pmatrix}20\\-8\\12\end{pmatrix}$

(b) (i) $\begin{pmatrix}2\\3\\9\end{pmatrix}$ (ii) $\begin{pmatrix}6\\-1\\5\end{pmatrix}$

(c) (i) $\begin{pmatrix}11\\-3\\8\end{pmatrix}$ (ii) $\begin{pmatrix}-3\\5\\6\end{pmatrix}$

(d) (i) $\begin{pmatrix}10\\-3\\11\end{pmatrix}$ (ii) $\begin{pmatrix}17\\6\\35\end{pmatrix}$

2. (a) (i) $-5\boldsymbol{i}+5\boldsymbol{k}$ (ii) $4\boldsymbol{i}+8\boldsymbol{j}$

(b) (i) $\boldsymbol{i}-3\boldsymbol{j}+3\boldsymbol{k}$ (ii) $2\boldsymbol{j}+\boldsymbol{k}$

(c) (i) $4\boldsymbol{i}+7\boldsymbol{k}$

(ii) $5\boldsymbol{i}-4\boldsymbol{j}+15\boldsymbol{k}$

3. (a) $-4\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$

(b) $-\frac{8}{3}\boldsymbol{i}+\frac{4}{3}\boldsymbol{j}-\frac{2}{3}\boldsymbol{k}$

(c) $4\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$

(d) $-\frac{1}{2}\boldsymbol{i}+\boldsymbol{j}-\frac{1}{2}\boldsymbol{k}$

4. $\begin{pmatrix} 2 \\ 0 \\ \frac{-3}{4} \end{pmatrix}$

5. -2

6. $-\frac{4}{3}$

7. -2

8. $p = \frac{3}{8}, q = \frac{1}{8}$

Exercise 13C

1. $|\boldsymbol{a}| = 2\sqrt{5}$ $|\boldsymbol{b}| = \sqrt{26}$ $|\boldsymbol{c}| = 2\sqrt{5}$ $|\boldsymbol{d}| = \sqrt{2}$

2. $|\boldsymbol{a}| = \sqrt{21}$ $|\boldsymbol{b}| = \sqrt{2}$ $|\boldsymbol{c}| = \sqrt{21}$ $|\boldsymbol{d}| = \sqrt{2}$

3. (a) (i) $\sqrt{29}$ (ii) $\sqrt{2}$
   (b) (i) $\sqrt{58}$ (ii) $\sqrt{5}$

4. (a) (i) $\sqrt{19}$ (ii) $\sqrt{38}$
   (b) (i) $\sqrt{74}$ (ii) $\sqrt{13}$

5. (a) $\sqrt{53}$ (b) $\sqrt{94}$
   (c) $\sqrt{53}$ (d) $\sqrt{2}$

6. (a) (i) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ (ii) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$
   (b) (i) $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
   (ii) $\frac{1}{5}\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$

7. $\pm 2\sqrt{6}$

8. $\frac{3}{2}$

9. $3, -\frac{5}{3}$

10. (a) $\begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$ (b) $\frac{\sqrt{6}}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

11. $-2, -\frac{23}{15}$

12. $t = \frac{1}{3}$, $d = \sqrt{\frac{14}{3}}$

Exercise 13D

1. (a) (i) 1.12 (ii) 1.17
   (b) (i) 1.88 (ii) 1.13
   (c) (i) 1.23 (ii) 1.77

2. (a) (i) $-\frac{5}{2\sqrt{21}}$
   (ii) $-\frac{20}{\sqrt{570}}$
   (b) (i) $-\frac{2}{\sqrt{102}}$
   (ii) $\frac{1}{\sqrt{35}}$
   (c) (i) $\frac{1}{\sqrt{50}}$ (ii) 0

3. (a) $61.0°, 74.5°, 44.5°$
   (b) $94.3°, 54.2°, 31.5°$

4. (a) (i) No (ii) Yes
   (b) (i) Yes (ii) No

5. $92.3°$

6. $40.0°$

7. (b) $107°, 73.2°$
   (c) $\frac{5}{4}$

8. (b) $41.8°, 48.2°$
   (c) $6\sqrt{5}$

Exercise 13E

1. (a) (i) 16 (ii) -56
   (b) (i) 16 (ii) -16
   (c) (i) 9 (ii) 9
   (d) (i) -4 (ii) 0

2. (a) (i) $\frac{7}{3\sqrt{6}}$ (ii) $\frac{5}{\sqrt{39}}$
   (b) (i) $\frac{2}{3}$ (ii) $\frac{1}{\sqrt{10}}$

3. (a) (i) $48.2°$ (ii) $98.0°$

4. (a) 19.2 (b) 3

6. (a) (i) $-\frac{1}{2}$ (ii) $\frac{2}{7}$
   (b) (i) $\frac{4}{5}$ (ii) $0, \frac{3}{2}$

7. (a) $x=-\frac{18}{11}$ $y=\frac{10}{11}$

(b) (i) $1+11x=-17$

(ii) $14=4+11y$

(iii) They are perpendicular to $\begin{pmatrix}3y\\y\end{pmatrix}$ and $\begin{pmatrix}2x\\-3x\end{pmatrix}$, respectively.

(c) (i) $\begin{pmatrix}1\\2\end{pmatrix}$ or $\begin{pmatrix}-1\\-2\end{pmatrix}$, and $\begin{pmatrix}3\\-5\end{pmatrix}$ or $\begin{pmatrix}-3\\5\end{pmatrix}$

(ii) $x=-\frac{13}{11}$ $y=-\frac{3}{11}$

8. (a) 19 (b) 7

(c) 32

9. (a) 2 (b) 6

10. (a) $\frac{52}{9}$

11. (a) 1.6

(b) 68.7°, 21.3°, 90°

(c) 88.7

12. **(a)** $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{b}-\boldsymbol{a}$

(b) $|\boldsymbol{b}|^2-|\boldsymbol{a}|^2$

13. (b) 2

(c) $4\sqrt{5}$

Exercise 13F

1. (a) (i) $\begin{pmatrix}-2\\6\\-1\end{pmatrix}$ (ii) $\begin{pmatrix}0\\0\\0\end{pmatrix}$

(b) (i) $\begin{pmatrix}-5\\-11\\-2\end{pmatrix}$ (ii) $\begin{pmatrix}12\\2\\6\end{pmatrix}$

2. (a) (i) $\frac{1}{2}\sqrt{153}$ (ii) $\sqrt{117}$

(b) (i) $\frac{15\sqrt{3}}{2}$ (ii) $\frac{9}{2}$

3. (a) $\begin{pmatrix}18\\-12\\72\end{pmatrix}, \begin{pmatrix}-18\\12\\-72\end{pmatrix}$

(b) $\boldsymbol{p}=-\boldsymbol{q}$

4. (a) $(11,2,0)$ (b) 16.8

5. (a) $C(5,4,0), F(5,0,2), G(5,4,2), H(0,4,2)$

(b) 11.9

Exercise 13G

1. (a) 0.775 (b) 0.128

(c) 0.630 (d) 0

2. (a) (i) $\begin{pmatrix}-1\\-10\\7\end{pmatrix}$ (ii) $\begin{pmatrix}-9\\-19\\2\end{pmatrix}$

(b) (i) $\begin{pmatrix}-23\\1\\8\end{pmatrix}$ (ii) $\begin{pmatrix}0\\4\\0\end{pmatrix}$

3. (a) $\frac{1}{\sqrt{14}}\begin{pmatrix}3\\1\\2\end{pmatrix}$

(b) $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$

4. 17.5

5. (b) $13\boldsymbol{a}\times\boldsymbol{b}$

6. (b) 0

Mixed examination practice 13

Short questions

1. −1

2. (a) $\frac{1}{2}\overrightarrow{AD}-\overrightarrow{AB}$

3. (a) $9\boldsymbol{i}+5\boldsymbol{j}+7\boldsymbol{k}$

(b) $\sqrt{155}$

4. (a) $\begin{pmatrix}-5\\-3p-1\\15\end{pmatrix}$

(b) $\frac{19}{3}$

5. 74.4°

6. $\frac{\pi}{2}-2\theta$

7. 0

Long questions

1. (a) $\begin{pmatrix}2\\0\\k-7\end{pmatrix}$ (c) $(3,6,1)$

(d) $-\frac{1}{\sqrt{10}}$

2. (a) $\begin{pmatrix} -1+\frac{3}{k+1} \\ \frac{3}{4+1} \\ -4 \end{pmatrix}$ (b) 5

(c) $\left(\frac{3}{2},\frac{3}{2},2\right)$ (d) $\sqrt{\frac{33}{2}}$

3. (a) (a,a^2)

(b) $\begin{pmatrix} -a \\ -a^2 \end{pmatrix},\begin{pmatrix} -a \\ 4-a^2 \end{pmatrix}$

(c) $\sqrt{3}$

(d) $2\sqrt{3}$

4. (a) $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$

(b) $|\boldsymbol{c}|\cos\theta$

(d) $\frac{1}{3}$

Chapter 14

Exercise 14A

1. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

(c) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(d) (i) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

2. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 7 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -9 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

3. (a) (i) Yes (ii) Yes
(b) (i) Yes (ii) No

4. (b) $(0,3,0)$

5. (a) $\boldsymbol{r}=\begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

(b) $(-5,-5,-11)$ or $(19,7,-7)$

6. (a) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

(b) 7

(c) $(-8,16,-26),(12,-14,34)$

Exercise 14B

1. (a) (i) 44.5° (ii) 56.5°
(b) (i) 26.6° (ii) 82.1°

2. (a) Perpendicular (b) Parallel
(c) Parallel (d) Same line

3. (a) (i) $(10,-7,-2)$
(ii) $(4.5,0,0)$
(b) (i) No intersection
(ii) No intersection

4. $\left(\frac{64}{9},\frac{4}{9},\frac{19}{9}\right)$

5. $\sqrt{\frac{6}{11}}$

6. (a) $(4,1,-2)$

(c) $(1,1,2)$ (d) $\frac{5\sqrt{26}}{2}$

7. (a) $\begin{pmatrix} 3t \\ 4t \end{pmatrix}$

(b) $\begin{pmatrix} 3t \\ 18-5t \end{pmatrix}$

(d) $t=2$

(e) 2 hours

8. (a) $\begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$

(d) 30 km

9. 3

10. (a) $\left(\frac{5}{6},\frac{19}{6},\frac{9}{2}\right)$ (b) 48.5°

(d) $\frac{11\sqrt{11}}{6}\ (=6.08)$

(e) 4.55

11. (a) $(9,-5,8)$

(c) $(3,4,-1)$

12. (b) $\left(2+\sqrt{6},-1-2\sqrt{6},2\sqrt{6}\right)$, or $\left(2-\sqrt{6},2\sqrt{6}-1,-2\sqrt{6}\right)$

Exercise 14C

1. (a) $\frac{x-1}{-1}=\frac{y-7}{1}=\frac{z-2}{2}$ (b) $x=-1, \frac{y-5}{-2}=\frac{z}{2}$

(c) $\boldsymbol{r}=\begin{pmatrix}3\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\5\end{pmatrix}$ (d) $\boldsymbol{r}=\begin{pmatrix}-1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\0\\-2\end{pmatrix}$

2. (a) Perpendicular (b) None
(c) None (d) Parallel

3. (a) $\frac{x-1}{3}=\frac{4-y}{2}=\frac{z+1}{3}$

(b) $\frac{1}{\sqrt{22}}\begin{pmatrix}3\\-2\\3\end{pmatrix}$

4. (a) $\boldsymbol{r}=\begin{pmatrix}1/2\\-2\\4/3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}$

(b) Yes at $\left(\frac{11}{6},0,0\right)$

(c) 61.0°

5. (a) 13.2° (b) No

6. (a) $(8,7,1)$

Exercise 14D

1. (a) (i) $\boldsymbol{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\5\\2\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\3\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\4\\-1\end{pmatrix}+\mu\begin{pmatrix}5\\3\\0\end{pmatrix}$

(b) (i) $\boldsymbol{r}=(\boldsymbol{j}+\boldsymbol{k})+\lambda(3\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})+\mu(\boldsymbol{i}-3\boldsymbol{j})$

(ii) $\boldsymbol{r}=(\boldsymbol{i}-6\boldsymbol{j}+2\boldsymbol{k})+\lambda(5\boldsymbol{i}-6\boldsymbol{j})+\mu(-\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k})$

2. (a) (i) $\boldsymbol{r}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}-1\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}-6\\2\\-4\end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix}9\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-11\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-8\\-1\\2\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}11\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}-10\\21\\-1\end{pmatrix}+\mu\begin{pmatrix}-16\\17\\-3\end{pmatrix}$

3. (a) (i) $\boldsymbol{r}=\begin{pmatrix}-1\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\2\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}11\\12\\13\end{pmatrix}+\lambda\begin{pmatrix}6\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}2\\15\\6\end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix}-3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}+\mu\begin{pmatrix}-7\\-3\\-1\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}4\\0\\2\end{pmatrix}$

4. (a) (i) $3x-5y+2z=-4$
(ii) $6x-y+2z=19$
(b) (i) $3x-y=-9$
(ii) $4x-5z=-10$

5. (a) (i) $\begin{pmatrix}10\\13\\-12\end{pmatrix}$ (ii) $\begin{pmatrix}3\\1\\1\end{pmatrix}$

(b) (i) $\begin{pmatrix}1\\5\\0\end{pmatrix}$ (ii) $\begin{pmatrix}1\\20\\7\end{pmatrix}$

6. (a) (i) $\boldsymbol{r}\cdot\begin{pmatrix}10\\13\\-12\end{pmatrix}=38$

(ii) $\boldsymbol{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=3$

(b) (i) $\boldsymbol{r}\bullet\begin{pmatrix}1\\5\\0\end{pmatrix}=22$

(ii) $\boldsymbol{r}\bullet\begin{pmatrix}1\\20\\7\end{pmatrix}=152$

7. (a) (i) $10x+13y-12z=38$

(i) $3x+y+z=1$

(b) (i) $x+5y=22$

(ii) $x+20y+7z=152$

8. (a) (i) $x+y+z=10$

(ii) $z=2$

(b) (i) $40x+5y+8z=580$

(ii) $x+y+z=1$

Exercise 14E

1. (a) (i) $(7,1,1)$

(ii) $(-19,-5,7)$

(b) (i) $\left(-\frac{4}{3},-\frac{7}{3},-4\right)$

(ii) $(8,-3,2)$

2. (a) (i) $46.4°$ (ii) $17.5°$

(b) (i) $47.6°$ (ii) $10.8°$

3. (a) $75.8°$ (b) $60°$

4. (a) (i) $\boldsymbol{r}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-13\\-7\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix}2\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\4\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}6\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\0\end{pmatrix}$

5. (a) $\begin{pmatrix}3\\-1\\1\end{pmatrix}$ (b) $57°$

6. $(5,3,1)$

8. (a) $(5,0,0),(0,-20,0),$
$(0,0,12)$

(b) 133

9. (b) $\frac{\sqrt{7}}{3}$

(c) $3\sqrt{5}$ (d) $\sqrt{35}$

10. (a) $\begin{pmatrix}-1\\4\\7\end{pmatrix}$

(d) $\boldsymbol{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\7\end{pmatrix}$

11. (a) $\begin{pmatrix}7\\-5\\-4\end{pmatrix}$

(b) (ii) $\boldsymbol{r}=\lambda\begin{pmatrix}7\\-5\\-4\end{pmatrix}$

(c) $\left(2,\frac{-10}{7},\frac{-8}{7}\right)$

Exercise 14F

1. $\left(\frac{5}{3},\frac{16}{3},-\frac{7}{3}\right)$

2. $(2,2,-3)$

3. $\boldsymbol{r}=\begin{pmatrix}0\\4\\-1\end{pmatrix}+t\begin{pmatrix}1\\-2\\2\end{pmatrix}$

4. $\left(\frac{3}{2},-\frac{11}{6},-\frac{1}{6}\right)$

6. (a) $d=-2$

(b) $\boldsymbol{r}=\begin{pmatrix}\frac{-1}{2}\\\frac{2}{5}\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\1\end{pmatrix}$

7. $\left(\frac{5+\boldsymbol{d}}{2},\frac{\boldsymbol{d}-3}{2},\frac{5-\boldsymbol{d}}{2}\right)$

8. (b) $2\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}$

9. (a) 2

(b) $x-5=y+1=z$

10. (a) $p=-4, a=2 \text{ or } -\frac{7}{5}$

(b) $\boldsymbol{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$

Exercise 14G

1. (a) $\mathbf{r}=\begin{pmatrix}-3\\-3\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$

   (b) $(3,3,1)$ (c) 9

2. (c) $\mathbf{r}=\lambda\begin{pmatrix}1\\-3\\1\end{pmatrix}$

   (d) $\frac{6\sqrt{11}}{11}$

3. (a) $\begin{pmatrix}-2\\3\\-1\end{pmatrix}$

   (b) (ii) $(1,-3,14)$

   (c) $2x-3y+z=25$

4. (b) $\mathbf{r}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=-5$

   (c) $\sqrt{30}$

   (d) $(8,-5,-1)$

5. (c) $\left(\frac{184}{11},-\frac{-32}{11},-\frac{-1}{11}\right)$

   (d) 6.99

6. (a) $(10,11,-6)$

   (b) $\begin{pmatrix}7\\-9\\-5\end{pmatrix}$

   (c) $7x-9y-5z=1$

7. (a) $\begin{pmatrix}-3\\-10\\2\end{pmatrix}$ (b) 5.32

   (c) $3x+10y-2z=16$

   (d) $\mathbf{r}=\begin{pmatrix}-7\\-28\\11\end{pmatrix}+\lambda\begin{pmatrix}3\\10\\-2\end{pmatrix}$

   (e) $(2,2,5)$; 31.9 (3*SF*)

   (f) 56.5

8. (a) $\mathbf{r}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}6\\1\\5\end{pmatrix}$

   (c) $\begin{pmatrix}-13\\38\\8\end{pmatrix}$

   (d) $-13x+38y+8z=83$

9. (a) $\left(\frac{96}{41},-\frac{32}{41},\frac{16}{41}\right)$

   (b) $\frac{16\sqrt{41}}{41}$

10. (b) $\mathbf{r}=\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$

    (c) (i) $(2,0,-2)$ (ii) $(-4,0,4)$

    (d) $6\sqrt{2}$

Mixed examination practice 14

Short questions

1. $\mathbf{r}=\begin{pmatrix}3\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\0\end{pmatrix}$

2. 5

3. (a) $3x+y-z=6$

   (b) $\frac{5}{4}$

4. $\left(\frac{3}{2},-\frac{11}{6},-\frac{1}{6}\right)$

5. $(11,13,8)$

6. $\left(\frac{11}{3},\frac{20}{3},\frac{2}{3}\right)$ or $\left(\frac{-29}{3},\frac{-20}{3},\frac{22}{3}\right)$

7. (a) $x=4+\lambda, y=1-3\lambda,$
   $z=12+2\lambda$

   (b) $\left(\frac{31}{14},\frac{89}{14},\frac{59}{7}\right)$

8. (c) $\frac{x-2}{3}=\frac{(y-2)}{7}=z-3$

9. $k=8$

11. $7x+2y-3z=3$

Long questions

1. (b) $\sqrt{33}$
   (c) 45.7°
   (d) 4.11

2. (b) (i) $\begin{pmatrix} \mu-2\lambda+8 \\ \mu+\lambda-3 \\ -\mu+8\lambda-16 \end{pmatrix}$

(iii) $3\mu-9\lambda+21=0$

(iv) $(1,1,2)$, $(4,-1,3)$

(v) $\sqrt{14}$

3. (a) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(b) $(6,-5,4)$

(c) $2\sqrt{6}$

4. (a) 12.6 km/min

(b) $(24+3t)\boldsymbol{i}+(66-8t)\boldsymbol{j}+(12-4t)\boldsymbol{k}$

(c) 22 km

(d) 5 km (when $t=2$)

5. (a) $\begin{pmatrix} -2 \\ 7 \\ -3 \end{pmatrix}$

(b) $(3,3,8)$

(c) $\begin{pmatrix} -2 \\ 7 \\ -3 \end{pmatrix}$

(d) $2x-7y+3z=9$

6. (a) $d=3$

(b) $\boldsymbol{r}=\begin{pmatrix} \frac{3}{7} \\ \frac{5}{7} \\ 0 \end{pmatrix}+t\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

(c) (i) $p=3$

(ii) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$

(iii) $\frac{\sqrt{34}}{15}$ (≈ 0.389)

7. (a) $\boldsymbol{r}=\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

(b) $\left(\frac{30}{17},-\frac{14}{17},-\frac{3}{17}\right)$

(c) 2.81

8. (b) $\sqrt{94}$

(c) 0.551

(d) 5.08

9. (a) (ii) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

(b) $3x-2y+z=5$

(c) $(2,1,1)$

10. (a) $\boldsymbol{r}=\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $\frac{1}{2}$

(c) $3\sqrt{2}$ (d) $\frac{3\sqrt{2}}{2}$

11. (b) $\boldsymbol{r}=\begin{pmatrix} 9 \\ -7 \\ 6 \end{pmatrix}+\lambda\begin{pmatrix} -4 \\ 8 \\ 3 \end{pmatrix}$

(c) $\boldsymbol{r}=\begin{pmatrix} 9 \\ -7 \\ 6 \end{pmatrix}+\mu\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

(d) $(7,1,2)$

(e) $\sqrt{84}$

12. (a) $\frac{x-3}{3}=\frac{y-1}{-1}=\frac{z+4}{-1}$

(b) $(0,2,-3)$

(c) $(-3,3,-2)$

(e) $3\sqrt{2}$

13. **(a)** $\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

(b) $2x-y+z=0$

(d) $\boldsymbol{r}=\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}+\lambda\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

(e) $(3,4,0)$ (f) 47.1°

Chapter 15

Exercise 15A

1. (a) (i) 5 (ii) −2
(b) (i) 1 (ii) −1
(c) (i) 2 (ii) 7
(d) (i) 0 (ii) −3
(e) (i) 0 (ii) 0
(f) (i) $a-1$ (ii) $-4-b$

2. (a) (i) 5i (ii) −8i
(b) (i) −5 (ii) −1
(c) (i) −9 (ii) −16
(d) (i) 5 − 2i (ii) −14 + 10i

3. (a) (i) −1 + i (ii) 15 + 6i
(b) (i) 5i (ii) 14 + 23i
(c) (i) 8 − i (ii) 16 + 2i
(d) (i) 8 + 6i (ii) 7 − 24i
(e) (i) 1 (ii) 13

4. (a) (i) 3 + 4i (ii) 3 − i
(b) (i) $\frac{1}{2}+\frac{1}{5}\text{i}$ (ii) $-\frac{1}{2}+\frac{1}{8}\text{i}$
(c) (i) $\frac{3}{2}+\frac{3}{2}\text{i}$ (ii) −3 + 11i

5. (a) (i) 2i (ii) 7i
(b) (i) $2\sqrt{2}\text{i}$ (ii) $5\sqrt{2}\text{i}$
(c) (i) $\frac{4}{3}-2\text{i}$ (ii) $\frac{1}{3}+\frac{5}{3}\text{i}$
(d) (i) $\frac{1}{3}+\frac{1}{2}\text{i}$ (ii) $\frac{5}{4}-\frac{\sqrt{5}}{2}\text{i}$

6. (a) (i) $x=\pm 3\text{i}$
(ii) $x=\pm 6i$
(b) (i) $x=\pm\sqrt{10}\text{i}$
(ii) $x=\pm\sqrt{13}\text{i}$
(c) (i) $x=1\pm 2\text{i}$
(ii) $x=\frac{1}{2}\pm\frac{\sqrt{39}}{2}\text{i}$
(d) (i) $x=1\pm\frac{\sqrt{51}}{3}\text{i}$
(ii) $x=-\frac{3}{5}\pm\frac{4}{5}\text{i}$

7. (a) (i) −i (ii) 1
(b) (i) 16 (ii) 125i
(c) (i) −8 (ii) 8i
(d) (i) i (ii) i

8. (a) (i) $a=-\frac{7}{13},\ b=\frac{7}{13}$
(ii) $a=\frac{12}{37},\ b=\frac{-12}{37}$
(b) (i) $a=8,\ b=-24$
(ii) $a=1,\ b=0$
(c) (i) $a=1,\ b=0$
(ii) $a=-6,\ b=6$

9. (a) (i) $z=\pm\left(\sqrt{2}-\sqrt{2}i\right)$
(ii) $z=\pm\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}\text{i}\right)$
(b) (i) $z=\pm\left(\sqrt{3}+\text{i}\right)$
(ii) $z=\pm\left(\sqrt{\frac{\sqrt{26}+5}{2}}+i\sqrt{\frac{\sqrt{26}-5}{2}}\right)$

10. $a=\pm\sqrt{3},\ b=\mp\frac{\sqrt{3}}{3}$

11. $a=8,\ b=1$

$a=-1,\ b=10$

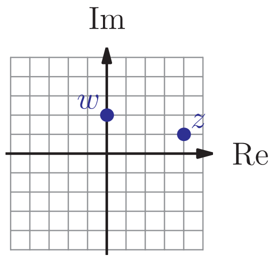
12. $z=-\frac{1}{2}+\frac{1}{2}i$

13. (a) $x=-\frac{1}{5},\ y=-\frac{2}{5}$
(b) $=-\frac{3}{5}-\frac{6}{5}\text{i}$

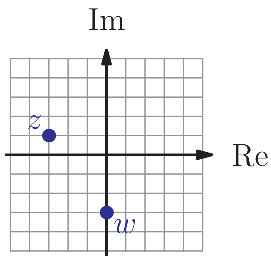
14. (a) $a=\pm 1,\ b=\mp 2$
(b) $z=\frac{1}{2}-\frac{\sqrt{3}+2}{2}\text{i},$
or $-\frac{1}{2}+\frac{2-\sqrt{3}}{2}\text{i}$

Exercise 15B

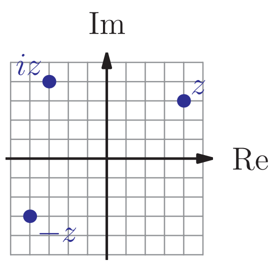
1. (a) (i)



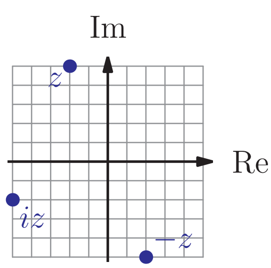
(ii)



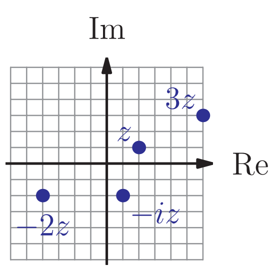
(b) (i)



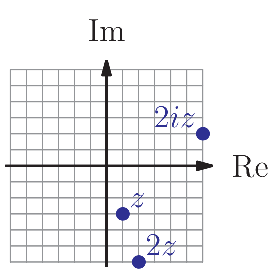
(ii)



(c) (i)



(ii)

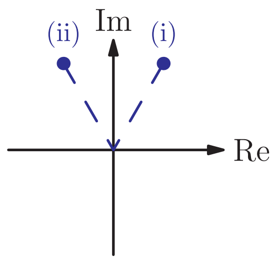


2. (a) (i) mod = 6, arg = 0
(ii) mod = 13, arg = 0
(b) (i) mod = 3, arg = π
(ii) mod = 1.6, arg = π
(c) (i) mod = 4, arg = $\frac{\pi}{2}$
(ii) mod = 0.5, arg = $\frac{\pi}{2}$
(d) (i) mod = 2, arg = $-\frac{\pi}{2}$
(ii) mod = 5, arg = $-\frac{\pi}{2}$
(e) (i) mod = $\sqrt{2}$ arg = $\frac{\pi}{4}$
(ii) mod = $\sqrt{7}$ arg = 0.714
(f) (i) mod = 2 arg = $-\frac{2\pi}{3}$
(ii) mod = $4\sqrt{2}$ arg = $\frac{\pi}{4}$

3. (a) (i) mod = 4, arg = $\frac{\pi}{3}$
(ii) mod = $\sqrt{7}$, arg = $\frac{3\pi}{7}$
(b) (i) mod = 1, arg = $\frac{\pi}{5}$
(ii) mod = 1, arg = $-\frac{\pi}{4}$
(c) (i) mod = 3, arg = $-\frac{\pi}{8}$
(ii) mod = 7, arg = $-\frac{4\pi}{5}$
(d) (i) mod = 10, arg = $-\frac{2\pi}{3}$
(ii) mod = 2, arg = $-\frac{5\pi}{6}$
(e) (i) mod = 6, arg = $\frac{\pi}{10}$
(ii) mod = $\frac{1}{2}$, arg = $\frac{\pi}{3}$

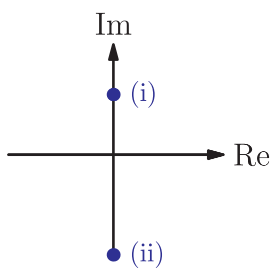
4. (a) (i) mod = $\sqrt{20}$, arg = 0.464
(ii) mod = 5, arg = −0.644
(b) (i) mod = 2, arg = $\frac{5\pi}{6}$
(ii) mod = $\sqrt{38}$, arg = 1.34
(c) (i) mod = $\sqrt{10}$, arg = −2.82
(ii) mod = $\sqrt{13}$, arg = 2.55

5. (a) (i) $2+2\sqrt{3}\text{i}$
(ii) $1+\text{i}$
(b) (i) $-\sqrt{2}+\sqrt{2}\text{i}$
(ii) $-1+\sqrt{3}\text{i}$
(c) (i) -3i
(ii) -4

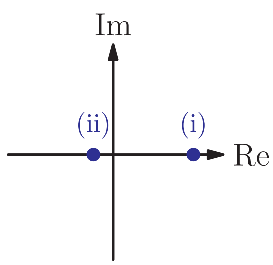
6. (a)


(i) $\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$

(ii) $\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\text{i}$

(b)


(i) 3i (ii) -5i

(c)


(i) 4 (ii) -1

7. (a) (i) $4\text{ cis}\left(\frac{\pi}{2}\right)$

(ii) $5\text{ cis}(\pi)$

(b) (i) $4\text{cis}\left(-\frac{\pi}{3}\right)$

(ii) $\frac{2}{3}\text{cis}\left(\frac{\pi}{6}\right)$

8. (a) (i) $|z|=\sqrt{x^2+y^2}$

$\tan(\arg(z))=-\frac{y}{x}$

(ii) $|z|=\sqrt{x^2+y^2}$

$\tan(\arg(z))=\frac{y}{x}$

(b) (i) $|z|=\sqrt{16x^2+9y^2}$

$\tan(\arg(z))=-\frac{3y}{4x}$

(ii) $|z|=9\sqrt{x^2+y^2}$

$\tan(\arg(z))=\frac{y}{x}$

(c) (i) $|z|=\sqrt{x^2+(y+3)^2}$

$\tan(\arg(z))=\frac{y+3}{x}$

(ii) $|z|=\sqrt{(x+2)^2+(y+1)^2}$

$\tan(\arg(z))=\frac{y+1}{x+2}$

(d) (i) $|z|=x^2+y^2$

$\tan(\arg(z))=\frac{2xy}{x^2-y^2}$

(ii) $|z|=x^2+y^2$

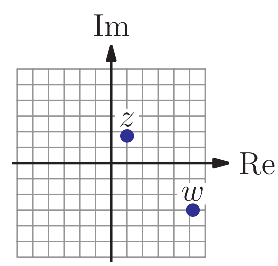
$\tan(\arg(z))=\frac{-2xy}{x^2-y^2}$

9. (a) $\frac{3\sqrt{2}}{2}-\frac{3\sqrt{2}}{2}\text{i}$

(b) $4\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

10. (a) $|z|=2,\ \arg(z)=\frac{\pi}{3}$

$|w|=6,\ \arg(w)=-\frac{\pi}{6}$

(b)


(c) $|zw|=12, \arg(z)=\frac{\pi}{6}$

$|zw|=|z||w|$

$\arg(zw)=\arg(z)+\arg(w)$

Exercise 15C

1. (a) (i) $2+3\text{i}$ (ii) $4-4\text{i}$
(b) (i) $-\text{i}-3$ (ii) $-3\text{i}+2$

(c) (i) $-3\mathrm{i}$ (ii) i
(d) (i) -45 (ii) 9

2. (a) (i) $-\frac{1}{5}-\frac{8}{5}\mathrm{i}$
(ii) $-\frac{10}{17}+\frac{6}{17}\mathrm{i}$
(b) (i) $-4\mathrm{i}$ (ii) i
(c) (i) $\frac{15}{17}+\frac{8}{17}\mathrm{i}$ (ii) $\frac{3}{5}-\frac{4}{5}\mathrm{i}$
(d) (i) $-1+\mathrm{i}$ (ii) $\frac{2}{5}-\frac{11}{5}\mathrm{i}$

3. (i) $z=\frac{7}{5}-\frac{3}{5}\mathrm{i}$ (ii) $z=6-\frac{1}{2}\mathrm{i}$

4. (i) $z=\frac{9}{2}-\frac{9}{2}\mathrm{i},\ w=-3-\frac{4}{3}\mathrm{i}$
(ii) $z=\frac{3}{10}+\frac{1}{10}\mathrm{i},\ w=\frac{3}{5}+\frac{1}{5}\mathrm{i}$

5. (i) $z=\frac{1}{2}-2\mathrm{i}$ (ii) $z=-\frac{2}{3}-3\mathrm{i}$

6. (i) $z=\frac{2}{3}+7\mathrm{i}$ (ii) $z=-\frac{5}{3}+\frac{1}{3}\mathrm{i}$

7. (a) (i) $z^*=3+x-\mathrm{i}y$
(ii) $z^*=x-2-\mathrm{i}y$
(b) (i) $z^*=x+2-\mathrm{i}(3y-1)$
(ii) $z^*=3-x-\mathrm{i}(y+3)$
(c) (i) $z^*=\frac{x(x^2+y^2+1)}{x^2+y^2}-\mathrm{i}\frac{y(x^2+y^2-1)}{x^2+y^2}$
(ii) $z^*=\frac{x(x^2+y^2-1)}{x^2+y^2}-\mathrm{i}\frac{y(x^2+y^2+1)}{x^2+y^2}$
(d) (i) $z^*=\mathrm{i}\frac{2xy}{x^2+y^2}$ (ii) $z^*=\frac{2x^2}{x^2+y^2}$

8. (a) (i) $z^*=1+2w^*$ (ii) $z^*=3w^*-1$
(b) (i) $z^*=1-3\mathrm{i}+w^*$ (ii) $z^*=2+\mathrm{i}-w^*$
(c) (i) $z^*=-\mathrm{i}w^*$ (ii) $z^*=(2+3\mathrm{i})w^*$
(d) (i) $z^*=(w^*)^2$ (ii) $z^*=\sqrt{w^*}$
(e) (i) $z^*=\frac{1}{w^*}$ (ii) $z^*=-\frac{3}{w^*}$
(f) (i) $z^*=(4-iw^*)^2$ (ii) $z^*=\frac{1}{1+w^*}$

9. (a) Re: $2x-3y$, Im: $3x-2$
(b) $z=-4-4\mathrm{i}$

10. $x=6,\ y=3$

12. $z=-\frac{1}{2}\mathrm{i}$

13. No solutions

14. (a) $2r\cos\theta$
(b) r^2
(c) $\mathrm{cis}(2.\ \theta)$

15. $z=\frac{6}{\sqrt{15}}-\mathrm{i}\frac{3}{\sqrt{15}}$

16. $z=\frac{9}{5}+\frac{12}{5}\mathrm{i}$

18. Re: $\frac{x(x+1)+y^2}{(x+1)^2+y^2}$
Im: $\frac{y}{(x+1)^2+y^2}$

19. Re: 0, Im: $\frac{\sin(\theta)}{1+\cos(\theta)}$

Exercise 15D

1. (a) (i) $(x-(1+\mathrm{i}))(x-(1-\mathrm{i}))$
(ii) $(x-(-3+4\mathrm{i}))(x-(-3-4\mathrm{i}))$
(b) (i) $\left(x-\left(-\frac{3}{2}+\frac{\sqrt{7}}{2}\mathrm{i}\right)\right)\left(x-\left(-\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)\right)$
(ii) $(x-(-1+2\mathrm{i}))(x-(-1-2\mathrm{i}))$
(c) (i) $\left(x-\left(\frac{1}{3}+\frac{\sqrt{29}}{3}\mathrm{i}\right)\right)\left(x-\left(\frac{1}{3}-\frac{\sqrt{29}}{3}\mathrm{i}\right)\right)$
(ii) $\left(x-\left(-\frac{2}{5}+\frac{\sqrt{6}}{5}\mathrm{i}\right)\right)\left(x-\left(-\frac{2}{5}-\frac{\sqrt{6}}{5}\mathrm{i}\right)\right)$

2. (a) (i) $(z-2\mathrm{i})(z+2\mathrm{i})$ (ii) $(z-5\mathrm{i})(z+5\mathrm{i})$
(b) (i) $(2z-7\mathrm{i})(2z+7\mathrm{i})$ (ii) $(3z-8\mathrm{i})(3z+8\mathrm{i})$
(c) (i) $(z-1)(z+1)(z-\mathrm{i})(z+\mathrm{i})$
(ii) $(2z-3)(2z+3)(2z-3\mathrm{i})(2z+3\mathrm{i})$

3. (a) (i) $a=0,\ b=25$ (ii) $a=0,\ b=9$
(b) (i) $a=-6,\ b=25$ (ii) $a=-2,\ b=5$

4. (a) (i) $z^2-(3\mathrm{i}+4)z+12\mathrm{i}$
(ii) $z^2+(\mathrm{i}-8)z-8\mathrm{i}$
(b) (i) $z^2-\mathrm{i}z+2$ (ii) $z^2-6\mathrm{i}z-5$
(c) (i) $z^2-6\mathrm{i}z-9$ (ii) $z^2+2\mathrm{i}z-1$
(d) (i) $z^2-(6+\mathrm{i})z+11-7\mathrm{i}$
(ii) $z^2-5z+7-\mathrm{i}$
(e) (i) $z^2-2(2+\mathrm{i})z+3+4\mathrm{i}$
(ii) $z^2-2(2-2\mathrm{i})z-8\mathrm{i}$

5. $-3i, 2$

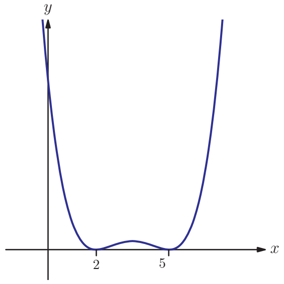
6. $1-2i, -3$

7. (a) $2+3i$
(b) $b=-2, c=5, d=26$

8. $z^4-10z^3+35z^2-90z+234$

9. $-2i$ and $3+i$ are also roots; there is only more root (because the order is 5), so it must be real.

10. (a) $f(x)=a(x-5)^2(x-2)^2$
(b)



11. (b) $-2i, \frac{-1\pm i\sqrt{3}}{2}$
(c) $(z^2+4)(z^2+z+1)$

Exercise 15E

1. (a) (i) sum $=-4$, product $=-1$
(ii) sum $=-6$, product $=3$
(b) (i) $0, -\frac{3}{4}$
(ii) $0, \frac{8}{5}$
(c) (i) $-1, -\frac{2}{3}$
(ii) $1, -\frac{3}{2}$
(d) (i) $\frac{5}{4}, -\frac{3}{4}$
(ii) $\frac{8}{9}, -\frac{5}{9}$

2. (a) (i) $a=-6, b=-6$ (ii) $a=-8, c=-4$
(b) (i) $a=3, b=12$ (ii) $b=-1, d=-5$

3. (a) $-3i, 3+i$
(b) $a=6, d=90$

4. $-\frac{1}{8}$

5. (i) 4 (ii) $\frac{1}{3}$

6. (a) 1

7. $x^2-\frac{3}{2}x+\frac{5}{2}=0$

8. $\frac{9}{4}$

10. (b) (i) $-\frac{b}{a}, -\frac{c}{a}$ (c) (ii) $\frac{25}{4}, 1$
(iii) $4x^3-20x^2+25x-4=0$

11. (b) $-2i, \frac{-1\pm\sqrt{3}i}{2}$ (c) $(z^2+4)(z^2+z+1)$

Exercise 15F

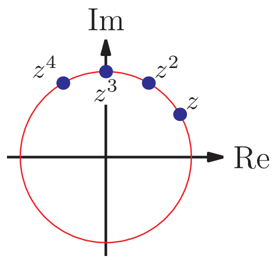
1. (a) (i) $21\text{cis}\left(\frac{11\pi}{30}\right)$ (ii) $4\text{cis}\left(\frac{2\pi}{9}\right)$
(b) (i) $4\text{cis}(4)$ (ii) $3\text{cis}\left(\frac{-5\pi}{14}\right)$
(c) (i) $64\text{cis}\left(\frac{-4\pi}{5}\right)$ (ii) $81\text{cis}\left(\frac{2\pi}{3}\right)$

2. (a) (i) $\cos\left(\frac{-11\pi}{12}\right)+i\sin\left(\frac{-11\pi}{12}\right)$
(ii) $\cos\left(\frac{13\pi}{20}\right)+i\sin\left(\frac{13\pi}{20}\right)$
(b) (i) $\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)$
(ii) $\cos\left(\frac{4\pi}{15}\right)+i\sin\left(\frac{4\pi}{15}\right)$
(c) (i) $\cos\left(\frac{7\pi}{20}\right)+i\sin\left(\frac{7\pi}{20}\right)$
(ii) $\cos\left(\frac{-5\pi}{12}\right)+i\sin\left(\frac{-5\pi}{12}\right)$
(d) (i) $\cos\left(\frac{9\pi}{20}\right)+i\sin\left(\frac{9\pi}{20}\right)$
(ii) $\cos\left(\frac{-13\pi}{20}\right)+i\sin\left(\frac{-13\pi}{20}\right)$
(e) (i) $\cos\left(\frac{-11\pi}{12}\right)+i\sin\left(\frac{-11\pi}{12}\right)$
(ii) $\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)$
(f) (i) $\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)$
(ii) $\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$

3. (a) $z^2 = \text{cis}\left(\frac{\pi}{3}\right), z^3 = \text{cis}\left(\frac{\pi}{2}\right),$

$z^4 = \text{cis}\left(\frac{2\pi}{3}\right)$

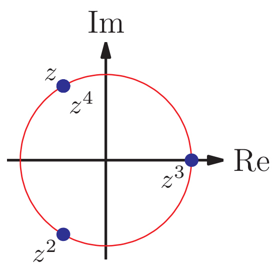
(b)



4. (a) $z^2 = \text{cis}\left(\frac{-2\pi}{3}\right), z^3 = \text{cis}(0),$

$z^4 = \text{cis}\left(\frac{2\pi}{3}\right)$

(b)



(c) $n = 1 + 3k, k \in \mathbb{Z}^+$

5. $\text{mod} = 2,\ \arg = \frac{\pi}{3}$

$32\text{cis}\frac{-\pi}{3}$

$16 - 16\sqrt{3}i$

6. (a) $2\text{cis}\left(\frac{3\pi}{4}\right)$

(b) 64i

8. (a) $4 - i$

(b) $B\left(\frac{\sqrt{3}-4}{2}, \frac{4\sqrt{3}+1}{2}\right),$

$C\left(-\frac{4+\sqrt{3}}{2}, -\frac{4\sqrt{3}-1}{2}\right)$

Exercise 15G

1. (a) (i) $\frac{3\sqrt{3}}{2} + \frac{1}{2}i$ (ii) $2\sqrt{2} + 2\sqrt{2}i$

(b) (i) -4 (ii) 5

(c) (i) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (ii) $-2i$

2. (a) (i) $5\sqrt{2}e^{i\frac{\pi}{4}}$ (ii) $4e^{-i\frac{\pi}{6}}$

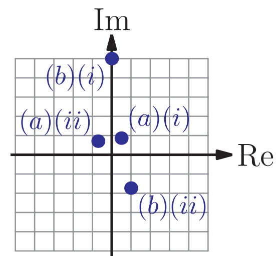
(b) (i) $\frac{1}{\sqrt{2}}e^{i\frac{3\pi}{4}}$ (ii) $\sqrt{13}e^{-0.983i}$

(c) (i) $4e^{-i\frac{\pi}{2}}$ (ii) $5e^{i\pi}$

3. (a) (i) $20e^{i\frac{5\pi}{12}}$ (ii) $\frac{1}{2}e^{i\frac{\pi}{2}}$

(b) (i) $\frac{8}{25}e^{i\frac{\pi}{12}}$ (ii) $2e^{-i\frac{\pi}{2}}$

4.



5. 3.76

6. $\cos(\ln 5) + i\sin(\ln 5)$

7. $9\cos(\ln 3) - 9i\sin(\ln 3)$

8. $-i\ln\left(2 \pm \sqrt{3}\right)$

9. (a) $e^{i\frac{\pi}{2}}$ (b) $e^{-\frac{\pi}{2}}$

Exercise 15H

1. (a) (i) $3,\ 3e^{\frac{2\pi i}{3}},\ 3e^{-\frac{2\pi i}{3}}$

(ii) $\sqrt[3]{100}, \sqrt[3]{100}e^{\frac{2\pi i}{3}},$

$\sqrt[3]{100}e^{-\frac{2\pi i}{3}}$

(b) (i) $2e^{\frac{\pi i}{6}},\ 2e^{\frac{5\pi i}{6}},\ 2e^{\frac{3\pi i}{2}}$

(ii) $e^{\frac{\pi i}{6}},\ e^{\frac{5\pi i}{6}},\ e^{\frac{3\pi i}{2}}$

(c) (i) $2^{\frac{1}{6}}e^{\frac{\pi i}{12}}, 2^{\frac{1}{6}}e^{\frac{3\pi i}{4}}, 2^{\frac{1}{6}}e^{\frac{17\pi i}{12}}$

(ii) $7^{\frac{1}{6}}e^{-0.238i}, 2^{\frac{1}{6}}$

$e^{1.86i}, 2^{\frac{1}{6}}e^{-2.33i}$

2. $\text{cis}(0), \text{cis}\left(\frac{\pi}{2}\right),$

$\text{cis}(\pi), \text{cis}\left(\frac{3\pi}{2}\right)$

3. $-2, 1+\sqrt{3}i, 1-\sqrt{3}i$

4. $-\frac{\sqrt{2}+\sqrt{6}}{2}+\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)i$,

$\frac{\sqrt{2}-\sqrt{6}}{2}-\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)i$,

$\sqrt{2}-\sqrt{2}i$

5. $3\text{cis}\left(\frac{3\pi}{8}\right), 3\text{cis}\left(\frac{7\pi}{8}\right)$,

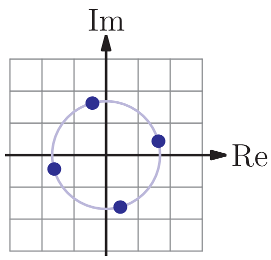
$3\text{cis}\left(\frac{-\pi}{8}\right), 3\text{cis}\left(\frac{-5\pi}{8}\right)$

6. (a) $8e^{i\frac{\pi}{3}}$

(b) $8^{\frac{1}{4}}e^{i\frac{\pi}{12}}, 8^{\frac{1}{4}}e^{i\frac{7\pi}{12}}$,

$8^{\frac{1}{4}}e^{i\frac{13\pi}{12}}, 8^{\frac{1}{4}}e^{i\frac{19\pi}{12}}$

(c)



7. (a) $\sqrt{2}\pm\sqrt{2}i, -\sqrt{2}\pm\sqrt{2}i$

(b) $(z^2+2\sqrt{2}z+4)$

$(z^2-2\sqrt{2}z+4)$

ANSWER HINT (7)

$(1+w_1+w_2+w_3+w_4+w_5)\times w$

9. a^2+b^2-ab

10. (a) $-1, e^{\frac{i\pi}{3}}, e^{-\frac{i\pi}{3}}$

(b) $x^3+6x^2+12x+8$

(c) $-3, -\frac{3}{2}\pm\frac{\sqrt{3}}{2}i$

Exercise 15I

1. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

2. (a) $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(b) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

3. (b) $A=2, B=10, C=20$

4. (b) $(z+z^{-1})^6 = z^6+6z^4+1$

$5z^2+20+15z^{-2}+6z^{-4}+z^{-6}$

$(z-z^{-1})^6 = z^6-6z^4+$

$15z^2-20+15z^{-2}-6z^{-4}+z^{-6}$

5. (a) $\text{Re}: \cos^5\theta$

$-10\cos^3\theta$

$\text{Im}: 5\cos^4\theta\sin\theta - 10\cos^2\theta$

$\sin^3\theta + \sin^5\theta$

(c) 5

Mixed examination practice 15

Short questions

1. $-\frac{\sqrt{3}}{2}+\frac{7}{2}i$

2. $w=5i,\ z=3+2i$

3. $a=-3, b=7, c=-5$

4. $-2-\frac{3}{8}i$

5. $-\frac{1}{64}$

6. $a=-9,\ b=33$

8. $64i$

9. $\frac{1}{\sqrt{3}}$

10. (a) $w=iz$

11. 2

12. 5

13. (a) -6

(b) -1

14. (b) $x^2+301x+8=0$

16. $\tan\theta$

17. $\frac{1}{2}$

Long questions

1. (a) $z_1=\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

$z_2=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$

(c) $\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$

2. (a) $\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$

(b) $c=1$

(c) $m=-6,\ n=4$

3. (a) (i) $Z^3\ \cos^3\theta-3\cos\theta\sin^2\theta+(3\cos^2\theta\sin\theta-\sin^3\theta)i$

(c) $\frac{23\sqrt{2}}{20}$

5. (a) (i) $x_1+x_2+x_3=-\frac{b}{a}, x_1+x_2+x_3=-\frac{d}{a}$

6. (c) (ii) $\frac{2}{3}\pm\frac{\sqrt{5}}{3}i,\ \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$

7. (a) $\omega^2=e^{\frac{4\pi}{5}i}, \omega^3=e^{-\frac{4\pi}{5}i}, \omega^4=e^{-\frac{2\pi}{5}i}$

(c) $4\cos^2\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{2\pi}{5}\right)-1=0$

8. (a) $\cos3\theta=4\cos^3\theta-3\cos\theta$

$\sin3\theta=3\sin\theta-4\sin^3\theta$

(f) $2-\sqrt{3}$

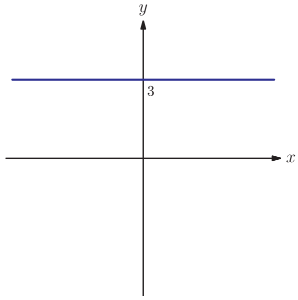
9. (b) (i) a (ii) $b(\cos\theta+i\sin\theta)$

(iii) $AB=\sqrt{b^2\sin^2\theta+(a-b\cos\theta)^2}$

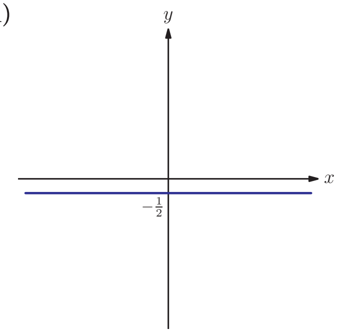
Chapter 16

Exercise 16A

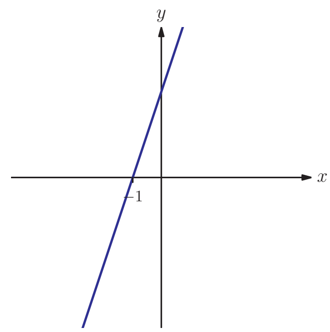
1. (a) (i)



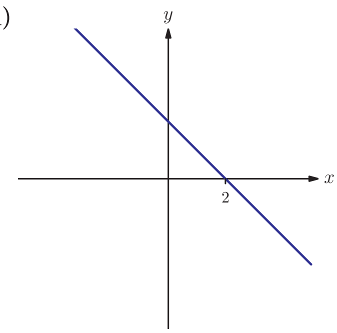
(ii)



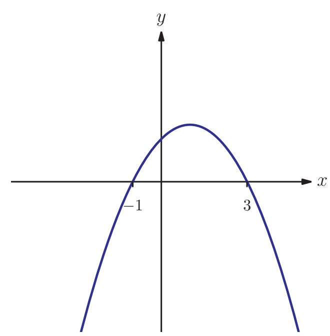
(b) (i)



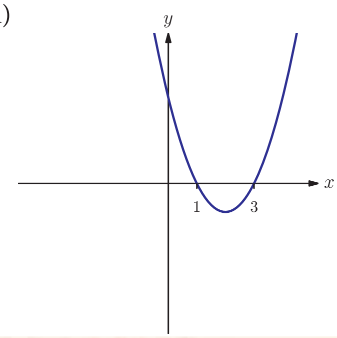
(ii)



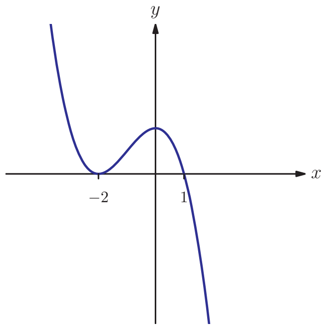
(c) (i)



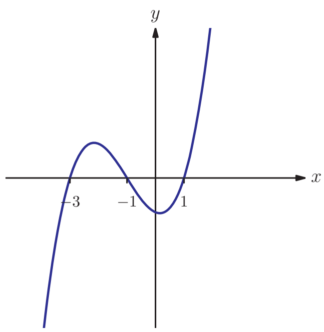
(ii)



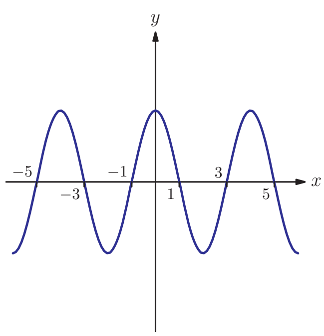
(d) (i)



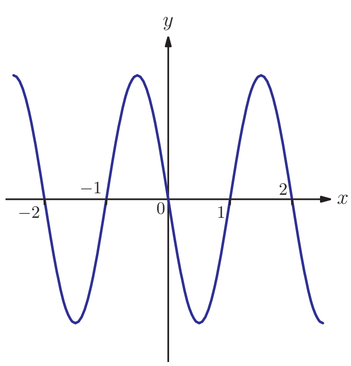
(ii)



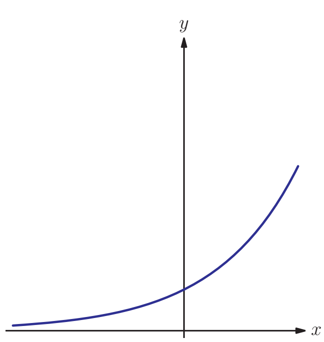
(e) (i)



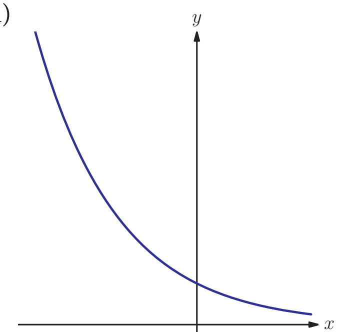
(ii)



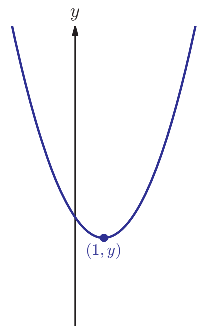
(f) (i)



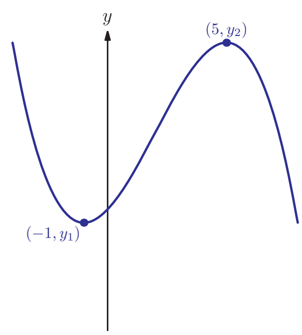
(ii)



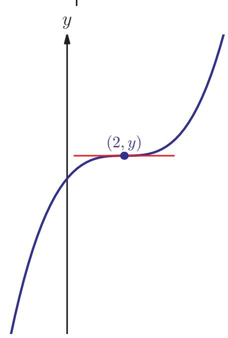
2. (a)



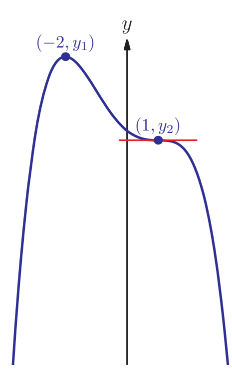
(b)



(c)



(d)



3. (a) Sometimes true
(b) Sometimes true
(c) Always true
(d) Sometimes true
(e) Sometimes true
(f) Sometimes true

Exercise 16B

1. (a) (i) $f'(x)=3x^2$ (ii) $f'(x)=4x^3$
(b) (i) $f'(x)=-4$ (ii) $f'(x)=6x$
(c) (i) $f'(x)=2x$ (ii) $f'(x)=2x-3$

Exercise 16C

1. (a) (i) $y'=4x^3$ (ii) $y'=1$
(b) (i) $y'=21x^6$ (ii) $y'=-20x^4$
(c) (i) $y'=0$ (ii) $y'=0$
(d) (i) $y'=12x^2-10x+2$
(ii) $y'=8x^3+9x^2-1$
(e) (i) $y'=2x^5$ (ii) $y'=-\frac{3}{2}x$
(f) (i) $y'=7-\frac{3}{2}x^2$ (ii) $y'=-20x^3+x^4$
(g) (i) $y'=\frac{3}{2}x^{\frac{1}{2}}$ (ii) $y'=\frac{2}{3}x^{-\frac{1}{3}}$
(h) (i) $y'=8x^{\frac{1}{3}}$ (ii) $y'=\frac{1}{2}x^{-\frac{1}{6}}$
(i) (i) $y'=12x^3-2x+6x^{-\frac{3}{5}}$
(ii) $y'=3x^2-x^{\frac{2}{3}}+\frac{2}{3}x^{-\frac{1}{2}}$
(j) (i) $y'=-x^{-2}$ (ii) $y'=3x^{-4}$
(k) (i) $y'=-\frac{1}{2}x^{-\frac{3}{2}}$ (ii) $y'=6x^{-\frac{7}{4}}$
(l) (i) $y'=5\times\frac{4}{3}x^{-\frac{7}{2}}$ (ii) $y'=x^{-\frac{10}{7}}-8x^{-7}$

2. (a) (i) $y'=\frac{1}{3}x^{-\frac{2}{3}}$ (ii) $y'=\frac{4}{5}x^{-\frac{1}{5}}$
(b) (i) $y'=-6x^{-3}$ (ii) $y'=4x^{-11}$
(c) (i) $y'=-\frac{1}{2}x^{-\frac{3}{2}}$ (ii) $y'=-2x^{-\frac{7}{4}}$
(d) (i) $y'=9x^2-8x$
(ii) $y'=\frac{7}{2}x^{\frac{5}{2}}-3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}$
(e) (i) $y'=\frac{4}{3}x^{\frac{1}{3}}+\frac{2}{3}x^{-\frac{2}{3}}-1$
(ii) $y'=2x-8x^{-3}$
(f) (i) $y'=9x^2+2x^{-2}$
(ii) $y'=\frac{15}{2}x^{\frac{2}{3}}-\frac{1}{2}x^{-\frac{4}{3}}$

3. (a) (i) -1 (ii) $\frac{3}{2}$
(b) (i) $-2x-1$ (ii) $4x^3+2$

Exercise 16D

1. (a) $\frac{dz}{dt}$ (b) $\frac{dQ}{dP}$
(c) $\frac{dR}{dm}$ (d) $\frac{dV}{dt}$
(e) $\frac{dy}{dx}$ (f) $\frac{d^2z}{dy^2}$
(g) $\frac{d^2H}{dm^2}$

2. (a) (i) $\frac{5}{3}x^{-\frac{2}{3}}$ (ii) $15q^4$
(b) (i) $3-7t^{-2}$ (ii) $1-c^{-2}$
(c) (i) $18+6x$ (ii) $6t^{-3}$

3. (a) (i) 30 (ii) $\frac{227}{36}$
Did you think about doing this on the calculator?
(b) (i) 7 (ii) –29999.8
(c) (i) 12 (ii) –10
(d) (i) 24 (ii) 32
(e) (i) 6 (ii) $\frac{7}{2\sqrt{6}}$

4. (a) (i) $2ax+1-a$ (ii) $3x^2$
(b) (i) $\frac{1}{2}\sqrt{\frac{b}{a}}$ (ii) $6a^2v$

5. (a) (i) 54 (ii) 384
(b) (i) 8 (ii) $\frac{1}{108}$
(c) (i) 0 (ii) 42

6. (a) (i) ± 2 (ii) 1
(b) (i) ± 17 (ii) 6

7. (a) (i) $x\in\left]-\infty,-\frac{2\sqrt{3}}{3}\right[\cup\left]\frac{2\sqrt{3}}{3},\infty3\right[$
(ii) $x\in]0,2[$
(b) (i) $x\in\left]-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right[$ (ii) $x>\frac{1}{4}$
(c) (i) $x>0$ (ii) $x>-2$
(d) (i) $1<x<2$ (ii) $-3<x<3$

8. $(-0.199, 0.913)$, $(1.29, -0.181), (3.91, 30.3)$

9. $x > \frac{-1+\sqrt{22}}{3}$ and $x < \frac{-1-\sqrt{22}}{3}$

10. $-3 < x < 1$ and $x > 1$

11. $n!$

Exercise 16E

1. (a) (i) $3\cos x$ (ii) $-2\sin x$
   (b) (i) $2+5\sin x$ (ii) $\sec^2 x$
   (c) (i) $\frac{\cos x - 2\sin x}{5}$ (ii) $\frac{1}{2}\sec^2 x - \frac{1}{3}\cos x$

2. π

3. $\frac{22-\pi^2}{12}$

4. $x = \frac{\pi}{4}, \frac{5\pi}{4}$

5. $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Exercise 16F

1. (a) (i) $3e^x$ (ii) $\frac{2e^x}{5}$
   (b) (i) $\frac{-2}{x}$ (ii) $\frac{1}{3x}$
   (c) (i) $\frac{1}{5x} - 3 + 4e^x$ (ii) $-\frac{e^x}{2} + \frac{3}{x}$

2. (i) $2 - \frac{7}{\ln 4}$
   (ii) $3 - \frac{1}{2\ln 3}$

3. $x = \ln 3$

4. $x = 3$

5. (a) (i) $\frac{3}{x}$ (ii) $\frac{1}{x}$
   (b) (i) e^{3+x} (ii) e^{x-3}
   (c) (i) $2x$ (ii) $3e^2x^2$
   (d) (i) $\frac{1}{x\ln 3}$ (ii) $\frac{4}{x\ln 6}$

Exercise 16G

1. (a) Tangent: $11x - 4y = 4$
   Normal: $4x + 11y = 126$
   (b) Tangent: $4x - y + 1 - \pi = 0$
   Normal: $4x + 16y - \pi - 16 = 0$
   (c) Tangent: $5x + y + 2 - 10\ln 5 = 0$
   Normal: $x - 5y - 10 - \ln 25 = 0$

2. $\left(\frac{1}{16}, \frac{7}{16}\right)$

3. $y = 3x - \ln 4 + 2$

4. -0.0541 or 2.05

5. $(-1, 4)$

6. $(0.410, 0.348)$

7. (a) $\sqrt{3}$ (b) $2.3°$

Exercise 16H

1. (a) (i) $(0,0)$ Local Maximum
   $\left(\frac{10}{3}, -\frac{500}{27}\right)$ Local Minimum
   In the exam you should give the value of $\frac{d^2y}{dx^2}$ to justify your choice.
   (ii) $(0,0)$ Local Maximum
   $(-2, 16)$ Local Minimum
   (b) (i) $(-2, -16)$
   Local Minimum
   $\left(\frac{2\pi}{3}, \frac{3\sqrt{3}+2\pi}{6}\right)$
   Local Maximum
   (ii) $\left(-\frac{2\pi}{3}, -\frac{3\sqrt{3}+2\pi}{6}\right)$
   Local Minimum
   $(0,3)$ Local Maximum
   (c) (i) $(\pi, -1)$ Local Minimum
   $(2\pi, 3)$ Local Maximun
   (ii) $(4, \ln 4 - 2)$ Local Maximum
   $(\ln(2.5), 5 - 5\ln(2.5))$
   Local Minimum

2. $(x_1, f(x_1))$ could be a point of inflexion.

3. $(-4, 92)$ Local Maximum
   $(2, -16)$ Local Minimum

4. $\left(\frac{1}{4}, -\frac{1}{4}\right)$ Local Minimum

5. $(0.245, 4.12)$ Local Maximum

$(3.39, -4.12)$ Local Minimum

7. $y \geq -21$

8. $y \geq 6 - 4\ln 4$

9. $(0,0)$ Local Minimum

$\left(-\frac{4}{k}, -\frac{32}{k^2}\right)$ Local Maximum

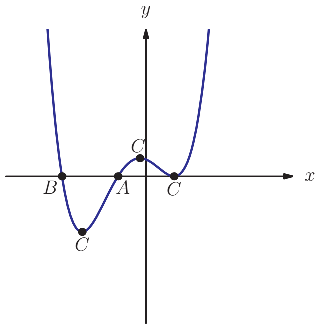
Exercise 16I

1. $\left(\ln 2, 2 - (\ln 2)^2\right)$

2. $(1,4), (-1,-10)$

4. $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$
As the question doesn't state how many points of inflexion there are, you need to show that both of these are by considering the gradient either side.

6.



Exercise 16J

1. Min 1, max e

2. (a) 225 m (b) 60 m

3. Min $-6\sqrt{3}$, max 80

4. Min $3 - 3\ln 3$, max $e^2 - 6$

5. Min 0, max 4π

6. 2

7. 2.25

8. Min 2, max 4

9. (a) 40 million litres (b) $\frac{\pi}{2}, \frac{3\pi}{2}$

10 (a) 2 g/tray (b) 0 g/tray (c) 1.94 g/tray

11. (a) 4 litres (b) 41.5 litres
(c) 20 seconds

12. (a) $2x - \frac{x^2}{10}$ (b) $10\,\text{m}^2$ (c) $0 < x < 10$

Mixed examination practice 16

Short questions

1. $y = e^{\frac{\pi}{2}}x - \frac{\pi}{2}e^{\frac{\pi}{2}} + e^{\frac{\pi}{2}} + 2$

2. $x = 2$

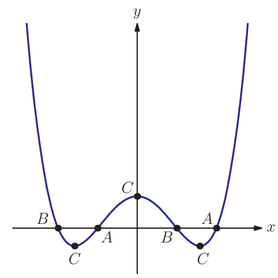
3. $b = 8, c = -7$

4. $\left(2, -\frac{2}{3}\right)$

5. $\frac{\pi}{6} \pm k\pi$ local minima $-\frac{\pi}{6} \pm k\pi$ local maxima $k \in \mathbb{Z}$

6. $-\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2$

7.



8. $a = \frac{1}{\sqrt[4]{3}}$

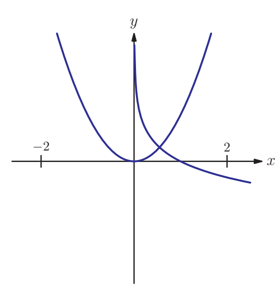
Long questions

1. (b) $3a + b = 6$ (c) $a = 1, b = 3$ (d) $(-7, -192)$

2. (a) (i) $\left(-\frac{1}{3}, \frac{86}{27}\right)$ (ii) $\left(\frac{1}{3}, \frac{70}{27}\right)$

(b) (i) $y = -\frac{8}{9}x + \frac{78}{27}$ (ii) $\frac{3 \pm 2\sqrt{3}}{9}$

3. (a)



(b) 0.548 (c) 0.302

4. (a) (i) 11 000 (ii) 2.42 h

(b) (i) $\frac{dP}{dt} = e^t - 3$ (ii) 8.7 h

(c) (i) $\frac{d^2P}{dt^2} = e^t$, the rate of change of the growth rate of the bacteria

(ii) 9704, $\frac{d^2P}{dt^2} = 3 > 0$, hence local minimum

Chapter 17

Exercise 17A

1. (a) $\frac{3}{4}x^4, \frac{3}{4}x^4+1; \frac{3}{4}x^4-1$ (b) $1, -1, 2$

2. (a) (i) $x^3(+c)$ (ii) $x^5(+c)$
   (b) (i) $\frac{1}{x}(+c)$ (ii) $\frac{1}{x^4}(+c)$
   (c) (i) $\sqrt{x}(+c)$ (ii) $\sqrt[3]{x}(+c)$
   (d) (i) $2x^5(+c)$ (ii) $4x^3(+c)$

Exercise 17B

1. (a) $\frac{3}{4}x^4, \frac{3}{4}x^4+1, \frac{3}{4}x^4-1$ (b) $1, -1, 2$

2. (a) (i) $\frac{7}{5}x^5+c$ (ii) $\frac{1}{9}x^3+c$
   (b) (i) $-\frac{1}{2t}+c$ (ii) $-\frac{4}{y^2}+c$

Exercise 17C

1. (a) (i) $x^9(+c)$ (ii) $x^{12}(+c)$
   (b) (i) $\frac{x^2}{2}(+c)$ (ii) $\frac{x^4}{4}(+c)$
   (c) (i) $9x(+c)$ (ii) $\frac{x}{2}(+c)$
   (d) (i) $\frac{x^6}{2}(+c)$ (ii) $\frac{9}{5}x^5(+c)$
   (e) (i) $2x^{\frac{3}{2}}(+c)$ (ii) $\frac{9}{4}x^{\frac{4}{3}}(+c)$
   (f) (i) $-\frac{5}{x}(+c)$ (ii) $-\frac{1}{x^2}(+c)$

2. (a) (i) $3t(+c)$ (ii) $7z(+c)$
   (b) (i) $\frac{q^6}{6}(+c)$ (ii) $\frac{r^{11}}{11}(+c)$
   (c) (i) $\frac{15}{2}g^{\frac{8}{5}}(+c)$ (ii) $\frac{10}{9}y^{\frac{9}{2}}(+c)$
   (d) (i) $-\frac{4}{h}(+c)$ (ii) $-\frac{1}{3p^3}(+c)$

3. (a) (i) $\frac{x^3}{3}+\frac{x^4}{4}+c$ (ii) $\frac{x^5}{5}-x^2+5x+c$
   (b) (i) $-\frac{1}{6t^2}-\frac{1}{12t^3}+c$ (ii) $-\frac{5}{v}+\frac{1}{v^4}+c$
   (c) (i) $\frac{2}{5}x^{\frac{5}{2}}+c$ (ii) $\frac{18}{7}x^{\frac{7}{6}}+c$
   (d) (i) $\frac{x^4}{4}+x^3+\frac{3x^2}{2}+x+c$
   (ii) $\frac{x^4}{4}+\frac{4x^3}{3}+2x^2+c$

4. $2\sqrt{x}+\frac{2}{3}x^{\frac{3}{2}}+c$

Exercise 17D

1. (a) (i) $2\ln x+c$ (ii) $3\ln x+c$
   (b) (i) $\frac{1}{2}\ln x+c$ (ii) $\frac{1}{3}\ln x+c$
   (c) (i) $\frac{x^2}{2}-\ln x+c$ (ii) $\frac{x^3}{3}+5\ln x+c$
   (d) (i) $3\ln x-\frac{2}{x}+c$ (ii) $\ln x+\frac{2}{\sqrt{x}}+c$

2. (a) (i) $5e^x+c$ (ii) $9e^x+c$
   (b) (i) $\frac{2e^x}{5}+c$ (ii) $\frac{7e^x}{11}+c$
   (c) (i) $\frac{e^x}{2}+\frac{3x^2}{4}+c$ (ii) $\frac{e^x}{5}+\frac{x^4}{20}+C$

Exercise 17E

1. (a) (i) $-\cos x-\sin x+c$
   (ii) $3\sin x-4\cos x+c$
   (b) (i) $x+\ln|\sec x|+c$
   (ii) $-\frac{1}{2}\cos x+\frac{1}{3}\ln|\sec x|+c$
   (c) (i) $\frac{x^2}{14}-\frac{\cos x}{7}+c$
   (ii) $\frac{1}{9}x^{\frac{3}{2}}+\frac{\sin x}{6}+c$
   (d) (i) $x-\sin x+\cos x+c$
   (ii) $-2\cos x-\sin x+c$

2. $\frac{1}{2}\ln|\sec x|+\frac{x}{2}+c$

3. $\sin x-\cos x+c$

ANSWER HINT (3)
Use an identity

Exercise 17F

1. (a) (i) $y=\frac{x^2}{2}+5$ (ii) $y=2x^3+5$
   (b) (i) $y=2\sqrt{x}+4$ (ii) $y=-\frac{1}{x}+4$
   (c) (i) $y=2e^x+2x-1-2e$ (ii) $y=e^x-5$
   (d) (i) $y=x+\ln x-1$ (ii) $y=\frac{1}{2}\ln x+4$
   (e) (i) $y=\sin x-\cos x$ (ii) $y=3\ln|\sec x|+4$

2. (a) $f(x)=\frac{1}{2}\ln x+c$

(b) $y=\frac{1}{2}\ln x-\frac{1}{2}\ln 2+7$

3. (a) $x=-2$, $y''=2x=4<0$, hence local maximum

4. $y=\ln\left|\frac{e^5}{x}\right|$

Exercise 17G

1. (a) (i) 320 (ii) 28.5
   (b) (i) 1 (ii) -2
   (c) (i) $e-1$ (ii) $3e-\frac{3}{e}$
2. (a) (i) 0.995 (3SF) (ii) 0.0997(3SF)
   (b) (i) 1.46(3SF) (ii) 1
3. $e^{\pi}+\pi+1$
5. 20
6. $a=16$

Exercise 17H

1. (a) (i) $\frac{7}{3}$ (ii) $\frac{1}{4}$
   (b) (i) $\frac{2}{3}$ (ii) $\frac{22}{3}$
   (c) (i) $\frac{11}{4}$ (ii) $13\frac{1}{6}$
2. 9
3. (a) 6
   (b) $\frac{22}{3}$
4. 2

ANSWER HINT(4)
There is an easier way than splitting this integral into two parts

5. $\frac{9}{2}$

Exercise 17I

1. (a) (i) 9.13 (ii) 2.50
   (b) (i) 0.828 (ii) 41.3
   (c) (i) 2.35 (ii) 5.38
2. 6
3. e^2-3
4. 25
5. $a=\sqrt{b}$; $a=\sqrt[3]{2b^{\frac{3}{2}}+2-3b}$;
   $a=1+\sqrt{3}$; Area $=3+2\sqrt{3}$

Exercise 17J

1. (a) (i) $\frac{32}{3}$ (ii) $\frac{1}{6}$
   (b) (i) 9 (ii) $\frac{1}{3}$
   (c) (i) $\frac{9}{8}$ (ii) $\frac{1}{3}$
2. $\frac{32}{3}$
3. $e^2-\frac{11}{3}$
4. 0.462
6. $2-\sqrt{2}$
7. 8
8. $m=4$

Mixed examination practice 17

Short questions

1. $f(x)=\frac{1}{2}-\cos x$
2. 0.201
3. $\frac{4k^3}{3}$
4. 3

ANSWER HINT(4)
You did not need to do anything with the red area to do this question

5. $\ln x+\frac{2}{5}x^{\frac{5}{2}}+c$
6. (a) $a=\sqrt{2}$ (b) $\frac{1}{2}$
7. $2\sqrt{3}-\frac{2}{3}\pi$
8. (a) (Local) Minimum
   (b) $x^3+3x^2-45x+100$

Long questions

1. (a) $(-a,0)$ and $(3a,8a^2)$

 (b) $\frac{64}{3}a^3$ (c) $\frac{15}{16}$

2. (b) $\arcsin a$ (c) $1-\sqrt{1-a^2}$

 (d) $a\arcsin a+\sqrt{1-a^2}-1$

Chapter 18

Exercise 18A

1. (a) (i) $7(2x-3)(x^2-3x+1)^6$

 (ii) $15x^2(x^3+1)^4$

 (b) (i) $(2x-2)e^{x^2-2x}$

 (ii) $-3x^2e^{4-x^3}$

 (c) (i) $-6e^x(2e^x+1)^{-4}$

 (ii) $20e^x(2-5e^x)^{-5}$

 (d) (i) $6x\cos(3x^2+1)$

 (ii) $-(2x+2)\sin(x^2+2x)$

 (e) (i) $-3\sin x\cos^2 x$

 (ii) $4\cos x\sin^3 x$

 (f) (i) $\frac{2-15x^2}{2x-5x^3}$ (ii) $\frac{8x}{4x^2-1}$

 (g) (i) $\frac{16}{x}(4\ln x-1)^3$ (ii) $-\frac{5}{x}(\ln x+3)^{-6}$

2. (a) (i) $10(2x+3)^4$ (ii) $32(4x-1)^7$

 (b) (i) $4(5-x)^{-5}$ (ii) $7(1-x)^{-8}$

 (c) (i) $4\sin(1-4x)$ (ii) $\sin(2-x)$

 (d) (i) $\frac{5}{5x+2}$ (ii) $\frac{1}{x-4}$

 (e) (i) $-3\csc^2(3x)$ (ii) $-5\csc(5x)\cot(5x)$

 (f) (i) $2\sec(2x+1)\tan(2x+1)$

 (ii) $-\sec^2(1-x)$

3. (a) (i) $6\sec^2(3x)\tan(3x)$

 (ii) $4\tan(2x)\sec^2(2x)$

 (b) (i) $6\sin(3x)\cos(3x)e^{\sin^2(3x)}$

 (ii) $\frac{2\ln(2x)}{x}e^{(\ln 2x)^2}$

 (c) (i) $-16\sin(2x)\cos(2x)(1-2\sin^2(2x))$

 (ii) $-24\sin 3x(4\cos 3x+1)$

 (d) (i) $\frac{6\sin 2x}{1-3\cos 2x}$ (ii) $\frac{5\sin 5x}{2-\cos 5x}$

4. $y=\frac{27\sqrt{2}}{8}x-\frac{77}{12}$

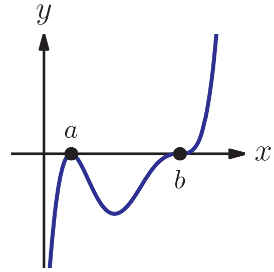
5. $\left(\frac{\pi}{2},e\right),\left(\frac{3\pi}{2},e^{-1}\right)$

6. (a) $-2\csc^2 x\cot x$ (b) $x=-\frac{\pi}{4},\frac{3\pi}{4}$

7. 7

8. (a) $\frac{qa+pb}{p+q}$

 (b)



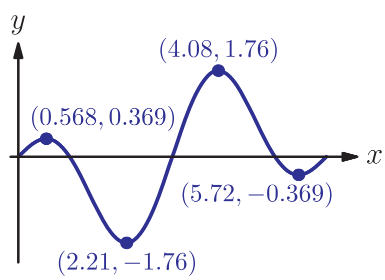
 (c) q is even.

9. (a) Left post $e^{-1}>e^{-4}$ (c) $\sqrt[3]{2}+\frac{1}{\sqrt[3]{4}}$

10. (a) $x=0,\frac{\pi}{3},\pi,\frac{5\pi}{3},2\pi$

 (b) $x=0.568, 2.21, 4.08, 5.72$

 (c)



Exercise 18B

1. (a) (i) $2x\cos x-x^2\sin x$

 (ii) $-x^{-2}\sin x+x^{-1}\cos x$

 (b) (i) $-2x^{-3}\ln x+x^{-3}$ (ii) $\ln x+1$

 (c) (i) $3x^2\sqrt{2x+1}+x^3(2x+1)^{-\frac{1}{2}}$

 (ii) $-x^{-2}\sqrt{4x}+2x^{-1}(4x)^{-\frac{1}{2}}$

 (d) (i) $2e^{2x}\tan x+e^{2x}\sec^2 x$

 (ii) $e^{x+1}\sec 3x+3e^{x+1}\sec 3x\tan 3x$

2. (a) (i) $3(x+1)^3(x-2)^4(3x-1)$

(ii) $(x-3)^6(x+5)^3(11x+23)$

(b) (i) $(2x-1)^3(1-3x)^2$ $(-42x+17)$

(ii) $(1-x)^4(4x+1)(-28x+3)$

3. $(6x^2+4x+3)e^{2x}$

4. $(9x^2+12x+2)e^{3x}$

5. $x=-\frac{1}{2},2$

6. $x=3,-\frac{1}{3},\frac{7}{4}$

7. (0.538, 0.473), (1.82, – 0.87), (3.29, 0.962), (4.81, –0.933)

8. $e^x(1+x)\cos(xe^x)$

9. (a) $\ln x+1$

(b) $x\ln x-x$

10. $\left(\frac{3\pi}{4},-\frac{\sqrt{2}}{2}e^{-\frac{3\pi}{4}}\right)$

11. $a=4, b=5$

12. (a) $y=e^{\ln x^x}$ (b) $(\ln x+1)x^x$

(c) $\left(e^{-1},e^{-e^{-1}}\right)$

Exercise 18C

1. (a) (i) $\frac{2}{(x+1)^2}$ (ii) $\frac{-5}{(x-3)^2}$

(b) (i) $\frac{x(2x+1)^{-\frac{1}{2}}-(2x+1)^{\frac{1}{2}}}{x^2}$

(ii) $\frac{2x(x-1)^{\frac{1}{2}}-\frac{1}{2}x^2(x-1)^{-\frac{1}{2}}}{x-1}$

(c) (i) $\frac{2(x^2-x-2)}{(x^2+2)^2}$ (ii) $-\frac{x^2+2x+4}{(1+x)^2}$

(d) (i) $\frac{1-\ln 3x}{x^2}$ (ii) $\frac{x-2x\ln 2x}{x^4}$

2. $y=\frac{\pi^2}{4}x+\frac{16-\pi^4}{8\pi}$

3. $(0,0),(1,1)$

4. $a=-1$

5. $\left(e,\frac{1}{e}\right)$ Local maximum

6. $x\in(0,2),\ x\neq 1$

7. $a=3,\ b=4,\ p=\frac{3}{2}$

Exercise 18D

1. (a) (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$

(b) (i) 0 (ii) −1

(c) (i) −1 (ii) $\frac{5}{3}$

(d) (i) −1 (ii) $-\frac{1}{2}$

2. (a) (i) $\frac{2x}{y^2}$ (ii) $\frac{-2x^3}{3y}$

(b) (i) $\frac{y(8x-y)}{2x(y-2x)}$ (ii) $\frac{y}{2y-x}$

(c) (i) $\frac{1-2y}{2x-4y-1}$ (ii) $\frac{y}{2y-x}$

(d) (i) $\frac{y(2x-e^y)}{xye^y-4}$ (ii) $\frac{\cos x-3\sin y}{3x\cos y-2\sin y}$

3. (i) $(3,2),(-3,-2)$

(ii) $(\sqrt{2},4\sqrt{2}),\ (-\sqrt{2},-4\sqrt{2})$

4. (a) (i) $3\ln 3$ (ii) $25\ln 5$

(b) (i) $-4\ln 2$ (ii) $-3\ln 3$

(c) (i) $\frac{3\ln 2}{8}$ (ii) $4\ln 4$

(d) (i) $-3\ln 3$ (ii) $-\frac{\ln 5}{5}$

5. (c) $\frac{d}{dx}(\ln kx)=\frac{1}{x},\frac{d}{dx}(\ln x^n)=\frac{n}{x}$

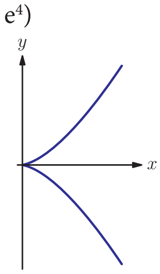
6. 4

7. $5x-2y=16$

8. $\frac{dy}{dx}=\frac{y2^y}{1-x2^y-\ln 2}$

9. $(2, e^4)$

10. (a)



(b) $y=3x-4$ (c) $(1\ -4)$

Exercise 18E

1. (a) (i) $\frac{-3}{\sqrt{1-9x^2}}$ (ii) $\frac{-2}{\sqrt{1-4x^2}}$

(b) (i) $\dfrac{2}{4+x^2}$ (ii) $\dfrac{10}{25+4x^2}$

(c) (i) $\arcsin x+\dfrac{x}{\sqrt{1-x^2}}$

(ii) $2x\arccos x-\dfrac{x^2}{\sqrt{1-x^2}}$

(d) (i) $\dfrac{2x}{1+\left(x^2+1\right)^2}$

(ii) $\dfrac{-2x}{\sqrt{1-\left(1-x^2\right)^2}}$

2. $-\dfrac{3}{\sqrt{35}}$

4. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1+\tan^2\left(\dfrac{1}{x}\right)}{x^2}$

5. (a) $\arcsin x+\dfrac{x}{\sqrt{1-x^2}}$

(b) $x\arcsin x+\sqrt{1-x^2}+c$

Mixed examination practice 18

Short questions

1. (a) $2x\arcsin x+\dfrac{x^2}{\sqrt{1-x^2}}$ (b) $\dfrac{e^y}{8y-xe^y}$

2. $\dfrac{2x}{\sqrt{1-\left(1-x^2\right)^2}}$

3. $\dfrac{16}{225}$

4. $y=\dfrac{14}{9}x+\dfrac{88}{9}$

5. $\dfrac{2\left(1-3x^4\right)}{\left(1+x^4\right)^2}$

6. $-\dfrac{1}{7}$

7. $\dfrac{5}{2}$

8. (b) $-\ln\dfrac{b}{c},\dfrac{a}{2b}$

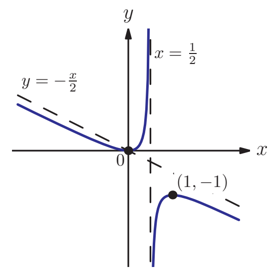
9. $\left(1,\dfrac{2\sqrt{e}-2}{\sqrt{e}}\right),\left(1,\dfrac{2\sqrt{e}+2}{\sqrt{e}}\right)$

Long questions

1. (a) $x=\dfrac{1}{2}$ (b) $(0,0),(1,-1)$

(c) $(0,0)$ local min $(1,-1)$ local max

(d)



2. (a) (ii) $\dfrac{(\ln 2)^2x^2-4x\ln 2+2}{2^x}$

(b) (i) $\dfrac{2}{\ln 2}$ (c) $\dfrac{2\pm\sqrt{2}}{\ln 2}$

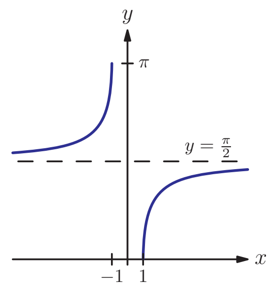
3. (c) $k=3, p=1$

4. (a) $(2,4),(-2,-4)$

(c) $(2,4)$ local max; $(-2,-4)$ local min

5. (a) $x\geq 1, x\leq -1$

(b)



(d) $\dfrac{1}{x\sqrt{x^2-1}}$

Chapter 19

Exercise 19A

1. (a) (i) $(x+3)^5+c$ (ii) $\dfrac{1}{6}(x-2)^6+c$

(b) (i) $\dfrac{1}{32}(4x-5)^8+c$ (ii) $2\left(\dfrac{1}{8}x+1\right)^4+c$

(c) (i) $-\dfrac{8}{7}\left(3-\dfrac{1}{2}x\right)^7+c$ (ii) $-\dfrac{1}{9}(4-x)^9+c$

(d) (i) $\dfrac{1}{3}(2x-1)^{\frac{3}{2}}+c$ (ii) $-\dfrac{4}{5}(2-5x)^{\frac{7}{4}}+c$

(e) (i) $4\left(2+\dfrac{x}{3}\right)^{\frac{3}{4}}+c$ (ii) $2(4-3x)^{-1}+c$

2. (a) (i) $e^{3x}+c$ (ii) $\frac{1}{2}e^{2x+5}+c$

(b) (i) $6e^{\frac{2x-1}{3}}+c$ (ii) $2e^{\frac{1}{2}x}+c$

(c) (i) $2e^{-3x}+c$ (ii) $-\frac{1}{4}e^{-4x}+c$

(d) (i) $8e^{-\frac{x}{4}}+c$ (ii) $-\frac{3}{2}e^{-\frac{2}{3}x}+c$

3. (a) (i) $\ln|x+4|+c$ (ii) $\ln|5x-2|+c$

(b) (i) $\frac{2}{3}\ln|3x+4|+c$ (ii) $-4\ln|2x-5|+c$

(c) (i) $\frac{3}{4}\ln|1-4x|+c$ (ii) $-\frac{1}{2}\ln|7-2x|+c$

(d) (i) $x+3\ln|5-x|+c$ (ii) $3x-\ln|3-x|+c$

4. (a) $\csc x+c$ (b) $\tan 3x+c$

(c) $\frac{1}{3}\cos(2-3x)+c$ (d) $-4\cot\left(\frac{1}{4}x\right)+c$

(e) $\frac{1}{2}\sin 4x+c$ (f) $2\sec\frac{x}{2}+c$

5. Both; $\frac{1}{3}\ln(3x)+c=\frac{1}{3}\ln x+\frac{1}{3}\ln 3+c=\frac{1}{3}\ln x+c'$

6. 0.492

Exercise 19B

1. (a) (i) $\frac{1}{8}(x^2+3)^4+c$ (ii) $\frac{1}{4}(x^2-1)^6+c$

(b) (i) $\frac{1}{15}(3x^2-15x+4)^5+c$

(ii) $\frac{1}{12}(x^3+3x^2-5)^4+c$

(c) (i) $\ln|x^2+3|+c$

(ii) $2\ln|x^3-6x+1|+c$

(d) (i) $-\frac{2}{9}\cos^6 3x+c$

(ii) $\frac{1}{8}\sin^4 2x+c$

(e) (i) $\frac{1}{2}e^{3x^2-1}+c$

(ii) $\frac{3}{2}e^{x^2}+c$

(f) (i) $\ln\sqrt{e^{2x+3}+4}+c$ (ii) $\ln\sqrt[4]{3+4\sin x}+c$

(g) (i) $4\tan^4 2x+c$ (ii) $4\sec^6\left(\frac{x}{4}\right)+$

(h) (i) $-\frac{1}{4}\csc x^4+c$

(ii) $-\ln\left(\sqrt{3+\cot 2x}\right)+c$

(i) (i) $-\sqrt{3-x^2}+c$

(ii) $-\frac{1}{3}\left(e^{-4x+1}\right)^{\frac{3}{2}}+c$

3. (a) (i) $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$

(ii) $\frac{2}{7}(x-2)^{7/2}+\frac{8}{5}(x-2)^{5/2}+$
$\frac{8}{3}(x-2)^{3/2}+c$

(b) (i) $\frac{2}{9}(x-5)^9+\frac{5}{4}(x-5)^8+c$

(ii) $\frac{1}{7}(x+3)^7-\frac{1}{2}(x+3)^6+c$

4. (a) (i) $\frac{1}{24}(2x-1)^6+\frac{1}{20}(2x-1)^5+c$

(ii) $\frac{1}{7}(3x+2)^7-\frac{1}{3}(3x+2)^6+c$

(b) (i) $\frac{2}{5}(x-3)^{\frac{5}{2}}+2(x-3)^{\frac{3}{2}}+c$

(ii) $\frac{2}{75}(5x-6)^{5/2}+$
$\frac{22}{75}(5x-6)^{3/2}+c$

(c) (i) $\frac{2}{5}(x-5)^{5/2}+\frac{20}{3}(x-5)^{3/2}+50$
$(x-5)^{1/2}+c$

(ii) $-\frac{1}{(2x-3)}-\frac{13}{2(2x-3)^2}+c$

5. (a) (i) $9-8\ln 2$

(ii) $12\ln\frac{5}{3}-5\frac{1}{15}$

(b) (i) $\ln\left(\frac{3}{2}\right)$ (ii) $\ln\left(\frac{3}{2}\right)$

(c) (i) $\frac{\pi}{18}$ (ii) $\frac{\pi}{12}$

6. e^5-e^{-1}

7. $\ln 8$

8. $\frac{2}{3}(x-2)^{3/2}+4(x-2)^{1/2}+c$

9. (b) $\ln|x^2+x+1|+c$

10. $\frac{1}{4}\tan(\ln(x^2))+c$

11. $-\frac{1}{4\sin^4 x}+c$

12. $2\sqrt{3}-2$

Exercise 19C

1. (a) $\frac{1}{3}\sec 3x+c$ (b) $-\cot x+c$

(c) $-\frac{1}{4}\cos 4x+c$

(d) $\frac{1}{2}(-3\cot 2x+\csc 2x)+c$

(e) $\sin x+\cos x+c$

2. (a) $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c$ (b) $-\frac{1}{\sin x}-\sin x+c$

(c) $-\frac{1}{4}e^{\cos 2x}+c$ (d) $\frac{1}{15}\tan^5 3x+c$

(e) $-\frac{1}{4}\sqrt{1+\cos 4x}+c$

3. (a) (i) $x+\frac{1}{2}\sin 2x+c$

(ii) $\frac{1}{2}\left(\frac{1}{6}\sin 6x+x\right)+c$

(b) (i) $4\tan\left(\frac{x}{2}\right)-2x+c$

(ii) $\frac{1}{3}\tan 3x-x+c$

4. (a) (i) $\frac{\pi}{2}$ (ii) $6\sqrt{3}-2\pi$

(b) (i) $1-\ln 2$ (ii) $\frac{9\pi}{8}-1$

5. They are all right, with different $+c$:

$\frac{1}{2}\sin^2 x=-\frac{1}{2}\cos^2 x+\frac{1}{2}=-\frac{1}{4}\cos 2x+\frac{1}{4}$

6. $\frac{1}{2}x-\frac{3}{4}\sin\left(\frac{2x}{3}\right)+c$

7. (b) $\frac{1}{2}\tan^2 x-\ln|\sec x|+c$

8. 3

9. (b) $-\frac{2}{3}\cos^3 x+\cos x+c$

10. 4

11. (b) (i) $z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}$

(ii) $a=1, b=-5, c=10$

(c) $\frac{1}{16}\left(-\frac{1}{10}\cos^5 2x+\frac{1}{3}\cos^3 2x-\frac{1}{2}\cos 2x\right)+c$

Exercise 19D

1. (a) (i) $\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}x\right)$

(ii) $\frac{1}{\sqrt{5}}\arctan\left(\sqrt{5}x\right)$

(b) (i) $\frac{1}{\sqrt{3}}\arcsin\sqrt{3}x+c$

(ii) $\frac{1}{2}\arcsin 2x+c$

(c) (i) $3\arctan\left(\frac{x}{3}\right)+c$

(ii) $\sqrt{10}\arctan\left(\frac{x}{\sqrt{10}}\right)+c$

(d) (i) $2\arcsin\left(\frac{x}{5}\right)+c$

(ii) $5\arcsin\left(\frac{x}{2}\right)+c$

2. (a) (i) $\arctan(x+2)+c$

(ii) $\arctan(x-3)+c$

(b) (i) $\arcsin(x-4)+c$

(ii) $\arcsin(x-1)+c$

(c) (i) $6\sqrt{2}\arctan\left(\frac{x+5}{\sqrt{2}}\right)+c$

(ii) $\frac{5}{2}\arcsin\left(\frac{2x+3}{3}\right)+c$

3. $\frac{\pi}{2}$

4. (a) $2(x+1)^2+9$

(b) $\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}(x+1)}{3}\right)+c$

5. (a) $2^2-3(x-1)^2$

(b) $\frac{\sqrt{3}\pi}{9}$

7. $\frac{\pi}{12}$

Exercise 19E

1. (a) $\frac{1}{6}(2x-3)^3+c$ (b) $-\frac{1}{5}\ln|2-5x|+c$

2. (a) $\frac{5}{2}\ln(x^2+6)+\frac{1}{\sqrt{6}}\arctan\left(\frac{x}{\sqrt{6}}\right)+c$

(b) $-\sqrt{4-x^2}-3\arcsin\left(\frac{x}{2}\right)+c$

(c) $4\ln|x^2+8x+25|-3\arctan\left(\frac{x+4}{3}\right)+c$

(d) $-\sqrt{-x^2+6x-7}-2\arcsin\left(\frac{x-3}{\sqrt{2}}\right)+c$

3. (a) (i) $x-\ln|x+2|+c$ (ii) $2x+5\ln|x-1|+c$

(b) (i) $\frac{1}{2}x^2+3x+11\ln|x-3|+c$ (ii) $\frac{1}{2}x^2-3x+14\ln|x+5|+c$

(c) $x+\frac{5}{2}\ln|x^2+3|-\frac{2}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+c$

4. (a) $\frac{1}{x-2}-\frac{1}{x+3}$ (b) $\ln\left|\frac{x-2}{x+3}\right|+c$

5. $\frac{\pi}{4}$

6. (a) $\frac{2}{x+1}+\frac{2}{1-x}$ (b) 8

7. $-4\sqrt{1-x^2}+5\arcsin x+c$

8. (a) $2(x-2)^2+9$

(b) $\frac{1}{2}\ln|(x-2)^2+\frac{9}{2}|+2\sqrt{2}\arctan\left(\frac{\sqrt{2}(x-2)}{3}\right)+c$

Exercise 19F

1. (a) (i) $\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$

(ii) $-2x\cos\left(\frac{x}{2}\right)+4\sin\left(\frac{x}{2}\right)+c$

(b) (i) $-2xe^{-2x}+e^{-2x}+c$

(ii) $\frac{1}{4}xe^{4x}-\frac{1}{16}e^{4x}+c$

(c) (i) $x^2\ln 5x-\frac{1}{2}x^2+c$

(ii) $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c$

(d) (i) $\frac{1}{3}x^2\sin 3x+\frac{2}{9}x\cos 3x-\frac{2}{27}\sin 3x+c$

(ii) $-x^2\cos x+2x\sin x+2\cos x+c$

(e) $x^2e^{\frac{x}{4}}-8xe^{\frac{x}{4}}-32e^{\frac{x}{4}}+c$

(f) $-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+c$

(g) $x(\ln x)^2-2x\ln x+2x+c$

2. (a) $x\arctan x-\frac{1}{2}\ln(1+x^2)+c$

(b) $x\ln(2x+1)-x+\frac{1}{2}\ln(2x+1)+c$

3. (a) $\frac{\pi}{2}-1$ (b) $\frac{1}{2}(1-\ln 2)$

(c) $\frac{1}{2}(1-\ln 2)$

4. No answers required

5. $-\frac{2}{3}xe^{-3x}-\frac{2}{9}e^{-3x}+c$

6. $\frac{5e^6+1}{36}$

7. (a) Proof

(b) $x\tan x-\ln|\sec x|+c$

8. 0.360 [7]

9. e^2+1

Mixed examination practice 19

Short questions

1. $\frac{\pi}{2}$

2. $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

3. 6.36

4. $\frac{\sqrt{3}}{4}$

5. (a) $-\frac{1}{3}\ln|1-3x| + c$

   (b) $-\frac{1}{2}(2x+3)^{-1} + c$

6. $x\ln x - x + c$

7. (a) $\frac{e^{-2x}}{e^{-2x}+3}$

   (b) $\frac{1}{2}\ln\left|\frac{1}{e^{-2x}+3}\right| + c$

8. $3\ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right) + c$

9. (b) $\ln\frac{7}{4}$

10. $\ln(\ln|x|) + c$

11. $\frac{8}{3}\left(\frac{1}{2}x-1\right)^{3/2} + 8\left(\frac{1}{2}x-1\right)^{1/2} + c$

12. $3 - 3\ln\left(\frac{7}{4}\right)$

13. $x\arctan x - \frac{1}{2}\ln|1+x^2| + c$

14. $\frac{e-1}{e+1}$

Long questions

1. (a) A = 2 B = 2

   (b) $2\ln|x+2| - \ln|x^2+1| + \arctan x + c$ (c) $\frac{4\pi}{3} - \frac{3}{2}$

2. (a) $x + c$ (b) $\ln|\sin x + \cos x| + c$

   (c) $\frac{1}{2}(x - \ln|\sin x + \cos x|) + c$

3. (a) $\frac{1}{2}(1+t^2)$ (c) 1

4. (c) (i) $\frac{1}{32}\sin 4a + \frac{1}{4}\sin 2a + \frac{3}{8}a$

   (ii) 2.96

Chapter 20

Exercise 20A

1. (a) (i) $144x^3$ (ii) $6x^2(x^3+1)$
   (b) (i) $-6x\sin(3x^2)$ (ii) $2x\sec^2(x^2+1)$

2. (a) (i) 50 (ii) −12
   (b) (i) −6 (ii) 1
   (c) (i) $\pm\frac{1}{3}$ (ii) −2

3. (a) (i) 45 (ii) 176
   (b) (i) 0.24 (ii) 0.006 67

4. $113\ \text{cm}^2\text{s}^{-1}$

5. $2\ \text{cms}^{-1}$

6. $2\ \text{cms}^{-1}$

7. 8.92 cm

8. $19.1\ \text{kmh}^{-1}$

Exercise 20B

1. (a) (i) $v = -8e^{-2t}$, $a = 16e^{-2t}$
   (ii) $v = -6e^{3t}$, $a = -18e^{3t}$
   (b) (i) $v = \frac{5}{2}\cos\left(\frac{t}{2}\right)$, $a = -\frac{5}{4}\sin\left(\frac{t}{2}\right)$
   (ii) $v = 6\sin(2t)$, $a = 12\cos(2t)$

2. (a) (i) $t^3 - t$ (ii) $\frac{1}{2}t - \frac{1}{8}t^4$
   (b) (i) $2 - 2e^{-t}$ (ii) $t + \frac{1}{2}e^{2t} - \frac{1}{2}$
   (c) (i) $3\ln\left(\frac{t+2}{2}\right)$ (ii) $3t - \ln(t+1)$

3. (a) (i) 1.73 (ii) 3.16
   (b) (i) 2.22 (ii) 0.746
   (c) (i) 3.23 (ii) 7.06

4. (a) (i) $-\frac{2}{81}$ (ii) $\frac{9}{2}$

(b) (i) $\frac{6}{125}$ (ii) $60e^{\frac{-10}{3}}$

6. (a) $-\frac{t^2}{(t^2+1)^2}$ (b) $\frac{1}{2}\ln(26)=1.63$ (3SF)

7. 10.3 m

8. (a) 13.6 m (b) 16.4 m

9. 6.25

10. 0.0733

11. (a) 1.04 ms^{-1} (b) 0.319 ms^{-1}

Exercise 20C

1. (a) (i) 304.8π (ii) $\frac{18}{7}\pi$

(b) (i) $\left(\frac{e^4}{4}+e^2-\frac{1}{4}\right)\pi$ (ii) $\left(\frac{5}{2}-4e^{-2}-\frac{e^{-4}}{2}\right)\pi$

(c) (i) 2π (ii) 2π

2. (a) (i) 101 (ii) 134
(b) (i) 12.6 (ii) 45.7
(c) (i) 3.59 (ii) 0.771

3. 19.0

4. π

5. $\pi(e^4-1)$

6. (a) (i) $y=-\frac{h}{r}x+h$

7. 7.17

8. $\frac{243\pi}{10}$

9. $\sqrt[3]{\frac{4}{3}}$

10. 1.02

11. (a) $(1,4), (9,12)$ (b) 154

12. (a) $(0,3), (4,19)$ (b) 630

13. $\frac{5e^2-3}{6}\pi$

Exercise 20D

1. (a) (i) 2 (ii) $\frac{49}{12}$

(b) (i) $2\sqrt{3}$ (ii) $4\sqrt{2}$

(c) (i) $4\sqrt{2}$ (ii) $\frac{8}{3}$

2. (a) $210x-2x^2$ (b) $x=52.5, y=105$

3. (a) $x(12-2x)^2$ (b) $x=2$

4. 48

5. $\left(\frac{2\sqrt{3}}{3},\frac{8}{3}\right)$

6. (a) (i) $(\pi-x,\sin x)$ (ii) $(\pi-2x)\sin x$
(c) 1.12

7. 889.5 cm^3

8. 788.3 cm^3

9. (a) 3 and 3 (b) 0 and 6

10. (b) $r=5.56, h=7.86$

13. $\left(\sqrt{7/2},\frac{7}{2}\right)$

Mixed examination practice 20

Short questions

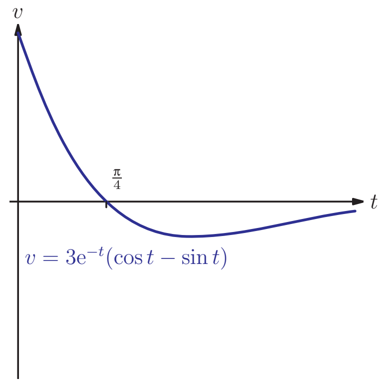
1. $\frac{\pi a^5}{30}$

2. (a) 6.25 m (b) 14.25 m

3. $x=y=4$

4. 20.9 cm

5. (a) −0.169 (b)



6. (a) $100h-\frac{1}{2}h^2$ (c) $k=6$

7. 240 km/h

8. (b) $\frac{2}{\sqrt{3}}$

Long questions

1. (a) (i) $4x$ (ii) $2\pi y$

   (b) $2-\frac{\pi y}{2}$ (d) 44.0%

2. (a) $(0,12)$ and $(4.5,-3.75)$ (b) 10.125

   (c) (i) $\pi\int_0^{4.5}(9x-2x^2)^2\,dx$ (ii) 773

3. (b) $\frac{\sqrt{5}}{2}$

4. (b) (i) $0.5\ \text{ms}^{-2}$ (ii) $3(t+1)\ln(t+1)-3t$

   (iii) 17.3 m

   (c) (i) 3.49 (ii) $8\ \text{ms}^{-1}$

   (iii) 7.47 s

5. (b) $h=30-2t$ (c) 25 – 6t

   (d) 13cms^{-1}

6. (b) $\frac{e+1}{2}$

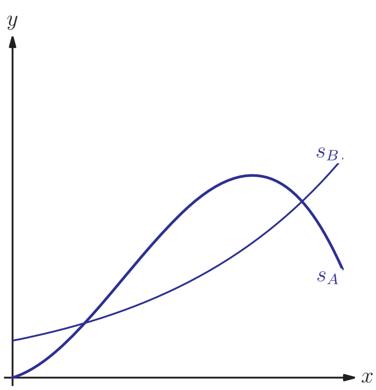
7. (a) $6\ \text{ms}^{-1}$ $\frac{d^2v}{dt^2}=-1$, hence local maximum

   (b) $0.445\ \text{ms}^{-2}$

   (c) $s_A=-\frac{1}{6}t^3+\frac{3}{2}t^2+\frac{3}{2}t$,

   $s_B=5e^{0.2t}$

   (d) (i)



   (ii) 1.95, 7.81

8. (b) (ii) 7.69 km

9. (a) (i) $AX=\frac{9}{\sin\theta}, BX=\frac{\sqrt{3}}{\cos\theta}$

   (ii) $8\sqrt{3}$

   (b) $8\sqrt{3}$

Chapter 21

Exercise 21A

1. (a) Discrete (b) Discrete
   (c) Continuous (d) Continuous
   (e) Discrete (f) Discrete

2. (a) Only shoppers' opinions found.
   (b) Truants would not be there.
   (c) People who partake in internet surveys may not be representative.
   (d) If you live in a big household your chances of being sampled are lower than if you live by yourself.

Exercise 21B

1. (a) (i) 7.23 (ii) 6.57

   (b) (i) 11.1 (ii) 35.3

2. (i) $\frac{\sqrt{56}}{5}$ (ii) $\frac{\sqrt{68}}{3}$

3. (a) 12 (b) 2.92

4. 18

5. 4.96

6. 11.7

7. 118 888.2

Exercise 21C

1. (a) $\bar{x}=1, s_n=\frac{2}{\sqrt{15}}$ (b) $\bar{x}=0, s_n=\frac{\sqrt{28}}{5}$

2. (a) $\bar{x}=12.1, s_n=1.90$

   (b) $\bar{x}=0.263, s_n=0.137$

3. (a) 17, 21 (b) 17, 20
   (c) 16.5, 20.5 (d) 16, 20

4. (a) (i) $\bar{x}=26.1, s_n=20.4$ (ii) $\bar{x}=253, s_n=151$

   (b) (i) $\bar{x}=6.38, s_n=5.23$

ANSWER HINT (4)

the first group starts at 0, not –0.5!

   (ii) $\bar{x}=102, s_n=5.78$

   (c) (i) $\bar{x}=6.58, s_n=5.11$ (ii) $\bar{x}=15.3, s_n=9.85$

5. (a) 1.5 (b) 1.84

6. $p = 45, q = 5$ and $p = 10, q = 40$

Mixed examination practice 21

Short questions

1. (a) 4.08 (b) 2.97

2. (a) $x = 6, y = 4$ (b) 1367

3. $\frac{79}{400}, \frac{\sqrt{390}}{400}$

4. $2\sqrt{k}$

Long questions

1. (a) 11 (b) 8,5,4
   (c) 11.8

2. (a) The first group starts at 39.5
   (b) 18, 6, 0, 1
   (c) 53.7, 8.50
   (d) Natural variation

3. (a) $\frac{a+2b}{a+b}$ (c) 1.25, 0.433

Chapter 22

Exercise 22A

1. (a) 1,2,3,4,5,6
   (b) RED, RDE, ERD, EDR, DRE, DER
   (c) BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGB
   (d) 1,2,3,4

2. (a) (i) $\frac{1}{2}$ (ii) $\frac{1}{4}$
   (b) (i) $\frac{1}{13}$ (ii) $\frac{4}{13}$
   (c) (i) $\frac{9}{26}$ (ii) $\frac{1}{13}$
   (d) (i) $\frac{3}{4}$ (ii) $\frac{9}{13}$
   (e) (i) $\frac{25}{52}$ (ii) $\frac{11}{13}$
   (f) (i) $\frac{5}{26}$ (ii) $\frac{3}{13}$

3. (a) (i) $\frac{2}{5}$ (ii) $\frac{1}{3}$
   (b) (i) $\frac{1}{3}$ (ii) $\frac{3}{5}$
   (c) (i) $\frac{3}{5}$ (ii) $\frac{2}{3}$
   (d) (i) 0 (ii) 1
   (e) (i) $\frac{4}{15}$ (ii) $\frac{2}{5}$
   (f) (i) $\frac{2}{5}$ (ii) $\frac{11}{15}$
   (g) (i) 0 (ii) 0

5. (a) $\frac{5}{36}$ (b) $\frac{11}{18}$
   (c) $\frac{1}{6}$ (d) $\frac{7}{36}$
   (e) $\frac{1}{3}$ (f) $\frac{7}{18}$

6. (a) $\frac{1}{8}$ (b) $\frac{19}{32}$
   (c) $\frac{5}{32}$ (d) $\frac{7}{32}$
   (e) $\frac{7}{16}$ (f) $\frac{1}{2}$

7. 115

8. $\frac{5}{108}$

Exercise 22B

1. (a) (i) 0.5 (ii) 1
   (b) (i) $\frac{1}{6}$ (ii) 0.05
   (c) (i) $\frac{4}{15}$ (ii) 0.2
   (d) (i) 0.7 (ii) 0.11

2. (a) (i) $\frac{7}{20}$ (ii) all (100%)
   (b) (i) 27.5% (ii) $\frac{7}{30}$
   (c) (i) 55% (ii) 18.3%

3. (a) $P(x > 4)$ (b) $P(y \le 3)$
   (c) O (d) $P(a \in \mathbb{R})$
   (e) P(fruit) (f) P(apple)
   (g) P(multiple of 4) (h) P(rectangle)
   (i) P(blue) (j) $P(\text{blue} \cap \text{red})$

4. 5%

5. 0.4

6. 0.3

7. (a) 0.166 (b) 0.041

Exercise 22C

1. (a) (i) 0.12 (ii) 0
   (b) (i) 0.24 (ii) 0.24
   (c) (i) $\frac{13}{20}$ (ii) $\frac{3}{4}$
2. 0.165
3. $\frac{1}{6}$
4. (a) $\frac{19}{27}$ (b) $\frac{25}{27}$
5. 0.048
6. 0.627
7. 15 or 21

Exercise 22D

1. (a) (i) 0.21 (ii) $\frac{1}{15}$
   (b) (i) $\frac{15}{28}$ (ii) 0.556
   (c) (i) 0.496 (ii) 0.4
   (d) (i) 0.333 (ii) 0.444
2. (a) (i) Yes (ii) No
   (b) (i) Yes (ii) No
3. (a) 0.3 (b) 0.54
4. (a) 62.6% (b) 2.56%
   (c) 97.4%
5. (a) 0.410 (b) 0.684

ANSWER HINT(5)

Did you consider finding the complement?

6. 14/15

Exercise 22E

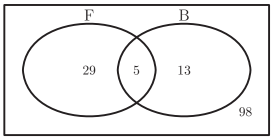
1. $\frac{1}{8}$
2. $\frac{4}{7}$
3. (a) 0.359 (b) 0.375
4. (a) 0.0476 (b) 0.119
5. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$
6. (a) 1.03×10^{-4} (b) 0.0102

Exercise 22F

1. (a) P(prime $\cap$ odd)
   (b) P(Senegal $\cup$ Taiwan)
   (c) P(French|IB)
   (d) P(heart|red)
   (e) P(lives in Munich | German)
   (f) P(not black $\cap$ not white)
   (g) P(potato|not cabbage)
   (h) P(red|red $\cup$ blue)
2. (a) (i) $\frac{2}{3}$ (ii) $\frac{15}{28}$
   (b) (i) $\frac{5}{14}$ (ii) $\frac{5}{6}$
3. (a) (i) 0.1 (b) $\frac{1}{6}$
4. (a) $\frac{1}{3}$ (b) $\frac{7}{15}$
5. $\frac{1}{6}$

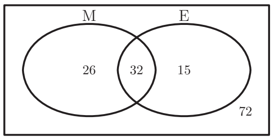
Exercise 22G

1. (a)



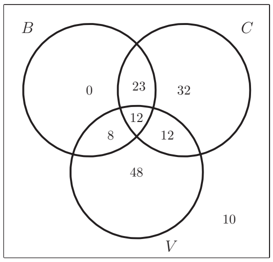
   (b) 98 (c) $\frac{18}{145}$
   (d) $\frac{5}{34}$

2. (a)



   (b) 32 (c) $\frac{16}{29}$

3. (a)

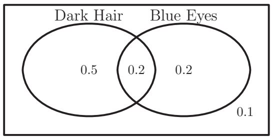


(b) 0 (c) 79

(d) $\frac{16}{29}$ (e) $\frac{3}{5}$

(f) $\frac{11}{29}$

4. (a)



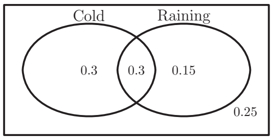
(b) 0.1 (c) $\frac{2}{7}$

(d) $\frac{2}{3}$

(e) No: $P(B \cap D) = 0.2, P(B) \times P(D) = 0.28$

5. (a) 0.3

(b)



(c) $\frac{1}{3}$ (d) $\frac{3}{8}$

(e) No: $P(C \cap R) = 0.3, P(C) \times P(R) = 0.27$

Exercise 22H

1. (a) 0.621 (b) 0.839

2. (a) 0.853 (b) 0.156

3. (a) 0.45 (b) 0.205

4. (a) $\frac{2}{3}$ (b) $\frac{4}{5}$

5. $\frac{4}{9}$

6. $\frac{16}{41}$

7. $\frac{10}{11}$

8. 0.02

9. $\frac{15}{47}$

10. $\frac{1}{3}$

11. 0.5

12. 0.0277

13. $\frac{1}{3}$

Mixed examination practice 22

Short questions

1. 0.320

2. 0.8

3. 0.1111

4. (a) $\frac{1}{120}$ (b) $\frac{1}{20}$

5. $\frac{1}{3}$

Long questions

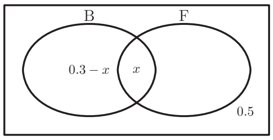
1. (a) $\frac{14}{95}$ (c) 0.138

(d) 0.356

2. (a) $0 \le P(X) \le 1$

(b) $P(A) - P(B|A)P(A)$

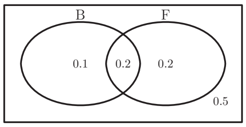
3. (a)



(b) 0.2

(c) 0.2

(d)



, 0.1

(e) $\frac{4}{13}$

Chapter 23

Exercise 23A

1. (a)

w	0	1	2	3	4
$P(W = w)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(b)

d	0	1	2	3	4	5
$P(D=d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

(c)

x	1	2	3	4	6
$P(X=x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(d)

g	1	2	3
$P(G=g)$	$\frac{6}{27}$	$\frac{36}{72}$	$\frac{30}{72}$

(e)

c	1	2	3	4
$C=c$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{27}{64}$

(f)

x	1	2	3	4	6	8	9	12	16
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

2. (a) (i) $\frac{1}{8}$ (ii) 0
(b) (i) $\frac{1}{10}$ (ii) $\frac{12}{25}$
(c) (i) $\frac{9}{10}$ (ii) $\frac{1}{10}$

3. 0.207

Exercise 23B

1. (a) (i) $E(X)=2$, med $=2$, $Var(X)=1$
(ii) $E(X)=9$, med $=9$, $Var(X)=4.2$
(b) (i) $E(X)=2.57$, med $=3$, $Var(X)=0.388$
(ii) $E(X)=3$, med $=2.5$, $Var(X)=2$

2. (b) 4.4

3. $k=\frac{40}{3}$, med = 3.5

4. (b) $\frac{32}{18}$

5. (a) $\frac{1}{10}$ (b) 2

6. $E(X)=\frac{13}{6}$, $Var(X)=\frac{65}{36}$

7. (a) $p=0.5, q=0.2$ (b) 0.35

8. 40

9. (a)

Profit ($)	$-n$	$1-n$	$2n$	$3n$
Probability	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

(b) $0.82

Exercise 23C

1. (a) (i) 0.147 (ii) 0.0459
(b) (i) 0.944 (ii) 0.797
(c) (i) 0.0563 (ii) 0.0104
(d) (i) 0.990 (ii) 0.797
(e) (i) 0.203 (ii) 0.832
(f) (i) 0.0562 (ii) 0.776

2. (a) (i) $\frac{5}{32}$ (ii) $\frac{1}{32}$
(b) (i) $\frac{31}{32}$ (ii) $\frac{6}{32}$
(c) (i) $\frac{1}{32}$ (ii) $\frac{26}{32}$

3. (a) (i) $E(X)=10, \sigma(X)=3$
(ii) $E(X)=8$, $\sigma(X)=2$
(b) (i) $E(X)=4.5, \sigma(X)=1.77$
(ii) $E(X)=7, \sigma(X)=2.13$
(c) (i) $E(X)=\frac{n-1}{n}, \sigma(X)=\left(\frac{n-1}{n}\right)$
(ii) $E(X)=2$, $\sigma(X)=\sqrt{\frac{2(n-2)}{n}}$

4. (a) 0.238
(b) 0.181
(c) 0.5

5. (a) Probability is not constant (effectively sampling without replacement)
(b) 0.103

6. (a) 2700 (b) 30

7. (a) 0.0231 (b) 2
(c) 1.22 (d) 0.0273

8. (a) 0.108
(b) 0.0267
(c) The probability that a person going to the doctor has the virus is the same as for the whole country.
OR
Doctors' patients have colds independently of their chance of visiting.

9. (a) 4.8 (b) 5

10. The second one (0.0165>0.0154)

11. (a) $\frac{2}{9}(n^2 - n) \times 0.6^n$
(b) 10

12. $p = 0.56, n = 30$

13. 0.14 or 0.20

14. $p = \frac{2}{3}$, $n = 18$

15. 0.560 or 0.891

16. $\frac{1}{5}$ or $\frac{4}{5}$

Exercise 23D

1. (a) (i) Po(36) (ii) Po(9)
(b) (i) Po(2.4) (ii) Po(120)
(c) (i) B(10,0.04) (ii) B(20,0.96)
(d) (i) Po(3.6) (ii) Po(48)

2. (a) (i) 0.180 (ii) 0.271
(b) (i) 0.946 (ii) 0.592
(c) (i) 0.201 (ii) 0.729
(d) (i) 0.933 (ii) 0.981
(e) (i) 0.215 (ii) 0.525

3.

x	0	1	2	3	4	>4
$P(X = x)$	0.183	0.311	0.264	0.150	0.0636	0.0296

4. 0.0116

5. (a) 0.0595 (b) 0.0548

6. (a) 2.5 (b) 0.0821

7. (a) 0.298 (b) 0.973

8. (a) 0.616 (b) 0.351
(c) 0.825 (d) 0.570

9. (a) 0.189 (b) 0.372

10. (a) 0.0537 (b) 0.321

11. (a) 0.511 (b) 0.0935

12. (a) $m = 4.5914$ (b) 0.270

13. 0.430

14. (a) 0.161,0.0614 (b) 2.76×10^{-6}
(c) There are other ways to get 42 emails in a week than 6 every day.

15. (a) 0.475 (b) 1
(c) 0.00413

16. (a) 0.273 (b) 0.143
(c) 201

17. (a) 0.219 (b) 4
(c) 2.8 (d) 7

18. $1 + \sqrt{3}$

19. (b) 7

Mixed examination practice 23

Short questions

1. 0.261

2. 55

3. (a) 4.8 (b) 0.0834

4. (a) 0.175 (b) 0.0195

5. (a) 0.232 (b) 0.894
(c) 0.096

6. (a) 0.0412 (b) 2.41×10^{-5}

7. (a) 0.0296 (b) 5.02×10^{-4}

8. (a) 1.19 (b) 0.259, 2.54
(c) $w = \lambda - 1$

9. (a) 15 (b) 0.273

10. (a) 0.804

Long questions

1. (a) 2.5 (b) 0.0162
(c) 0.756 (d) 2
(e) It is constant, and equal to 0.25.

2. (a) 0.362 (b) 0.279
(c) 0.659 (d) 0.0464
(e) 1.68×10^6 litres (f) 0.247
(g) 0.641

3. (a) 0.460 (b) $1-(0.95)^n$
(c) 32

4. (a) (i) $\frac{5}{36}$ (ii) $\frac{25}{216}$
(iii) $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2n-2}$

(c) $\frac{5}{11}$ (d) 0.432

5. (a) (i) 0.547 (ii) 0.0894

(b) 3

6. (a) (i) $\frac{1}{9}$ (ii) $\frac{1}{81}$

(b) (i) 0.113 (ii) 0.444

(c) (ii)

x	1	2	3	4	5	6
$P(x=x)$	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{65}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$	$\frac{671}{1296}$

(iii) 5.24

7. (a) (i) 0.214 (ii) 0.190

(b) (i) 0.209 (ii) 0.818

(c) (i) 0.322

Chapter24

Exercise 24A

1. (a) (i) $\frac{4}{65}$ (ii) $\frac{3}{14}$

(b) (i) $-\frac{2}{3}$ (ii) -1.3

(c) (i) -0.797 (ii) 0.210, 1.38

(d) (i) $\frac{-1\pm\sqrt{5}}{2}$ (ii) 0.582

(e) (i) $\sqrt{2}$ (ii) $\frac{1\pm\sqrt{5}}{2}$

(f) (i) -0.418 (ii) 0.754

(g) (i) $3^{\frac{1}{4}}$ (ii) $\sqrt{\frac{2}{3}}$

(h) (i) 51 100 (ii) 0.277

(i) (i) No such k (ii) 0

2. (a) (i) 0.48 (ii) 0.75

(b) (i) $\frac{\sqrt{3}-\sqrt{2}}{2}$ (ii) 0.5

(c) (i) 0.301 (ii) 0.477

3. (a) (i) 0.632 (ii) 0.949

(b) (i) 1.26 (ii) 2.83

(c) (i) 0.4 (ii) 3.5

4. (a) 0.0968

(b) 370

5. e^{-6}

6. 0.399

7. (a) $b=e^k$ (b) $a=\frac{2(e^k-1)}{e^k+1}$

8. 0.560

Exercise 24B

1.

	$E(X)$	Median	Mode	$Var(X)$
(a) (i)	$\frac{1}{3}$	0.293	0	$\frac{1}{18}$
(a) (ii)	$\frac{8}{3}$	$\sqrt{8}$	8	$\frac{8}{9}$
(b) (i)	3.91	$\sqrt{10}$	1	6.22
(b) (ii)	1.39	$\frac{4}{3}$	1	0.0782
(c) (i)	0.571	$\frac{\pi}{6}$	0	0.0142
(c) (ii)	0.386	0.405	ln 2	0.0391
(d) (i)	1.5	1.26	1	0.75
(d) (ii)	1.33	1.19	1	0.222

2. (a) (i) 1.21 (ii) 11

(b) (i) 2.82 (ii) 4.18

3. (a) 1.44 (b) 2

4. (a) $\frac{3}{973}$ (b) 7.65

5. (b) $\frac{1}{k}$

6. (b) $\frac{5}{\sqrt{3}}$

7. 0

Exercise 24C

1. (a) (i) 0.885 (ii) 0.212

(b) (i) 0.401 (ii) 0.878

(c) (i) 0.743 (ii) 0.191

(d) (i) 0.807 (ii) 0.748

(e) (i) 0.997 (ii) 0.055

2. (a) (i) 0.5 (ii) 1

(b) (i) -1.67 (ii) -0.4

3. (a) (i) $P(Z<1.6)$ (ii) $P(Z<1.28)$

(b) (i) $P(Z\geq -0.68)$ (ii) $P(Z\geq -2.96)$

(c) (i) $P(-1.4 < Z < 0.2)$

(ii) $P(-2.36 \le Z \le -0.2)$

4. (a) 1.16 (b) 0.123

5. (a) 0.278

(b) (i) 0.127 (ii) 0.334

6. 1547

7. (a) 0.160 (b) 0.171

(c) 0.727

8. (a) 0.707 (b) 0.663

9. (a) 67.3% (b) 0.314

10. (a) 0.952 (b) 0.838

11. (a) 0.841 (b) 0.811

12. 0.639

13. 0.0228

14. $1-k$

Exercise 24D

1. (a) (i) 19.9 (ii) 13.3

(b) 32.4 (ii) 37.3

(c) 2.34 (ii) 4.44

2. (a) (i) 0.32 (ii) 7.32

(b) (i) 43.4 (ii) 15.3

3. (a) (i) 8.91, 2.27 (ii) 148, 111

(b) (i) –0.200, 1.19 (ii) 870, 202

4. 141

5. 3.61 kg

6. 1.97 cm

7. (a) 1.29 mm, 0.554 mm

(b) 1.36 mm, 0.764 mm

8. 1.18 hours

9. (a) 0.691 (b) 39.7

(c) 0.240

10. 99.7%

11. 11.2°

12. (a) 1 (b) 1.48

13. 0

14. (a) 0.0853, (b) 1.46 kN

15. $Z = \mu + \sigma\,\Phi^{-1}(x)$

Mixed examination practice 24

Short questions

1. (a) 2 (b) 0.0556

2. (a) 6.68% (b) 62

3. 31.4, 4.52

4. $a = \frac{3}{32},\ b = -\frac{1}{32}$

5. 0.149

Long questions

1. (a) 0.0192 (b) 3, 1

(c) 0.1808 (d) 1.10

2. (b) $\frac{e}{4} + e^{\frac{1}{4}} - e^{\frac{1}{2}}$

(c) $\frac{e}{2} - 1,\ 1 + \frac{e}{3} - \frac{e^2}{4}$ (d) 0.0290

(e) 0.0243 (f) 0.179

3. (a) (i) $E(X) = \frac{1}{12}\int_0^2 x(8x - x^3)\,dx$

(ii) $\frac{56}{45}$

(b) (ii) $\sqrt{8-\sqrt{40}}$ (c) $\sqrt{\frac{8}{3}}$

4. (a) 953 Pesos (b) 0.0578

ANSWER HINT(4)

If you got 0.0577 for (b) you probably used a rounded value for your standard deviation. This would cost you a mark!

(c) 0.193 (d) 0.784

(e) Probably skewed

Chapter 25

Exercise 25F

7. $N = 7$

Mixed examination practice 25

Short questions

6. 0

Long questions

1. (c) $2\sqrt{2}, 3\frac{\pi}{4}$

(d) $1+\mathrm{i}, -\left(\frac{1+\sqrt{3}}{2}+\frac{1-\sqrt{3}}{2}\mathrm{i}\right), -\left(\frac{1-\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}\mathrm{i}\right)$

3 (b) 12

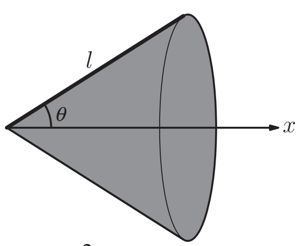
Chapter 26

Short questions

1. $\frac{5}{36}$

2. 4

3.



4. $a = \frac{3}{256}$ $b = 8$

5. $\frac{1}{3}x^3 + x^2 + c$

6. $\frac{a(r^n - 1)}{n(r-1)}$

7. $a = 4, b = -1$ or $a = -4, b = 3$

8. $1.02 < x < 1.57$ or $2.59 < x < 3.14$

9. $\frac{1}{2}$

10. $\frac{8}{3}$

11. $\frac{e}{24}$

12. $n = \frac{1+\sqrt{1+8k}}{2}$

13. Translation by $\begin{pmatrix} \frac{\pi}{2} \\ 1 \end{pmatrix}$

14. $\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

15. (a) $\frac{1}{k}$ (b) $\begin{pmatrix} 0 \\ \ln k \end{pmatrix}$

16. (a) $\frac{1023}{512}$ (b) $-45\ln 2$

17. (a) $\frac{\pi}{2}$ (b) $e^{i}\frac{\pi}{2}$

(c) $e^{-}\frac{\pi}{2}$

18. $x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

20. (a) $\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$ (b) $\lambda = -6, 1$

21. 5.10

22. $\frac{3}{64}$

23. (b) $cx^2 + bx + a = 0$

24. (a) $a = 1, b = 5$ (b) $x = -6, 1, -1, -4$

25. $x = 1.61$

26. $-0.618 < x < 1.62$

27. $\sqrt{13}$

29. $f = \frac{1}{6}\left(\frac{5}{6}\right)^{r-1}$

30. $\ln 2$

31. (a) $2\sin Bx\cos Ax$

(b) $\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x$

32. $\frac{2l^3}{9\sqrt{3}}$

33. (a) $x = \sin y$ (b) $\cos y$

34. (b) $\left(\frac{\pi}{6},2\right)$

35. (a) $\phi=\pi-2\theta$

(b) $4\cos^2\theta$

36. (a) $\ln\left(\frac{x-3}{x}\right)$ (b) $\frac{3}{1-e^x}$

38. (a) $e^{-\mu}\left(\mu+\frac{\mu^2}{2}\right)$

(b) $\sqrt{2}$

39. (a) $\frac{1}{k}e^{kx}+c$

(c) $\frac{1}{10}e^x\sin 3x-\frac{3}{10}e^x\cos 3x+c$

40. **(a)** Vertical stretch, scale factor 2

(b) Vertical stretch, scale factor $\log_{10} e$

41. (a) $\frac{1}{y}\frac{dy}{dx}$

(b) $4\ln x-\ln(2+5x)-\frac{1}{2}\ln(x^2+1)$

(c) $\left(\frac{4}{x}-\frac{5}{2+5x}+\frac{x}{x^2+1}\right)\left(\frac{x^4}{(2+5x)\sqrt{x^2+1}}\right)$

42. (a) $\frac{1}{\sqrt{2}}(1+i)$

(b) $\ln 2+i\frac{\pi}{4}$

43. $\frac{1}{5}$

44. $\frac{1}{3\sqrt{3}}$

46. (b) 0

47. $\frac{\theta}{1-\cos\theta}$

48. 4^n

49. (a) $d=-\frac{1}{2}$

(b) $a=\frac{1}{2}, b=\frac{1}{\pi}, c=-\frac{1}{2}$

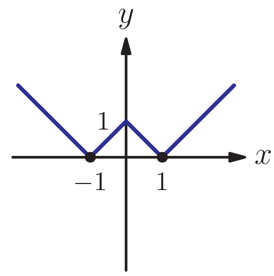
50. (a) $x=\ln\left(\frac{y\pm\sqrt{y^2-4}}{2}\right)$

51. (a) $z=\pm\left(\sqrt{\frac{\sqrt{2}-1}{2}}+i\sqrt{\frac{\sqrt{2}+1}{2}}\right)$

(b) $w=-i\pm\left(\sqrt{\frac{\sqrt{2}-1}{2}}+i\sqrt{\frac{\sqrt{2}+1}{2}}\right)$

52. (a) $(-y,x)$ (b) $y=f^{-1}(-x)$

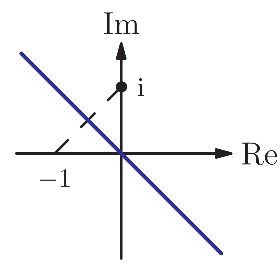
53. (a)



(b) 2

54. (a) $\sqrt{x^2+(y-1)^2}$

(b)



55. $\frac{1}{2}e^2-2e+\frac{5}{2}$

56. 3, 11

57. (a) $\frac{1-x^n}{1-x}$

58. $0, n$

59. (a) 5 + 5i (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{4}$

60. (a) $\text{Re}=0$, $\text{Im}=\frac{\sin 2\theta}{1-\cos 2\theta}$

62. (a) $(\sqrt{5})^n\cos\left(n\arctan\left(\frac{1}{2}\right)\right)$

63. (b) $\sum_{k=0}^{\infty}\binom{n}{k}x^{n-k}h^k$

64. $\frac{3}{2}$

Long questions

1. (b) (i) $-\frac{1+\sqrt{3}}{\sqrt{3}}$

(ii) $-\frac{2+\sqrt{3}}{\sqrt{3}}$

(c) 8.08

2. (a) $\frac{1}{5}$ (b) $\frac{4}{25}$

(c) $\frac{16}{125}$ (e) $\frac{4}{9}$

(f) $\frac{4}{5}$

3. (a) No (b) 6

(c) -7

4. (a) $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

(b) $-\frac{1}{2}e^{i\theta}$

(c) $\dfrac{1}{1+\frac{1}{2}e^{i\theta}}$

5. (a) $2+\cos 2x - \sec^2 x$

6. (a) $|\cos x| < 1$ for $0 < x < \pi$ (c) $\frac{1}{\sqrt{3}}$

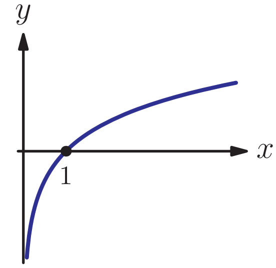
7. (a) $z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$

(c) $\frac{24}{45}$

8. (a) $\frac{1}{3}$ (b) $k=1$

(d) 0.095

9. (a)



(b) e

(c) $0 < k < \frac{1}{e}$

10. **(b)** $2(x-a)g(x) + (x-a)^2 g'(x) + m$

(d) $f(a)=0, f'(a)=0$

11. (b) $\frac{6}{17}$ (c) $\frac{5}{21}$

(d) 0.443

12. (a) $\dfrac{e^{i\theta}}{1-e^{2i\theta}}$ if $|e^{2i\theta}| < 1$

(c) $\theta = \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}$

14. (c) $\ln 2$

Index

Note: entries in **red bold** are key terms that are defined in the glossary on the CD-ROM. For an explanation in context, see the page number(s) also in **red bold**.

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